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UNIQUE SOLVABILITY OF SECOND ORDER NONLINEAR TOTALLY CHARACTERISTIC EQUATIONS

MICHAEL E. STA. BRIGIDA AND JOSE ERNIE C. LOPE*

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Abstract. We consider a second order singular nonlinear partial differential equation of the form $(t\partial_t)^2u = F(t,x,u,\partial_x u,\partial_x^2 u,t\partial_t u,t\partial_t\partial_x u)$, where F is assumed to be continuous in t and holomorphic with respect to the other variables. Under certain conditions, we prove that the equation has a unique solution that is continuous in t and holomorphic in x.

1. Introduction

Let $(t,x) \in \mathbb{R}\times\mathbb{C}$ and let $T_0, R_0, R_1 > 0$. Consider the singular partial differential equation

$$t\partial_t u = F(t,x,u,\partial_x u), \tag{1.1}$$

where the nonlinear function $F(t,x,u,v)$ has the following properties:

(A_1) $F(t,x,u,v)$ is continuous on $[0,T_0]\times D_{R_0}\times D_{R_1}^2$ and is holomorphic with respect to (x,u,v) for each fixed t;

(A_2) $F(0,x,0,0) \equiv 0$ on D_{R_0};

(A_3) $\partial_v F(0,x,0,0) = x\gamma(x)$, with $\gamma(0) \neq 0$.

Here, D_s is the open ball in $\mathbb{C}$ centered at the origin with radius $s > 0$. Bacani and Tahara [2] studied this equation as the partially holomorphic version of totally characteristic type equations that were introduced by Chen and Tahara [7, 8] in the late 1990s. We are using the term *partially holomorphic* in the sense of Miyake [15] because only continuity is assumed with respect to t. (In contrast, the term was also used in [18] and [23] when referring to functions that are of class C^∞ in one variable but are analytic with respect to the other variables.)

The assumptions on F allow us to write it as

$$F(t,x,u,v) = a(t,x) + \lambda(t,x)u + [b(t) + xc(t,x)]v + G(t,x,u,v)$$

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* Corresponding author.

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where $a(t,x) = F(t,x,0,0)$, $\lambda(t,x) = \partial_u F(t,x,0,0)$, $b(t)+xc(t,x) = \partial_v F(t,x,0,0)$, and $G(t,x,u,v)$ denotes all the nonlinear terms with respect to u and v. We will further write $\lambda(t,x) = \lambda(0,0)+\tilde{\lambda}(t,x)$ and $c(t,x) = \gamma(0)+\tilde{c}(t,x)$ and introduce the linear operator $\mathscr{L}_1 = t\partial_t - \lambda(0,0) - \gamma(0)x\partial_x$, thus allowing us to write (1.1) as:

$$\mathscr{L}_1 u = a(t,x)+\tilde{\lambda}(t,x)u+\tilde{d}(t,x)\partial_x u+G(t,x,u,\partial_x u). \tag{1.2}$$

From our assumptions, we know that $c(0,x) = \gamma(x)$ with $c(0,0) \neq 0$, the functions $a(t,x)$ and $c(t,x)$ are continuous on $[0,T_0]\times D_{R_0}$ and holomorphic in x for each fixed t, and the function $b(t)$ is continuous on $[0,T_0]$. Moreover, both $a(t,x)$ and $b(t)$ vanish at $t=0$, and so there exists a continuous increasing function $\psi : [0,T_0] \to [0,\infty)$ such that $\psi(0)=0$, and for any $t\in[0,T_0]$, for any $x\in D_{R_0}$, we have

$$|a(t,x)| \leqslant \psi(t) \quad\text{and}\quad |b(t)| \leqslant \psi(t). \tag{1.3}$$

Ona and Lope [17] proved the following unique solvability result in the class of partially holomorphic functions, which was a slight improvement of the existence and uniqueness theorem given in [2].

THEOREM 1. *If* $\operatorname{Re}\lambda(0,0)<0$ *and* $\operatorname{Re}\gamma(0)<0$*, then there exist* $T>0$ *and* $R>0$ *such that* (1.1) *has a solution* $u(t,x)$ *that is continuous on* $[0,T]\times D_R$*, holomorphic in* x *for each fixed* t*, and satisfies the estimates*

$$|u(t,x)| \leqslant C^*\psi(t) \quad \textit{and} \quad |\partial_x u(t,x)| \leqslant \frac{C^*}{R}\psi(t) \quad \textit{on} \quad [0,T]\times D_R,$$

for some $C^*>0$*. This solution is unique in the sense that if* $v(t,x)$ *is another solution of* (1.1) *having the same properties, then* $u(t,x)=v(t,x)$ *on* $[0,T^*]\times D_R$ *for some* $T^*\in(0,T]$.

The above theorem is also said to be of Nagumo type because it assumes holomorphy with respect to the 'space' variable x but only continuity with respect to the 'time' variable t, as was the case in Nagumo's seminal work [16]. Baouendi and Goulaouic's pioneer work on Fuchsian linear PDEs [4] present an existence and uniqueness theorem of Nagumo type, as well as the generalizations that followed [21, 10, 11]. As for nonlinear Fuchsian or Briot-Bouquet type equations, solvability in the class of partially holomorphic solutions have been established in [5, 12, 1].

If the function $F(t,x,u,v)$ is assumed to be holomorphic with respect to all its variables, then the setting above corresponds to PDEs of totally characteristic type introduced by Chen and Tahara [7, 8]. Further investigations on totally characteristic equations have been done by Tahara [22] and Shirai [19, 20]. The case when the singularity at $x=0$ is irregular, that is, when $\partial_v F(0,x,0,0) = x^p\gamma(x)$ with $\gamma(0)\neq 0$ and $p\geqslant 2$, has also been explored [6, 14, 3].

In this article, we extend the work of Ona and Lope [17] to the second-order case. Specifically, we shall consider the second-order singular partial differential equation

$$(t\partial_t)^2 u = F(t,x,u,\partial_x u,\partial_x^2 u,t\partial_t u,t\partial_t\partial_x u) \tag{1.4}$$

where the nonlinear function $F(t,x,u,v,w,y,z)$ has the following properties:

(B_1) $F(t,x,u,\mathbf{v})$ is continuous on $[0,T_0]\times D_{R_0}\times D_{R_1}\times D_{R_1}^4$ and is holomorphic with respect to $(x,u,\mathbf{v})$ for each fixed t where $\mathbf{v}=(v_{01},v_{02},v_{10},v_{11})$;

(B_2) $F(0,x,0,\mathbf{0})\equiv 0$ on D_{R_0};

(B_3) $\partial_{v_{ij}}F(0,x,0,\mathbf{0})=x^j\gamma_{ij}(x)$ for $0\leqslant i\leqslant j$.

Under these assumptions, we can write

$$F(t,x,u,\mathbf{v})=a(t,x)+g_{00}(t,x)u+[b_{01}(t)+xg_{01}(t,x)]\partial_x u+[b_{02}(t)+x^2g_{02}(t,x)]\partial_x^2 u \\ +g_{10}(t,x)t\partial_t u+[b_{11}(t)+xg_{11}(t,x)]t\partial_t\partial_x u+G(t,x,u,\mathbf{v}) \qquad (1.5)$$

where the coefficients are given by $a(t,x)=F(t,x,0,\mathbf{0})$, $g_{00}(t,x)=\partial_u F(t,x,0,\mathbf{0})$, $g_{10}(t,x)=\partial_{v_{10}}F(t,x,0,\mathbf{0})$, $b_{ij}(t)+x^jg_{ij}(t,x)=\partial_{v_{ij}}F(t,x,0,\mathbf{0})$ for $(i,j)=(0,1)$, $(0,2)$ or $(1,1)$, and $G(t,x,u,\mathbf{v})$ denotes all the nonlinear terms with respect to $(u,\mathbf{v})$.

Finally, putting $a_{11}=-g_{11}(0,0)$, $a_{02}=-g_{02}(0,0)$, $a_{10}=-g_{10}(0,0)$, $a_{01}=g_{02}(0,0)-g_{01}(0,0)$, $a_{00}=-g_{00}(0,0)$, and introducing the second order linear operator

$$\mathscr{L}_2=(t\partial_t)^2+a_{11}t\partial_t x\partial_x+a_{02}(x\partial_x)^2+a_{10}t\partial_t+a_{01}x\partial_x+a_{00} \qquad (1.6)$$

with constant coefficients, we see that (1.4) can now be written as:

$$\mathscr{L}_2 u=a(t,x)+b_{01}(t)\partial_x u+b_{02}(t)\partial_x^2 u+b_{11}(t)t\partial_t\partial_x u+c_{00}(t,x)u \\ +c_{01}(t,x)x\partial_x u+c_{02}(t,x)(x\partial_x)^2u+c_{10}(t,x)t\partial_t u \\ +c_{11}(t,x)t\partial_t x\partial_x u+G(t,x,u,\partial_x u,\partial_x^2 u,t\partial_t u,t\partial_t\partial_x u) \qquad (1.7)$$

where all the functions $c_{ij}(t,x)$ vanish at $(0,0)$. This rewriting of the linear part is being done because it is easier to handle the operator $(x\partial_x)^2$ compared to the operator $x^2\partial_x^2$.

Observe that $a(t,x)$ and $b_{ij}(t,x)$ for $(i,j)=(0,1)$, $(0,2)$ or $(1,1)$ vanish at $t=0$, and so there exists a continuous increasing function $\psi:[0,T_0]\to[0,\infty)$ such that $\psi(0)=0$, and for any $(t,x)\in[0,T_0]\times D_{R_0}$,

$$|a(t,x)|\leqslant\psi(t) \text{ and } |b_{ij}(t)|\leqslant\psi(t). \qquad (1.8)$$

Let $a,b,c,$ and d be positive real numbers satisfying: $1-a^2-4c\geqslant 0$, $1-b^2-4d\geqslant 0$, $b-2c\geqslant 0$, $a-2d\geqslant 0$, $(1-b^2-4d)^2+(b-2c)^2<1$. We shall assume that the coefficients a_{ij} in (1.6) satisfy the following inequalities:

(C_1) $\operatorname{Re}a_{11},\operatorname{Re}a_{10}>0$;

(C_2) $a<\dfrac{\operatorname{Im}a_{1k}}{\operatorname{Re}a_{1k}}<b$ for $k=0,1$;

(C_3) $c<\dfrac{\operatorname{Im}a_{0k}}{\binom{2}{k}(\operatorname{Re}a_{11})^k(\operatorname{Re}a_{10})^{2-k}},\ \dfrac{\operatorname{Re}a_{0k}}{\binom{2}{k}(\operatorname{Re}a_{11})^k(\operatorname{Re}a_{10})^{2-k}}<d$ for $k=0,1,2$.

These three inequalities are a bit technical but are being assumed to ensure the invertibility of an ordinary differential operator derived from $\mathscr{L}_2$, as can be seen later in the proof.

Here is our main result:

THEOREM 2. *Assume that* (B_1)–(B_3) *and* (C_1)–(C_3) *are all satisfied. Then there exist* $T>0$ *and* $R>0$ *such that* (1.4) *has a solution* $u(t,x)$ *that is continuous on* $[0,T]\times D_R$*, holomorphic in* x *for each fixed* t*, and satisfies the estimates*

$$|u(t,x)| \leqslant C\psi(t) \quad \text{and} \quad |\partial_x u(t,x)| \leqslant \frac{C}{R}\psi(t) \quad \text{on} \quad [0,T]\times D_R,$$

for some $C>0$*. This solution is unique in the sense that if* $v(t,x)$ *is another solution of* (1.4) *having the same properties, then* $u(t,x)=v(t,x)$ *on* $[0,T^*]\times D_{R^*}$ *for some* $T^*\in(0,T]$ *and* $R^*\in(0,R)$*.*

2. Preliminaries

Given two formal power series $f(z)=\sum_\alpha f_\alpha z^\alpha$ and $g(z)=\sum_\alpha g_\alpha z^\alpha$, we say that $f\ll g$ if $|f_\alpha|\leqslant g_\alpha$ for all multi-indices α. We also recall Lax's [9] majorant function

$$\phi(x)=\frac{1}{4S}\sum_{n=0}^{\infty}\frac{x^n}{(n+1)^2} \tag{2.1}$$

where $S=\pi^2/6$. The constant $1/4S$ is introduced in [13] to facilitate computations. Let us note that the power series converges for all $|x|<1$. Moreover, it satisfies the following majorant relations:

$$\phi^2(x)\ll\phi(x) \quad \text{and} \quad x\phi(x)\ll 4\phi(x). \tag{2.2}$$

Given a convergent series $f(t,x)=\sum_{i\geqslant 0} f_i(t)x^i$, whose coefficients f_i depend continuously on a parameter t, and a positive real number ρ we define the formal norm

$$|||f(t)|||_\rho=\sum_{i=0}^{\infty}|f_i(t)|\rho^i. \tag{2.3}$$

We list down some properties of the ρ-norm which are useful in estimating the norms of complicated expressions.

LEMMA 1. *Suppose* $f(t,x)$*,* $g(t,x)$*, and* $h(t,x,\theta)$ *are continuous in* t *and holomorphic in the other variables for each fixed* t*, and* ρ *is a positive real number. The following hold:*

(i) $|||f(t)g(t)|||_\rho \ll |||f(t)|||_\rho\,|||g(t)|||_\rho$

(ii) $\left|\left|\left|\partial_\rho f(t)\right|\right|\right|_\rho \ll \partial_\rho |||f(t)|||_\rho$

(iii) $\left|\left|\left|\int_0^1 h(t,x,\theta)\,d\theta\right|\right|\right|_\rho \ll \int_0^1 |||h(t,x,\theta)|||_\rho\,d\theta$

We can use ϕ to majorize the norm of a function $g(t,x)$ that is continuous in t and holomorphic in x for each fixed t. If $g(t,x)$ is bounded by M on $[0,T_0]\times D_{R_0}$ and $\rho\in(0,R_0)$. Then for all $t\in[0,T_0]$,

$$|||g(t)|||_\rho \ll \frac{M}{1-\rho/R_0}. \tag{2.4}$$

Since $4S\phi \gg 1$ and if we restrict $R\leqslant \frac{1}{2}R_0$, we obtain

$$|||g(t)|||_\rho \ll C_{R_0}\phi\left(\frac{\rho}{R}\right), \tag{2.5}$$

where the constant $C_{R_0}>0$ depends on M and R_0 but not on R.

The existence and uniqueness theorem for the solution of (1.1) is proved by first considering the following first-order linear equation:

$$\mathscr{L}_1 u = f(t,x) \tag{2.6}$$

where $\mathscr{L}_1$ is the operator in (1.2) and $f(t,x)$ is continuous on $[0,T_0]\times D_{R_0}$, holomorphic in x for each fixed t. It is shown in [17] that:

LEMMA 2. *Suppose that* $\max\{\operatorname{Re}\lambda(0,0),\operatorname{Re}\gamma(0)\}\leqslant -L$ *for some positive constant L. If* $|||f(t)|||_\rho \ll M\psi(t)\phi(\rho/R)$, *where* $\psi(t)$ *is the one in* (1.3) *and* $R\in(0,R_0)$, *then* (2.6) *has a unique solution $u(t,x)$ that is continuous on* $[0,T_0]\times D_{R_0}$, *holomorphic in x for each fixed t, and satisfies*

$$\left\{|||u(t,x)|||_\rho, R|||\partial_x u(t,x)|||_\rho, |||x\partial_x u(t,x)|||_\rho\right\} \ll M\psi(t)\phi\left(\frac{\rho}{R}\right).$$

This is the key lemma in solving (1.1) using the method of successive approximations. Thus, to solve (1.4), we need to formulate a second order version of this key lemma. The assumptions on the real parts of the coefficients are essential. In the second order case, this translates to the concept of stability of polynomials, i.e., polynomials whose roots have negative real parts.

We consider the second-order partial differential equation

$$\mathscr{L}_2 u = f(t,x), \tag{2.7}$$

where the operator $\mathscr{L}_2$ is the one defined in (1.6). Since the functions above are assumed to be holomorphic in x, we can expand both sides with respect to x. Writing $f(t,x)=\sum_{n\geqslant 0} f_n(t)x^n$, $u(t,x)=\sum_{n\geqslant 0} u_n(t)x^n$, and comparing the coefficient of x^n on both sides of (2.7), we obtain this family of equations:

$$(tD_t)^2 u_n(t) + (a_{11}n+a_{10})tD_t u_n(t) + (a_{02}n^2+a_{01}n+a_{00})u_n(t) = f_n(t). \tag{2.8}$$

Note that for all $n\geqslant 0$, (2.8) is an ordinary second-order differential equation of Fuchs type, and we want this to be uniquely solvable so that (2.7) is uniquely solvable as well.

To this end, let us now consider the quadratic polynomials in the variable p:

$$E_n(p) = p^2 + (a_{11}n + a_{10})p + (a_{02}n^2 + a_{01}n + a_{00}) \text{ for all } n \geqslant 0. \tag{2.9}$$

Put $\alpha_n = a_{11}n + a_{10}$ and $\beta_n = a_{02}n^2 + a_{01}n + a_{00}$. We now present the following lemma which gives estimates for the real and imaginary parts of α_n, β_n, and $\alpha_n^2 - 4\beta_n$, respectively.

LEMMA 3. *Assume that (C_1)–(C_3) are satisfied. The following inequalities hold:*

(i) $a\mathrm{Re}\,\alpha_n < \mathrm{Im}\,\alpha_n < b\mathrm{Re}\,\alpha_n$.

(ii) $c(\mathrm{Re}\,\alpha_n)^2 < \mathrm{Re}\,\beta_n < d(\mathrm{Re}\,\alpha_n)^2$.

(iii) $c(\mathrm{Re}\,\alpha_n)^2 < \mathrm{Im}\,\beta_n < d(\mathrm{Re}\,\alpha_n)^2$.

(iv) $(2a - 4d)(\mathrm{Re}\,\alpha_n)^2 < \mathrm{Im}\,(\alpha_n^2 - 4\beta_n) < (2b - 4c)(\mathrm{Re}\,\alpha_n)^2$.

(v) $(1 - b^2 - 4d)(\mathrm{Re}\,\alpha_n)^2 < \mathrm{Re}\,(\alpha_n^2 - 4\beta_n) < (1 - a^2 - 4c)(\mathrm{Re}\,\alpha_n)^2$.

Proof. Since $\mathrm{Im}\,\alpha_n = n\mathrm{Im}\,a_{11} + \mathrm{Im}\,a_{10}$, the first inequality immediately follows from (C_2). Now note that $\mathrm{Re}\,\beta_n = n^2\mathrm{Re}\,a_{02} + n\mathrm{Re}\,a_{01} + \mathrm{Re}\,a_{00}$ and $\mathrm{Im}\,\beta_n = n^2\mathrm{Im}\,a_{02} + n\mathrm{Im}\,a_{01} + \mathrm{Im}\,a_{00}$. Multiplying both sides of the inequalities in (C_3) by n^k, we see that

$$c\binom{2}{k}(n\mathrm{Re}\,a_{11})^k(\mathrm{Re}\,a_{10})^{2-k} < n^k\mathrm{Re}\,a_{0k},\, n^k\mathrm{Im}\,a_{0k} < d\binom{2}{k}(n\mathrm{Re}\,a_{11})^k(\mathrm{Re}\,a_{10})^{2-k},$$

for $k = 0, 1, 2$. Adding these inequalities (for $k = 0, 1, 2$), we obtain the inequalities in (ii) and (iii).

Let us note that

$$\alpha_n^2 - 4\beta_n = (\mathrm{Re}\,\alpha_n)^2 - (\mathrm{Im}\,\alpha_n)^2 - 4\mathrm{Re}\,\beta_n + i\big[2\mathrm{Re}\,\alpha_n\mathrm{Im}\,\alpha_n - 4\mathrm{Im}\;\beta_n\big].$$

Applying (i)–(iii) above, we have

$$\begin{aligned}\mathrm{Im}\,(\alpha_n^2 - 4\beta_n) &= 2\mathrm{Re}\,\alpha_n\mathrm{Im}\,\alpha_n - 4\mathrm{Im}\,\beta_n \\ &< 2(\mathrm{Re}\,\alpha_n)(b\mathrm{Re}\,\alpha_n) - 4c(\mathrm{Re}\,\alpha_n)^2 \\ &= (2b - 4c)(\mathrm{Re}\,\alpha_n)^2,\end{aligned}$$

and similarly,

$$\begin{aligned}\mathrm{Im}\,(\alpha_n^2 - 4\beta_n) &> 2(\mathrm{Re}\,\alpha_n)(a\mathrm{Re}\,\alpha_n) - 4d(\mathrm{Re}\,\alpha_n)^2 \\ &= (2a - 4d)(\mathrm{Re}\,\alpha_n)^2,\end{aligned}$$

which proves the fourth inequality.

Finally, applying (i) and (ii), we obtain these estimates for the real part of $\alpha_n^2 - 4\beta_n$:

$$\begin{aligned}\operatorname{Re}(\alpha_n^2 - 4\beta_n) &< (\operatorname{Re}\alpha_n)^2 - a^2(\operatorname{Re}\alpha_n)^2 - 4c(\operatorname{Re}\alpha_n)^2 \\ &= (1 - a^2 - 4c)(\operatorname{Re}\alpha_n)^2\end{aligned}$$

and

$$\begin{aligned}\operatorname{Re}(\alpha_n^2 - 4\beta_n) &> (\operatorname{Re}\alpha_n)^2 - b^2(\operatorname{Re}\alpha_n)^2 - 4d(\operatorname{Re}\alpha_n)^2 \\ &= (1 - b^2 - 4d)(\operatorname{Re}\alpha_n)^2,\end{aligned}$$

which completes the proof. $\square$

The following lemma provides a sufficient condition for the stability of E_n for all n.

LEMMA 4. *Assume that (C_1)–(C_3) hold. Then the polynomial $E_n(p)$ is stable for all $n \geqslant 0$. More precisely, if its roots are denoted by $-c_n$ and $-d_n$, then there exists a constant $L > 0$ such that*

$$\operatorname{Re} c_n, \operatorname{Re} d_n > L(n+1). \tag{2.10}$$

Proof. Let us note that $c_n = \frac{1}{2}(\alpha_n + \sqrt{\alpha_n^2 - 4\beta_n})$ and $d_n = \frac{1}{2}(\alpha_n - \sqrt{\alpha_n^2 - 4\beta_n})$. To estimate the real parts of c_n and d_n, we shall use the estimates in Lemma 3. Let us recall that for any nonzero complex number ω, we have

$$\operatorname{Re}\sqrt{\omega} = \pm\sqrt{\tfrac{1}{2}(|\omega| + \operatorname{Re}\omega)}. \tag{2.11}$$

Set $A = \min\{\operatorname{Re} a_{11}, \operatorname{Re} a_{10}\} > 0$, in view of (C_1). We first consider the positive square root in (2.11). By Lemma 3(iv), we have

$$\begin{aligned}\operatorname{Re}\sqrt{\alpha_n^2 - 4\beta_n} &= \sqrt{\tfrac{1}{2}\left[|\alpha_n^2 - 4\beta_n| + \operatorname{Re}(\alpha_n^2 - 4\beta_n)\right]} \\ &\geqslant \sqrt{\operatorname{Re}(\alpha_n^2 - 4\beta_n)} \\ &> \sqrt{1 - b^2 - 4d}\,\operatorname{Re}\alpha_n,\end{aligned}$$

and consequently, because $\alpha_n = a_{11}n + a_{10}$,

$$\begin{aligned}\operatorname{Re} c_n &> \tfrac{1}{2}\left(\operatorname{Re}\alpha_n + \sqrt{1 - b^2 - 4d}\,\operatorname{Re}\alpha_n\right) \\ &> \tfrac{1}{2}\left[A(n+1) + A(n+1)\sqrt{1 - b^2 - 4d}\right] \\ &= k_1(n+1),\end{aligned}$$

where $k_1 = \frac{A}{2}\left(1 + \sqrt{1 - b^2 - 4d}\right) > 0$.

To estimate the negative square root in (2.11), we apply (iv) and (v) of Lemma 3. We have

$$\begin{aligned}\operatorname{Re}\sqrt{\alpha_n^2-4\beta_n} &\geqslant -\sqrt{|\alpha_n^2-4\beta_n|}\\ &= -\sqrt[4]{\left[\operatorname{Re}(\alpha_n^2-4\beta_n)\right]^2+\left[\operatorname{Im}(\alpha_n^2-4\beta_n)\right]^2}\\ &> -\sqrt[4]{(1-b^2-4c)^2+(2b-4c)^2}\operatorname{Re}(\alpha_n)\end{aligned}$$

and so,

$$\begin{aligned}\operatorname{Re} c_n &= \tfrac{1}{2}\left(\operatorname{Re}\alpha_n+\operatorname{Re}\sqrt{\alpha_n^2-4\beta_n}\right)\\ &> \frac{1}{2}\left[1-\sqrt[4]{(1-b^2-4d)^2+(b-2c)^2}\right]A(n+1)\\ &= k_2(n+1),\end{aligned}$$

where $k_2=\frac{A}{2}\left(1-\sqrt[4]{(1-b^2-4d)^2+(b-2c)^2}\right)>0$. The estimates for $\operatorname{Re} d_n$ can be shown in a similar manner. Finally, we simply take $L=\min\{k_1,k_2\}>0$. $\square$

Without loss of generality, we may assume that the positive constant L in Lemma 4 is less than 1. We now state and prove the existence and uniqueness of the solution of the second-order linear differential equation (2.7).

LEMMA 5. *Assume that* (C_1)–(C_3) *hold and* $|||f(t)|||_\rho \ll M\psi(t)\phi(\rho/R)$. *Then* $\mathscr{L}_2 u=f(t,x)$ *has a unique solution* $u(t,x)$ *that is continuous on* $[0,T_0]\times D_{R_0}$ *and holomorphic in* x *for each fixed* t. *Moreover, there exists a constant* $K>0$ *such that the solution* $u(t,x)$ *satisfies the following estimates:*

(i) $\left|\left|\left|\partial_x^j u(t)\right|\right|\right|_\rho \ll \dfrac{M}{L^2R^k}\psi(t)\phi\left(\dfrac{\rho}{R}\right)$ *for j=0,1,2;*

(ii) $|||(x\partial_x)^j u(t)|||_\rho \ll \dfrac{M}{L^2}\psi(t)\phi\left(\dfrac{\rho}{R}\right)$ *for j=1,2;*

(iii) $|||(t\partial_t)(x\partial_x)^j u(t)|||_\rho \ll \dfrac{KM}{L^2}\psi(t)\phi\left(\dfrac{\rho}{R}\right)$ *for j=0,1; and*

(iv) $|||t\partial_t\partial_x u(t)|||_\rho \ll \dfrac{KM}{L^2R}\psi(t)\phi\left(\dfrac{\rho}{R}\right)$.

Proof. Let us note that we can write (2.8) as

$$(tD_t+c_n)(tD_t+d_n)u_n(t)=f_n(t). \tag{2.12}$$

From the theory of Fuchsian linear differential equations, this has a unique solution if the characteristic exponents have negative real parts, which is guaranteed by Lemma 4. The unique solution of (2.12) is given by:

$$u_n(t)=\int_0^t\left(\frac{s}{t}\right)^{d_n}\left(\int_0^s\left(\frac{\sigma}{s}\right)^{c_n}f_n(\sigma)\,\frac{d\sigma}{\sigma}\right)\frac{ds}{s}.$$

To simplify the calculation of the estimate for $u_n(t)$, let us first call the inner integral above as

$$w_n(s) = \int_0^s \left(\frac{\sigma}{s}\right)^{c_n} f_n(\sigma)\,\frac{d\sigma}{\sigma}.$$

Since $\operatorname{Re} c_n > L(n+1)$, the function $\psi(t)$ is increasing, and $|f_n(\sigma)| \leqslant \dfrac{M\psi(\sigma)}{R^n(n+1)^2}$ for all $n \geqslant 0$ by assumption, we see that $w_n(s)$ satisfies the estimate

$$\begin{aligned}
|w_n(s)| &\leqslant \frac{M\psi(s)}{R^n(n+1)^2}\int_0^s \left(\frac{\sigma}{s}\right)^{L(n+1)}\frac{d\sigma}{\sigma}\\
&= \frac{M\psi(s)}{R^n L(n+1)^3}.
\end{aligned}$$

Likewise, since $\operatorname{Re} d_n > L(n+1)$, the above estimate for $w_n(s)$ implies

$$\begin{aligned}
|u_n(t)| &\leqslant \int_0^t \left(\frac{s}{t}\right)^{L(n+1)} |w_n(s)|\,\frac{ds}{s}\\
&\leqslant \frac{M\psi(t)}{R^n L(n+1)^3}\,\frac{1}{L(n+1)}\\
&\leqslant \frac{M\psi(t)}{L^2}\,\frac{1}{R^n(n+1)^2},
\end{aligned}$$

which proves (i) for $j=0$. Since $\partial_x u(t,x) = \sum_{n\geqslant 0}(n+1)u_{n+1}(t)x^n$ and $\partial_x^2 u(t,x) = \sum_{n\geqslant 0}(n+1)(n+2)u_{n+2}(t)x^n$, the sharper estimate for $|u_n(t)|$ implies that (i) also holds for $j=1$ and 2. Similarly, by applying $(x\partial_x)^j$ (for $j=1,2$) on the expansion of $u(t,x)$, we see that the estimates in (ii) also follow.

We now prove (iii) and (iv). Setting $B = \max\{|a_{11}|,|a_{10}|\} > 0$, then from (iv) and (v) in Lemma 3, we have the following estimates:

$$\begin{aligned}
|d_n| \leqslant \frac{1}{2}\left(|\alpha_n| + \sqrt{|\alpha_n^2 - 4\beta_n|}\right) &= \frac{1}{2}\left(|\alpha_n| + \sqrt[4]{\left[\operatorname{Re}(\alpha_n^2-4\beta_n)\right]^2 + \left[\operatorname{Im}(\alpha_n^2-4\beta_n)\right]^2}\right)\\
&< \frac{|\alpha_n|}{2}\left(1+\sqrt[4]{(1-a^2-4c)^2+(2b-4c)^2}\right)\\
&< \frac{B(n+1)}{2}\left(1+\sqrt[4]{(1-a^2-4c)^2+(2b-4c)^2}\right)\\
&= C'(n+1),
\end{aligned}$$

where $C' = \frac{B}{2}\left[1+\sqrt[4]{(1-a^2-4c)^2+(2b-4c)^2}\right]$.

Let us note that $t\partial_t u(t,x) = \sum_{n\geqslant 0} t u_n'(t)x^n$. Applying the Leibniz integral rule, we see that

$$\begin{aligned}
tu_n'(t) &= t\left(\frac{w_n(t)}{t} - d_n\int_0^t \left(\frac{s}{t}\right)^{d_n-1}\frac{s}{t^2}\,\frac{w_n(s)}{s}\,ds\right)\\
&= w_n(t) - d_n\int_0^t \left(\frac{s}{t}\right)^{d_n}\frac{w_n(s)}{s}\,ds.
\end{aligned}$$

Letting $K = 1 + C'$, then the estimates for d_n and $w_n(s)$ computed earlier imply that

$$\begin{aligned} |tu_n'(t)| &\leqslant \frac{M\psi(t)}{LR^n(n+1)^3} + C'(n+1)\frac{M\psi(t)}{LR^n(n+1)^3}\frac{1}{L(n+1)} \\ &\leqslant \left(\frac{M}{L^2} + \frac{C'M}{L^2}\right)\psi(t)\frac{1}{R^n(n+1)^3} \\ &\leqslant \frac{KM}{L^2}\psi(t)\frac{1}{R^n(n+1)^3}, \end{aligned}$$

since $L \leqslant 1$. This shows that (iii) holds for $j = 0$. The rest of the estimates can be shown by applying ∂_x and $x\partial_x$ on $t\partial_t u(t,x)$, as was done in (i) and (ii).

The majorant relation in (i) for $j = 0$ implies that $u(t,x)$ converges absolutely and uniformly on $[0,T_0] \times D_{R_0}$. Consequently, $u(t,x)$ is continuous on $[0,T_0] \times D_{R_0}$. Finally, $u(t,x)$ being a power series in x is holomorphic in x for each fixed t. $\square$

3. Proof of the main result

We recall that the function $c_{ij}(t,x)$ is holomorphic in x for each fixed t and vanishes at $(0,0)$ for all i and j. These allow as to expand $c_{ij}(t,x)$ as

$$c_{ij}(t,x) = c_{ij}(t,0) + x\tilde{c}_{ij}(t,x) \quad \text{for} \quad 0 \leqslant i+j \leqslant 2,\ i < 2,$$

where $c_{ij}(t,0)$ and $\tilde{c}_{ij}(t,x)$ are continuous in the variable t, and $\tilde{c}_{ij}(t,x)$ is holomorphic in x for each fixed t. We can now write (1.7) as

$$\begin{aligned} \mathscr{L}_2 u = a(t,x) &+ b_{01}(t)\partial_x u + b_{02}(t)\partial_x^2 u + b_{11}(t)t\partial_t\partial_x u + [c_{01}(t,0) + x\tilde{c}_{01}(t,x)]u \\ &+ [c_{00}(t,0) + x\tilde{c}_{00}(t,x)]x\partial_x u + [c_{02}(t,0) + x\tilde{c}_{02}(t,x)](x\partial_x)^2 u \\ &+ [c_{10}(t,0) + x\tilde{c}_{10}(t,x)]t\partial_t u + [c_{11}(t,0) + x\tilde{c}_{11}(t,x)]t\partial_t x\partial_x u \\ &+ G(t,x,u,\partial_x u,\partial_x^2 u,t\partial_t u,t\partial_t\partial_x u). \end{aligned}$$

We shall use successive approximations to establish the existence of the solution of (1.4). We define the sequence of approximate solutions $\{u_n(t,x\}_{n=0}^{\infty}$ by the recursion:

$$\begin{aligned} \mathscr{L}_2 u_0 &= a(t,x) \\ \mathscr{L}_2 u_n &= a(t,x) + \big\{b_{01}(t)\partial_x + b_{02}(t)\partial_x^2 + b_{11}(t)t\partial_t\partial_x + [c_{01}(t,0) + x\tilde{c}_{01}(t,x)] \\ &\quad + [c_{00}(t,0) + x\tilde{c}_{00}(t,x)]x\partial_x + [c_{02}(t,0) + x\tilde{c}_{02}(t,x)](x\partial_x)^2 \\ &\quad + [c_{10}(t,0) + x\tilde{c}_{10}(t,x)]t\partial_t + [c_{11}(t,0) + x\tilde{c}_{11}(t,x)]t\partial_t x\partial_x\big\}u_{n-1} \\ &\quad + G(t,x,u_{n-1},\partial_x u_{n-1},\partial_x^2 u_{n-1},t\partial_t u_{n-1},t\partial_t\partial_x u_{n-1}), \quad \text{for } n \geqslant 1. \end{aligned} \tag{3.1}$$

We shall now show that $\{u_n(t,x)\}_{n=0}^{\infty}$ converges to some function $u(t,x)$ which is a solution of (1.4). To achieve this goal, we define a new sequence $\{v_n(t,x)\}_{n=0}^{\infty}$ as follows:

$$v_0(t,x) = u_0(t,x) \text{ and } v_n(t,x) = u_n(t,x) - u_{n-1}(t,x), \text{ for } n \geqslant 1. \tag{3.2}$$

It is easily seen that $u_n(t,x) = \sum_{k=0}^{n} v_k(t,x)$. Thus, the convergence of $\{u_n(t,x)\}_{n=0}^{\infty}$ is equivalent to the convergence of the series $\sum_{k=0}^{\infty} v_k(t,x)$. We will follow the technique in [24] and in [17] to estimate each $v_n(t,x)$. Let us define $\sigma : [0,T_0] \to [0,\infty)$ by

$$\sigma(t) = \max_{\tau\in[0,t]} \{|c_{ij}(\tau,0)| : 0 \leqslant i+j \leqslant 2,\ i<2\}.$$

Evidently, $\sigma(t)$ is an increasing, continuous function, and $\lim_{t\to 0}\sigma(t) = 0$ since each $c_{ij}(t,x)$ vanishes at $(0,0)$. Let $0<\rho<R\leqslant \frac{1}{2}R_0$. Since we are only constructing a local solution, we may assume that $R<1$. In view of (2.5), we can choose $C>0$ large enough such that for all $t\in[0,T_0]$,

$$\left|\left|\left|\tilde{c}_{ij}(t)\right|\right|\right|_\rho \ll C\phi\left(\frac{\rho}{R}\right) \text{ and } |||a(t)|||_\rho \ll \frac{1}{2}C\psi(t)\phi\left(\frac{\rho}{R}\right). \tag{3.3}$$

In dealing with the nonlinear term in (3.1), we shall use the following lemma, which follows from properties of ϕ (see also [24, 17]).

LEMMA 6. *Let $G(t,x,Z) = \sum_{\alpha\geqslant 2} G_\alpha(t,x)Z^\alpha$, with $Z=(z_1,z_2,z_3,z_4,z_5)$ be convergent for $t\in[0,T]$, $x\in D_{R_0}$ and $Z\in D_{R_1}^5$. Suppose that $0<\rho<R<R_0$, $T>0$ is small enough so that $Q\psi(T)<\frac{R_1}{2}$, and for all i we have*

$$|||z_i(t)|||_\rho \ll Q\psi(t)\phi\left(\frac{\rho}{R}\right),$$

then there exists $C_0>0$ independent of T and Q such that for all $t\in[0,T]$ we have:

(i) $|||G(t,X)|||_\rho \ll C_0[Q\psi(t)]^2\phi\left(\frac{\rho}{R}\right)$

(ii) $|||\partial_{x_i}G(t,X)|||_\rho \ll C_0Q\psi(t)\phi\left(\frac{\rho}{R}\right)$.

To avoid repeatedly writing long sums, we introduce the index sets $\Delta_2 = \{(i,j)\in\mathbb{N}^2 : 0\leqslant i+j\leqslant 2, i<2\}$ and $\Delta_0 = \Delta_2 - \{(0,0),(1,0)\}$. We now present the following proposition which gives estimates concerning v_n and the approximate solutions as well.

PROPOSITION 1. Let $D=\max\{K,C\}$, where the constants K and C are the ones in Lemma 5 and in (3.3), respectively. Then there exist $T,R>0$ such that for all $n\geqslant 0$, the following majorant relations hold on $[0,T]\times D_R$:

(i) $\left|\left|\left|(t\partial_t)^i\partial_x^j v_n(t)\right|\right|\right|_\rho \ll \dfrac{D^{i+1}}{2^{n+1}L^2R^j}\psi(t)\phi\left(\dfrac{\rho}{R}\right)$ for all $(i,j)\in\Delta_0$;

(ii) $\left|\left|\left|(t\partial_t)^i(x\partial_x)^j v_n(t)\right|\right|\right|_\rho \ll \dfrac{D^{i+1}}{2^{n+1}L^2}\psi(t)\phi\left(\dfrac{\rho}{R}\right)$ for all $(i,j)\in\Delta_2$;

(iii) $\left|\left|\left|(t\partial_t)^i\partial_x^j u_n(t)\right|\right|\right|_\rho \ll \left(1-\dfrac{1}{2^{n+1}}\right)\dfrac{D^{i+1}}{L^2R^j}\psi(t)\phi\left(\dfrac{\rho}{R}\right)$ for all $(i,j)\in\Delta_0$;

(iv) $\left|\left|\left|(t\partial_t)^i(x\partial_x)^j u_n(t)\right|\right|\right|_\rho \ll \left(1-\frac{1}{2^{n+1}}\right)\frac{D^{i+1}}{L^2}\psi(t)\phi\left(\frac{\rho}{R}\right)$ for all $(i,j)\in\Delta_2$.

Proof. We shall prove these estimates by induction. First note that $|||a(t)|||_\rho \ll \frac{1}{2}C\psi(t)\phi(\frac{\rho}{R})$, and so by Lemma 5, the equation $\mathscr{L}_2 u_0 = a(t,x)$ has a unique solution $u_0(t,x)$ that is continuous on $[0,T_0]\times D_{R_0}$, holomorphic in x for each fixed t. Moreover, there exists $K>0$ such that u_0 satisfies the following estimates:

1. $\left|\left|\left|(t\partial_t)^i\partial_x^j u_0(t)\right|\right|\right|_\rho \ll \frac{K^iC}{2L^2R^j}\psi(t)\phi\left(\frac{\rho}{R}\right)$ for all $(i,j)\in\Delta_0$;

2. $\left|\left|\left|(t\partial_t)^i(x\partial_x)^j u_0(t)\right|\right|\right|_\rho \ll \frac{K^iC}{2L^2}\psi(t)\phi\left(\frac{\rho}{R}\right)$ for all $(i,j)\in\Delta_2$.

Since $D=\max\{K,C\}$ and $u_0=v_0$, the estimates in (1) and (2) above imply that (i)–(iv) hold for $n=0$. We now show that the proposition above holds for $n=1$. By definition and linearity of $\mathscr{L}_2$, we see that v_1 satisfies

$$\begin{aligned}\mathscr{L}_2 v_1 = &\sum_{(i,j)\in\Delta_0} b_{ij}(t)(t\partial_t)^i\partial_x^j v_0 + \sum_{(i,j)\in\Delta_2}[c_{ij}(t,0)+x\tilde{c}_{ij}(t,x)](t\partial_t)^i(x\partial_x)^j v_0 \\ &+ G\left(t,x,\{(t\partial_t)^i\partial_x^j v_0(t)\}_{(i,j)\in\Delta_2}\right)\end{aligned} \tag{3.4}$$

To obtain the estimates for v_1, we first need to find the estimate for the nonlinear term above. Since $L,R<1$ and $D>1$, the estimates for v_0 we obtained earlier imply that for $(i,j)\in\Delta_2$,

$$|||(t\partial_t)^i\partial_x^j v_0(t)|||_\rho \ll \frac{D^2}{2L^2R^2}\psi(t)\phi\left(\frac{\rho}{R}\right).$$

Because of Lemma 6(i), there exists $C_0>0$ such that the nonlinear term in (3.4) satisfies the majorant relation

$$\left|\left|\left|G\left(t,x,\{(t\partial_t)^i\partial_x^j v_0\}_{(i,j)\in\Delta_2}\right)\right|\right|\right|_\rho \ll \frac{C_0D^4}{4L^4R^4}\psi^2(t)\phi\left(\frac{\rho}{R}\right).$$

Now we need to find $T,R>0$ so that on $[0,T]\times D_R$, the estimate in the proposition above holds for each n. To do this, we first choose R small enough such that

$$\frac{20D^2R}{L^2}<\frac{1}{8}.$$

We then fix this R and choose T small enough such that these are satisfied:

$$\begin{aligned}\left(\frac{3D}{L^2R^2}+\frac{5C_0D^3}{L^4R^4}\right)\psi(T) &< \frac{1}{8}\\ \frac{5D}{L^2}\sigma(T) &< \frac{1}{4}\\ \frac{D^2}{2L^2R^2}\psi(T) &< \frac{R_1}{2}.\end{aligned}$$

If we denote the right-hand side of (3.4) by I_1. Then by (1.3), (2.2), Lemma 1 (i), and the fact that the functions $\psi(t), \sigma(t)$ are increasing, we obtain the following estimates:

$$\begin{aligned}
\|\|I_1\|\|_\rho &\ll \sum_{(i,j)\in\Delta_0} \frac{D^{i+1}}{2L^2R^j}\psi^2(t)\phi\left(\frac{\rho}{R}\right) + \sum_{(i,j)\in\Delta_2}\left\{\sigma(t)+\rho D\phi\left(\frac{\rho}{R}\right)\right\}\frac{D^{i+1}}{2L^2}\psi(t)\phi\left(\frac{\rho}{R}\right) \\
&\qquad + \frac{C_0D^4}{4L^4R^4}\psi^2(t)\phi\left(\frac{\rho}{R}\right) \\
&\ll \frac{D}{2}\psi(t)\phi\left(\frac{\rho}{R}\right)\left\{\sum_{(i,j)\in\Delta_0}\frac{D^i\psi(T)}{L^2R^j} + \sum_{(i,j)\in\Delta_2}\frac{D^i\sigma(T)}{L^2}\right. \\
&\qquad \left. + \sum_{(i,j)\in\Delta_2}\frac{4D^{i+1}R}{L^2} + \frac{C_0D^3}{2L^4R^4}\psi(T)\right\} \\
&\ll \frac{D}{2}\psi(t)\phi\left(\frac{\rho}{R}\right)\left\{\frac{3D}{L^2R^2}\psi(T) + \frac{5D}{L^2}\sigma(T) + \frac{20D^2R}{L^2} + \frac{C_0D^3}{2L^4R^4}\psi(T)\right\} \\
&= \frac{D}{2}\psi(t)\phi\left(\frac{\rho}{R}\right)\left\{\frac{20D^2R}{L^2} + \left(\frac{3D}{L^2R^2} + \frac{C_0D^3}{2L^4R^4}\right)\psi(T) + \frac{5D}{L^2}\sigma(T)\right\} \\
&\ll \frac{D}{2^2}\psi(t)\phi\left(\frac{\rho}{R}\right).
\end{aligned}$$

By Lemma 5, we see that the estimates in (i)–(ii) hold for v_1. Applying the triangle inequality on $u_1 = v_0 + v_1$, we see that the estimates in (iii)–(iv) clearly hold for u_1.

Let us note that by definition, for $n \geqslant 2$, v_n satisfies

$$\begin{aligned}
\mathscr{L}_2 v_n &= \sum_{(i,j)\in\Delta_0} b_{ij}(t)(t\partial_t)^i\partial_x^j v_{n-1} + \sum_{(i,j)\in\Delta_2}[c_{ij}(t,0)+x\tilde{c}_{ij}(t,x)](t\partial_t)^i(x\partial_x)^j v_{n-1} \\
&\quad + G\left(t,x,\{(t\partial_t)^i\partial_x^j v_{n-1}(t)\}_{(i,j)\in\Delta_2}\right) - G\left(t,x,\{(t\partial_t)^i\partial_x^j v_{n-2}(t)\}_{(i,j)\in\Delta_2}\right).
\end{aligned} \tag{3.5}$$

Suppose now that for $n \geqslant 3$, the functions $u_2,\dots,u_{n-1}$ and $v_2,\dots,v_{n-1}$ satisfy the induction hypothesis. To estimate the right-hand side of (3.5), we first provide an estimate for the difference of the nonlinear terms above. Note that we can write

$$\begin{aligned}
H &= G\left(t,x,\{(t\partial_t)^i\partial_x^j v_{n-1}(t)\}_{(i,j)\in\Delta_2}\right) - G\left(t,x,\{(t\partial_t)^i\partial_x^j v_{n-2}(t)\}_{(i,j)\in\Delta_2}\right) \\
&= \sum_{(i,j)\in\Delta_2}(t\partial_t)^i\partial_x^j v_{n-1}\int_0^1 \partial_{w_{ij}}G\left(t,x,\{w_{ij}\}_{(i,j)\in\Delta_2}\right)d\theta,
\end{aligned}$$

where, $w_{ij} = (t\partial_t)^i\partial_x^j(\theta v_{n-1} + u_{n-2})$ for all $(i,j)\in\Delta_2$ and $\theta\in(0,1)$. By the induction hypothesis, we obtain the following estimates:

$$\begin{aligned}
\left\|\left\|w_{ij}\right\|\right\|_\rho &= \left\|\left\|(t\partial_t)^i\partial_x^j(\theta v_{n-1}+u_{n-2})\right\|\right\|_\rho \\
&\ll \frac{D^{i+1}}{2^nL^2R^j}\psi(t)\phi\left(\frac{\rho}{R}\right) + \left(1-\frac{1}{2^{n-1}}\right)\frac{D^{i+1}}{L^2R^j}\psi(t)\phi\left(\frac{\rho}{R}\right) \\
&\ll \frac{D^{i+1}}{L^2R^j}\psi(t)\phi\left(\frac{\rho}{R}\right)
\end{aligned}$$

for all $(i,j) \in \Delta_2$. Since $D > 1$ and the function $\psi(t)$ is increasing, then by (2.2), Lemma 1(iii), Lemma 6(ii), and the induction hypothesis, the norm of H satisfies

$$\begin{aligned}
\|\!|H|\!\|_\rho &\ll \sum_{(i,j)\in\Delta_2} \psi(t)\frac{D^{i+1}}{2^n L^2 R^j}\psi(t)\phi\left(\frac{\rho}{R}\right)\frac{C_0 D^{i+1}}{L^2 R^j}\psi(t)\phi\left(\frac{\rho}{R}\right)\\
&\ll \sum_{(i,j)\in\Delta_2} \frac{C_0 D^{2i+2}}{2^n L^4 R^{2j}}\psi^2(t)\phi\left(\frac{\rho}{R}\right)\\
&\ll \frac{5C_0 D^5}{2^n L^4 R^4}\psi^2(t)\phi\left(\frac{\rho}{R}\right)\\
&\ll \frac{D}{2^n}\psi(t)\phi\left(\frac{\rho}{R}\right)\frac{5C_0 D^4}{L^4 R^4}\psi(T).
\end{aligned}$$

We can now obtain the estimate for the right-hand side of (3.5) which we shall denote by I_n. Since $D > 1$ and the functions $\psi(t), \sigma(t)$ are increasing, then by (1.3), (2.2), Lemma 1(i), and the induction hypothesis we get the following estimates:

$$\begin{aligned}
\|\!|I_n|\!\|_\rho &\ll \sum_{(i,j)\in\Delta_0} \frac{D^{i+1}}{2^n L^2 R^j}\psi^2(t)\phi\left(\frac{\rho}{R}\right) + \sum_{(i,j)\in\Delta_2}\left\{\sigma(t) + \rho D\phi\left(\frac{\rho}{R}\right)\right\}\frac{D^{i+1}}{2^n L^2}\psi(t)\phi\left(\frac{\rho}{R}\right)\\
&\quad + \frac{D}{2^n}\psi(t)\phi\left(\frac{\rho}{R}\right)\frac{5C_0 D^4}{L^4 R^4}\psi(T)\\
&\ll \frac{D}{2^n}\psi(t)\phi\left(\frac{\rho}{R}\right)\left\{\sum_{(i,j)\in\Delta_0}\frac{D^i\psi(T)}{L^2 R^j} + \sum_{(i,j)\in\Delta_2}\frac{D^i\sigma(T)}{L^2}\right.\\
&\quad \left. + \sum_{(i,j)\in\Delta_2}\frac{4D^{i+1}R}{L^2} + \frac{5C_0 D^3}{L^4 R^4}\psi(T)\right\}\\
&\ll \frac{D}{2^n}\psi(t)\phi\left(\frac{\rho}{R}\right)\left\{\frac{3D}{L^2R^2}\psi(T) + \frac{5D}{L^2}\sigma(T) + \frac{20D^2R}{L^2} + \frac{5C_0D^3}{L^4R^4}\psi(T)\right\}\\
&= \frac{D}{2^n}\psi(t)\phi\left(\frac{\rho}{R}\right)\left\{\frac{20D^2R}{L^2} + \left(\frac{3D}{L^2R^2} + \frac{5C_0D^3}{L^4R^4}\right)\psi(T) + \frac{5D}{L^2}\sigma(T)\right\}\\
&\ll \frac{D}{2^{n+1}}\psi(t)\phi\left(\frac{\rho}{R}\right)
\end{aligned}$$

for all $t \in [0,T]$. Thus, by Lemma 5, we see that the estimates (i)–(ii) hold for v_n. Using the fact that $u_n = \sum_{k=0}^n v_k$, the estimates (iii)–(iv) hold for u_n. This completes the induction and the proof of Proposition 1. $\square$

The estimates in Proposition 1 imply the convergence of the approximate solutions to the desired solution $u(t,x)$ on $[0,T]\times D_R$. Moreover, this function $u(t,x)$ satisfies

$$\begin{aligned}
&|(t\partial_t)^i\partial_x^j u(t,x)| \leqslant \frac{A}{R^j}\psi(t) \text{ for all } (i,j)\in\Delta_0,\\
&|(t\partial_t)^i(x\partial_x)^j u(t,x)| \leqslant A\psi(t) \text{ for all } (i,j)\in\Delta_2
\end{aligned}$$

for some constant $A > 0$. This completes the existence part of the proof.

4. Uniqueness of the solution

We now show the uniqueness of the solution. Let v be another solution of (1.4) having the same properties as the solution u we have constructed earlier. Putting $w = u - v$ and using the estimates satisfied by u and v, we obtain

$$\left|\left|\left|(t\partial_t)^i \partial_x^j u(t)\right|\right|\right|_\rho \leqslant \frac{A^*}{R^j}\psi(t) \text{ for all } (i,j) \in \Delta_0,$$
$$\left|\left|\left|(t\partial_t)^i (x\partial_x)^j u(t)\right|\right|\right|_\rho \leqslant A^*\psi(t) \text{ for all } (i,j) \in \Delta_2$$

on $[0,T]$ for some $A^* > 0$. Now, by definition and the linearity of $\mathscr{L}_2$, we see that w satisfies

$$\mathscr{L}_2 w = \sum_{(i,j)\in\Delta_0} b_{ij}(t)(t\partial_t)^i\partial_x^j w + \sum_{(i,j)\in\Delta_2} [c_{ij}(t,0) + x\tilde{c}_{ij}(t,x)](t\partial_t)^i(x\partial_x)^j w \qquad (4.1)$$
$$+ G\left(t,x,\{(t\partial_t)^i\partial_x^j u(t)\}_{(i,j)\in\Delta_2}\right) - G\left(t,x,\{(t\partial_t)^i\partial_x^j v(t)\}_{(i,j)\in\Delta_2}\right),$$

which is of the same form as (3.5). Thus, if we also apply the steps and techniques used earlier in obtaining the estimates for v_n, we can find $B > 0$, $T^* \in (0,T)$ and $R^* \in (0,R)$ such that

$$\left|\left|\left|(t\partial_t)^i (x\partial_x)^j w(t)\right|\right|\right|_\rho \ll \frac{B^{i+1}}{2^{n+1}L^2R^j}\psi(t)\phi\left(\frac{\rho}{R}\right) \text{ for } (i,j) \in \Delta_2 \text{ and for all } n \in \mathbb{N}$$

on $[0,T^*] \times D_{R^*}$. Taking $(i,j) = (0,0)$, we see that w satisfies the majorant relation

$$|||w(t)|||_\rho \ll \frac{B}{2^{n+1}L^2}\psi(t)\phi\left(\frac{\rho}{R}\right) \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$, we conclude that $w(t,x) \equiv 0$, or equivalently, $u \equiv v$ on $[0,T^*] \times D_{R^*}$.

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Michael E. Sta. Brigida
Institute of Mathematics
C .P. Garcia St., U. P. Campus, Quezon City, Philippines
e-mail: mstabrigida@math.upd.edu.ph

Jose Ernie C. Lope
Institute of Mathematics
C. P. Garcia St., U. P. Campus, Quezon City, Philippines
e-mail: jclope@up.edu.ph