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ON THE STABILITY OF SYSTEMS OF TWO LINEAR FIRST–ORDER ORDINARY DIFFERENTIAL EQUATIONS

GEORG A. GRIGORIAN

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Abstract. The Riccati equation method is used to establish some new stability criteria for systems of two linear first-order ordinary differential equations. It is shown that two of these criteria in the two dimensional case imply the Routh-Hurwitz's criterion.

1. Introduction

Let $a(t)$, $b(t)$, $c(t)$ and $d(t)$ be complex-valued continuous functions on $[t_0,+\infty)$. Consider the linear system

$$\begin{cases} \phi' = a(t)\phi + b(t)\psi, \\ \psi' = c(t)\phi + d(t)\psi, \ \ t \geqslant t_0. \end{cases} \tag{1.1}$$

DEFINITION 1.1. A normal linear system of ordinary differential equations (in particular the system (1.1)) is called asymptotically stable if all its solutions tend to zero for t tending to $+\infty$.

Study of the stability behavior of the system (1.1), in general, of linear systems of ordinary differential equations is an important problem of Qualitative theory of differential equations, and many works are devoted to it (see [1] and cited works therein, [2–4]). The fundamental thorem of R. Bellman (see [5], pp. 168, 169) reduces the study of boundedness conditions of solutions of a wide class of nonlinear systems of ordinary differential equations to the study of stability conditions of linear systems of ordinary differential equations. There exist various methods of detection of stable and (or) unstable linear systems of ordinary differential equations. Among them notice the Lyapunov, Bogdanov, and Wazevski's methods, the method involving estimates of solutions in the Lozinski's logarithmic norms, and the freezing method (see [1], pp. 40–98). These and other methods (see e.g.; [6–10]) permit to carry out wide classes of stable and (or) unstable linear systems.

In this paper on the basis of results of works [11] and [12] by the use of Riccati equation method new stability criteria for the system (1.1) are obtained. It is shown that

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in the two dimensional case of linear systems the Routh-Hurwitz's stability criterion is a consequence of the obtained results.

REMARK 1.1. It should be noticed that the results of the paper [11] are based on I. M. Sobol's result from [10] (the work [10] is devoted specially to deep study of the stability problem for second order linear ordinary differential equations). Notice also that Theorems 3.1, 3.2 and 3.4 of this paper (see below) are based on the results of [11]. Hence, I. M. Sobol's result underlies in mentioned theorems.

2. Auxiliary propositions

Let $p(t)$ and $q(t)$ be complex-valued continuous functions on $[t_0,+\infty)$. Consider the second order linear ordinary differential equation

$$\phi'' + p(t)\phi' + q(t)\phi = 0, \qquad t \geqslant t_0. \tag{2.1}$$

The substitution $\phi' = \psi$ in this equation reduces it into the linear system

$$\begin{cases} \phi' = \psi, \\ \psi' = -q(t)\phi - p(t)\psi, \ t \geqslant t_0. \end{cases} \tag{2.2}$$

DEFINITION 2.1. Eq. (2.1) is called Lyapunov (asymptotically) stable if the corresponding system (2.2) is Lyapunov (asymptotically) stable.

REMARK 2.1. It follows from Definition 2.1 that Eq. (2.1) is Lyapunov (asymptotically) stable if and only if its all solutions $\phi(t)$ with $\phi'(t)$ are bounded (vanish at $+\infty$).

Set: $G(t) \equiv q(t) - \frac{p'(t)}{2} - \frac{p^2(t)}{4}$, $\mathscr{L}_0(t) \equiv \frac{1}{\sqrt[4]{G(t)}} \int\limits_{t_0}^{t} \frac{|(\sqrt{G(\tau)})'|}{\sqrt[4]{G(\tau)}} d\tau$, $t \geqslant t_0$. Hereafter we will assume that $p(t)$ and $G(t)$ are continuously differentiable on $[t_0,+\infty)$, and $G(t) \neq 0$, $t \geqslant t_0$.

THEOREM 2.1. *Let the following conditions be satisfied.*
$G(t) > 0$, $t \geqslant t_0$, $\lim\limits_{t\to+\infty} \frac{G'(t)}{G^{3/2}(t)} = \alpha$, $|\alpha| < 4$, $\mathscr{L}_0(t)$ *and* $Var_{t_0}^{t} \frac{G'(t)}{G^{3/2}(t)}$ *are bounded. Then all solutions of Eq.* (2.1) *are bounded (vanish at* $+\infty$*) if and only if*

$$\inf_{t\geqslant t_0}\left\{\int\limits_{t_0}^{t} \mathfrak{Re}\, p(\tau)d\tau + \frac{1}{2}\ln G(t)\right\} > -\infty \qquad \left(\lim_{t\to+\infty}\left\{\int\limits_{t_0}^{t} \mathfrak{Re}\, p(\tau)d\tau + \frac{1}{2}\ln G(t)\right\} = +\infty\right).$$

See the proof in [11].

THEOREM 2.2. *Let the conditions of Theorem* 2.1 *be satisfied. Then Eq.* (2.1) *is Lyapunov (asymptotically) stable if and only if*

$$\begin{cases} \inf\limits_{t\geqslant t_0}\left\{\int\limits_{t_0}^{t} \mathfrak{Re}\, p(\tau)d\tau - 2\ln(1+|p(t)|) + \frac{1}{2}\ln G(t)\right\} > -\infty \\ \inf\limits_{t\geqslant t_0}\left\{\int\limits_{t_0}^{t} \mathfrak{Re}\, p(\tau)d\tau - \frac{1}{2}\ln G(t)\right\} > -\infty \end{cases}$$

$$\left(\begin{cases}\lim\limits_{t\to+\infty}\left\{\int\limits_{t_0}^{t}\mathfrak{Re}\,p(\tau)d\tau - 2\ln(1+|p(t)|)+\frac{1}{2}\ln G(t)\right\}=+\infty\\[2ex] \lim\limits_{t\to+\infty}\left\{\int\limits_{t_0}^{t}\mathfrak{Re}\,p(\tau)d\tau - \frac{1}{2}\ln G(t)\right\}=+\infty\end{cases}\right)$$

See the proof in [11].

For any positive and continuously differentiable on $[t_0,+\infty)$ function $x(t)$ denote

$$R_x(t_1;t)\equiv\frac{1+\sqrt{x(t_0)}(t_1-t_0)}{1+\sqrt{x(t_0)}(t-t_0)}\exp\left\{-\int\limits_{t_1}^{t}\sqrt{x(s)}ds\right\}\sup_{\xi\in[t_0,t_1]}\frac{|(\sqrt{x(\xi)})'|}{\sqrt{x(\xi)}} + \sup_{\xi\in[t_1,t]}\frac{|(\sqrt{x(\xi)})'|}{\sqrt{x(\xi)}},$$

$t_0\leqslant t_1\leqslant t$. Set $\rho_x(t)\equiv\inf\limits_{t_1\in[t_0,t]}R_x(t_1;t),\ \ t\geqslant t_0$.

THEOREM 2.3. *Let the conditions*

A) $G(t)<0,\ t\geqslant t_0,\ p(t)$ and $G(t)$ are continuously differentiable, and one of the following groups of conditions

B) $G(t)$ is non increasing; for some $\varepsilon>0$ the function $\frac{G'(t)}{|G(t)|^{3/2-\varepsilon}}$ is bounded;

C) $-G(t)\geqslant\varepsilon>0$; the function $\frac{G'(t)}{G(t)}$ is bounded and $\int\limits_{t_0}^{+\infty}\rho_{|G|}(\tau)\frac{|G'(\tau)|}{|G(\tau)|^{3/2}}d\tau<+\infty$

be satisfied. Then all solutions of Eq. (2.1) are bounded (tend to zero for t tending to $+\infty$) if and only if

$$\inf_{t\geqslant t_0}\left[\int\limits_{t_0}^{t}\left(\mathfrak{Re}\,p(\tau)-2\sqrt{|G(\tau)|}\right)+\frac{1}{2}\ln|G(t)|\right]>-\infty$$

$$\left(\lim_{t\to+\infty}\left[\int\limits_{t_0}^{t}\left(\mathfrak{Re}\,p(\tau)-2\sqrt{|G(\tau)|}\right)+\frac{1}{2}\ln|G(t)|\right]=+\infty\right).$$

See the proof in [12].

THEOREM 2.4. *Let the conditions A) and the group of conditions C) or the group of conditions*

D) $G(t)$ is non increasing, $\frac{G'(t)}{G(t)}$ is bounded

be satisfied. Then Eq. (2.1) is Lyapunov (asymptotically) stable if and only if

$$\inf_{t\geqslant t_0}\left[\int\limits_{t_0}^{t}\left(\mathfrak{Re}\,p(\tau)-2\sqrt{|G(\tau)|}\right)d\tau+\frac{1}{2}\ln|G(t)|-2\ln(1+|p(t)-2\sqrt{|G(t)|}|)\right]>-\infty$$

$$\left(\lim_{t\to+\infty}\left[\int\limits_{t_0}^{t}\left(\mathfrak{Re}\,p(\tau)-2\sqrt{|G(\tau)|}\right)d\tau+\frac{1}{2}\ln|G(t)|-2\ln(1+|p(t)-2\sqrt{|G(t)|}|)\right]=+\infty\right).$$

See the proof in [12].

COROLLARY 2.1. *Assume* $-G(t) \geqslant \varepsilon > 0,\ t \geqslant t_0;\ \left|\frac{G'(t)}{G(t)}\right| \leqslant \frac{M}{(1+t-t_0)^{\alpha}},\ t \geqslant t_0,$ $M > 0,\ \alpha > 0,\ \int\limits_{t_0}^{+\infty} \frac{d\tau}{\sqrt{|G(\tau)|}(1+\tau-t_0)^{2\alpha}} < +\infty$ *and let the conditions A) be satisfied. Then the following statements are valid.*

$A_1)$ *All solutions of Eq.* (2.1) *are bounded (tend to zero for* t *tending to* $+\infty$*) if and only if*

$$\inf_{t\geqslant t_0}\left[\int\limits_{t_0}^{t}\left(\mathfrak{Re}\, p(\tau) - 2\sqrt{|G(\tau)|}\right) + \frac{1}{2}\ln|G(t)|\right] > -\infty$$

$$\left(\lim_{t\to+\infty}\left[\int\limits_{t_0}^{t}\left(\mathfrak{Re}\, p(\tau) - 2\sqrt{|G(\tau)|}\right) + \frac{1}{2}\ln|G(t)|\right] = +\infty\right);$$

$B_1)$ *Eq.* (2.1) *is Lyapunov (asymptotically) stable if and only if*

$$\inf_{t\geqslant t_0}\left[\int\limits_{t_0}^{t}\left(\mathfrak{Re}\, p(\tau)-2\sqrt{|G(\tau)|}\right)d\tau+\frac{1}{2}\ln|G(t)|-2\ln(1+|p(t)-2\sqrt{|G(t)|}\,|)\right] > -\infty$$

$$\left(\lim_{t\to+\infty}\left[\int\limits_{t_0}^{t}\left(\mathfrak{Re}\, p(\tau)-2\sqrt{|G(\tau)|}\right)d\tau+\frac{1}{2}\ln|G(t)|-2\ln(1+|p(t)-2\sqrt{|G(t)|}\,|)\right] = +\infty\right).$$

See the proof in [12].

Consider the Riccati equations

$$y' + b(t)y^2 + A(t)y - c(t) = 0, \qquad t \geqslant t_0, \tag{2.3}$$

$$z' + c(t)z^2 - A(t)z - a(t) = 0, \qquad t \geqslant t_0, \tag{2.4}$$

where $A(t) \equiv a(t) - d(t),\ t \geqslant t_0$. It is not difficult to verify that the solutions $y(t)\ \ (z(t))$ of Eq. (2.3) (Eq. (2.4)), existing on an interval $[t_1,t_2)\ \ (t_0 \leqslant t_1 < t_2 \leqslant +\infty)$ are connected with solutions $(\phi(t),\psi(t))$ of the system (1.1) by relations (see e.g.; [2])

$$\phi(t) = \phi(t_1)\exp\left\{\int\limits_{t_1}^{t}\left[b(\tau)y(\tau) + a(\tau)\right]d\tau\right\},\ \ \phi(t_1) \neq 0,\ \ \psi(t) = y(t)\phi(t),\ \ t \in [t_1,t_2) \tag{2.5}$$

$$\left(\psi(t) = \psi(t_1)\exp\left\{\int\limits_{t_1}^{t}\left[c(\tau)z(\tau) + d(\tau)\right]d\tau\right\},\ \ \psi(t_1) \neq 0,\ \ \phi(t) = z(t)\psi(t),\right) \tag{2.6}$$

$t \in [t_1,t_2)$. Hereafter we will assume that $a(t),\ b(t),\ c(t)$ and $d(t)$ are continuously differentiable on $[t_0,+\infty)$ and $a(t) \neq 0,\ c(t) \neq 0,\ t \geqslant t_0$. Set:

$$D_1(t) \equiv \frac{a(t)b'(t) - a'(t)b(t)}{b(t)} + a(t)d(t) - b(t)c(t),$$

$$D_2(t) \equiv \frac{d(t)c'(t) - d'(t)c(t)}{c(t)} + a(t)d(t) - b(t)c(t), \qquad t \geqslant t_0.$$

The substitution

$$u = b(t)y + a(t), \qquad t \geqslant t_0 \tag{2.7}$$

in Eq. (2.3) transforms that into the equation

$$u' + u^2 - \left[S(t) + \frac{b'(t)}{b(t)}\right]u + D_1(t) = 0, \qquad t \geqslant t_0, \tag{2.8}$$

where $S(t) \equiv a(t) + d(t),\ t \geqslant t_0$. Analogously the substitution

$$v = c(t)z + d(t), \qquad t \geqslant t_0 \tag{2.9}$$

in Eq. (2.4) transforms that into the equation

$$v' + v^2 - \left[S(t) + \frac{c'(t)}{c(t)}\right]v + D_2(t) = 0, \qquad t \geqslant t_0, \tag{2.10}$$

Consider the second order linear ordinary differential equations

$$\phi'' - \left[S(t) + \frac{b'(t)}{b(t)}\right]\phi' + D_1(t)\phi = 0, \qquad t \geqslant t_0, \tag{2.11}$$

$$\psi'' - \left[S(t) + \frac{c'(t)}{c(t)}\right]\psi' + D_2(t)\psi = 0, \qquad t \geqslant t_0. \tag{2.12}$$

It is not difficult to verify that the solutions $u(t)$ $(v(t))$ of Eq. (2.8) (Eq. (2.10)), existing on $[t_1, t_2)$, are connected with solutions $\phi_0(t)$, $(\psi_0(t))$ of Eq. (2.11) (Rq. (2.12)) by relations

$$\phi_0(t) = \phi_0(t_1)\exp\left\{\int_{t_1}^{t} u(\tau)d\tau\right\},\ \ \phi_0(t_1) \neq 0,\ \ t \in [t_1, t_2), \tag{2.13}$$

$$\psi_0(t) = \psi_0(t_1)\exp\left\{\int_{t_1}^{t} v(\tau)d\tau\right\},\ \ \psi_0(t_1) \neq 0,\ \ t \in [t_1, t_2), \tag{2.14}$$

On the other hand by (2.5)–(2.7) and (2.9) the same solutions $u(t)$ and $v(t)$ are connected with solutions $(\phi(t), \psi(t))$ of the system (1.1) by relations

$$\phi(t) = \phi(t_1)\exp\left\{\int_{t_1}^{t} u(\tau)d\tau\right\}, \qquad \psi(t) = \psi(t_1)\exp\left\{\int_{t_1}^{t} v(\tau)d\tau\right\}, \qquad t \in [t_1, t_2), \tag{2.15}$$

$\phi(t_1) \neq 0,\ \ \psi(t_1) \neq 0,\ \ \frac{u(t_1)-a(t_1)}{b(t_1)}\ \frac{v(t_1)-d(t_1)}{c(t_1)} = 1$. By (2.5)–(2.7) and (2.9) the last equality is equivalent to the following one

$$\left[\frac{\phi'(t_1)}{\phi(t_1)} - a(t_1)\right]\left[\frac{\psi'(t_1)}{\psi(t_1)} - d(t_1)\right] = b(t_1)c(t_1). \tag{2.16}$$

By the uniqueness theorem from (2.13)–(2.16) we immediately get

LEMMA 2.1. *Let $\phi_0(t)$ and $\psi_0(t)$ be solutions of Eq.* (2.11) *and* (2.12) *respectively such that* $\phi_0(t) \neq 0, \ \ \psi_0(t) \neq 0, \ \ t \in [t_1, t_2), \ \ \left[\frac{\phi_0'(t_1)}{\phi_0(t_1)} - a(t_1)\right]\left[\frac{\psi_0'(t_1)}{\psi_0(t_1)} - d(t_1)\right] = b(t_1)c(t_1)$. *Then* $(\phi_0(t), \psi_0(t))$ *is a solution of the system* (1.1) *on* $[t_1, t_2)$.

Hereafter we will assume that $S(t) + \frac{b'(t)}{b(t)}$ and $S(t) + \frac{c'(t)}{c(t)}$ are continuously differentiable on $[t_0, +\infty)$. Set:

$$G_1(t) \equiv D_1(t) + \frac{1}{2}\left[S(t) + \frac{b'(t)}{b(t)}\right]' - \frac{1}{4}\left[S(t) + \frac{b'(t)}{b(t)}\right]^2, \qquad t \geqslant t_0,$$

$$G_2(t) \equiv D_2(t) + \frac{1}{2}\left[S(t) + \frac{c'(t)}{c(t)}\right]' - \frac{1}{4}\left[S(t) + \frac{c'(t)}{c(t)}\right]^2, \qquad t \geqslant t_0.$$

LEMMA 2.2. *Assume* $\mathfrak{Im}\, G_1(t) \equiv 0 \ \ (\mathfrak{Im}\, G_2(t) \equiv 0), \ \ t \geqslant t_0$, *and* $\mathfrak{Im}\left[\lambda - \frac{1}{2}\left(S(t_0) + \frac{b'(t_0)}{b(t_0)}\right)\right] \neq 0 \ \ \left(\mathfrak{Im}\left[\lambda - \frac{1}{2}\left(S(t_0) + \frac{c'(t_0)}{c(t_0)}\right)\right] \neq 0\right)$ *for some complex* λ. *Then Eq.* (2.8) *(Eq.* (2.10)*) has a solution* $u(t) \ \ (v(t))$ *on* $[t_0, +\infty)$ *with* $u(t_0) = \lambda \ \ (v(t_0) = \lambda)$.

Proof. In Eq. (2.8) substitute

$$u = w + \frac{1}{2}\left(S(t) + \frac{b'(t)}{b(t)}\right), \qquad t \geqslant t_0. \tag{2.17}$$

We obtain

$$w' + w^2 + G_1(t) = 0, \qquad t \geqslant t_0.. \tag{2.18}$$

Show that this equation has a solution $w(t)$ on $[t_0, +\infty)$ with $w(t_0) = \lambda + \frac{1}{2}\left[S(t_0) + \frac{b'(t_0)}{b(t_0)}\right]$. Consider the second order linear ordinary differential equation

$$\chi'' + G_1(t)\chi = 0, \qquad t \geqslant t_0.$$

Let $\chi_1(t)$ and $\chi_2(t)$ be the solutions of this equation on $[t_0, +\infty)$ with $\chi_k(t_0) = 1, \ \ k = 1,2, \ \ \chi_1'(t_0) = \lambda_1 - \lambda_2, \ \ \chi_2'(t_0) = \lambda_1 + \lambda_2$, where $\lambda_1 \equiv \mathfrak{Re}\left[\lambda - \frac{1}{2}\left(S(t_0) + \frac{b'(t_0)}{b(t_0)}\right)\right], \ \ \lambda_2 \equiv \mathfrak{Im}\left[\lambda - \frac{1}{2}\left(S(t_0) + \frac{b'(t_0)}{b(t_0)}\right)\right] \neq 0$. Since $G_1(t)$ is a real-valued function $\chi_k(t), \ \ k = 1,2$ are also real-valued ones. Moreover, obviously, $\chi_k(t), \ \ k = 1,2$ are linearly independent. Consequently $\chi(t) \equiv \chi_1(t) + i\chi_2(t) \neq 0, \ \ t \geqslant t_0$ and $w(t) \equiv \frac{\chi'(t)}{\chi(t)}$ is a solution of Eq. (2.18) on $[t_0, +\infty)$ with $w(t_0) = \lambda - \frac{1}{2}\left(S(t_0) + \frac{b'(t_0)}{b(t_0)}\right)$. Then by (2.17) $u(t) \equiv v(t) + \frac{1}{2}\left(S(t) + \frac{b'(t)}{b(t)}\right)$ is a solution of Eq. (2.8) on $[t_0, +\infty)$ with $u(t_0) = \lambda..$ Existence of $v(t)$ can be proved by analogy. The lemma is proved. $\square$

THEOREM 2.5. *The following statements are valid.*

I. The system (1.1) *is Lyapunov (asymptotically) stable if and only if all solutions of Eq.* (2.11) *and Eq.* (2.12) *are bounded (vanish at* $+\infty$*).*

II. Assume $a(t)$, $b(t)$ *and* $\frac{1}{b(t)}$ *are bounded. Then the system* (1.1) *is Lyapunov (asymptotically) stable if and only if Eq.* (2.11) *is Lyapunov (asymptotically) stable.*

Proof. Obviously there exist $\lambda_1 \neq \lambda_2$ such that $\mathfrak{Im}\left[\lambda_k - \frac{1}{2}\left(S(t_0) + \frac{b'(t_0)}{b(t_0)}\right)\right] \neq 0$, $\mathfrak{Im}\left[\frac{b(t_0)c(t_0)}{\lambda_k - a(t_0)} + d(t_0) - \frac{1}{2}\left(S(t_0) + \frac{c'(t_0)}{c(t_0)}\right)\right] \neq 0$, $k=1,2$. Let $u_k(t)$ $(v_k(t))$, $k=1,2$ be solutions of Eq. (3.8) (Eq. (2.10)) with $u_k(t_0) = \lambda_k$ $(v_k(t_0) = \frac{b(t_0)c(t_0)}{\lambda_k - a(t_0)} + d(t_0))$, $k = 1,2$. Then by Lemma 2.2 $u_k(t)$ $(v_k(t)))$, $k=1,2$ exist on $[t_0,+\infty)$; moreover

$$[u_k(t_0) - a(t_0)][v_k(t_0) - d(t_0)] = b(t_0)c(t_0), \qquad k = 1,2. \tag{2.19}$$

Set: $\phi_k(t) \equiv \exp\left\{\int\limits_{t_0}^{t} u_k(\tau)d\tau\right\}$, $\psi_k(t) \equiv \exp\left\{\int\limits_{t_0}^{t} v_k(\tau)d\tau\right\}$, $t \geqslant t_0$, $k=1,2$. By (2.13) (by (2.14)) $\phi_k(t)$ $(\psi_k(t))$, $k=1,2$ are solutions of Eq. (2.11) (of Eq. (2.12)) on $[t_0,+\infty)$ and by (2.19) we have

$$\left[\frac{\phi_k'(t_0)}{\phi_k(t_0)} - a(t_0)\right]\left[\frac{\psi_k'(t_0)}{\psi_k(t_0)} - d(t_0)\right] = b(t_0)c(t_0), \qquad k = 1,2.$$

In virtue of Lemma 2.1 from here it follows that $(\phi_k(t),\psi_k(t))$, $k=1,2$ are solutions of the system (1.1) on $[t_0,+\infty)$. Let us prove statement I. Assume all solutions of Eq. (2.11) and (2.12) are bounded (vanish at $+\infty$). Then the linearly independent solutions $(\phi_k(t),\psi_k(t))$, $k=1,2$ are bounded (vanish at $+\infty$). Consequently the system (1.1) is Lyapunov (asymptotically) stable. Assume now the system (1.1) is Lyapunov (asymptotically) stable. Then the linearly independent solutions $\phi_k(t)$ $(\psi_k(t))$, $k=1,2$ of Eq. (2.11) (of Eq. (2.12)) are bounded (vanish at $+\infty$). Therefore all solutions of Eq. (2.11) and Eq. (2.12) are bounded (vanish at $+\infty$). The statement I is proved. Prove statement II. Assume Eq. (2.11) is Lyapunov (asymptotically) stable. Then the functions $\phi_k(t)$, $\phi_k'(t)$, $k=1,2$ are bounded (vanish at $+\infty$). Since by (1.1) $\psi_k(t) = -\frac{a(t)}{b(t)}\phi_k(t) + \frac{1}{b(t)}\phi_k'(t)$, $k=1,2$ and $\frac{a(t)}{b(t)}$, $\frac{1}{b(t)}$ are bounded the functions $\psi_k(t)$, $k=1,2$ are bounded (vanish at $+\infty$) as well. So the linearly independent solutions $(\phi_k(t),\psi_k(t))$, $k=1,2$ of the system (1.1) are bounded (vanish at $+\infty$). Therefore the system (1.1) is Lyapunov (asymptotically) stable. Let now the system (1.1) be Lyapunov (asymptotically) stable. Then the functions $\phi_k(t)$, $\psi_k(t)$, $k=1,2$ are bounded (vanish at $+\infty$). Since by (1.1) $\phi_k'(t) = a(t)\phi_k(t) + b(t)\psi_k(t)$, $t \geqslant t_0$ and the functions $a(t)$ and $b(t)$ are bounded the functions $\phi_k'(t)$, $k=1,2$ are also bounded (vanish at $+\infty$). Thus all solutions $\phi(t)$ of Eq. (2.11) with $\phi'(t)$ are bounded (vanish at $+\infty$). Therefore Eq. (2.11) is Lyapunov (asymptotically) stable. The theorem is proved. □

REMARK 2.2. From the proof of statement II is seen that the restrictions on $c(t)$ for that statement are not obligatory.

3. Main results

In this section we study the stability behavior of the system (1.1) in the following cases

I. $G_1(t) > 0, \quad G_2(t) > 0, \quad t \geqslant t_0;$
II. $G_1(t) > 0, \quad G_2(t) < 0, \quad t \geqslant t_0;$
III. $G_1(t) < 0, \quad G_2(t) < 0, \quad t \geqslant t_0;$
IV. $G_1(t) > 0, \quad t \geqslant t_0;$
V. $G_2(t) < 0, \quad t \geqslant t_0.$

The case VI. $G_1(t) < 0,\ G_2(t) > 0,\ t \geqslant t_0$ is reducible to the case III by simple transformation $\phi \to -\phi$.

REMARK 3.1. It is easy to study the trivial case $G_1(t) = G_2(t) \equiv 0,\ t \geqslant t_0$ separately.

Set:

$$\mathscr{L}_k(t) \equiv \frac{1}{\sqrt[4]{G_k(t)}} \int\limits_{t_0}^{t} \frac{|(\sqrt{G_k(\tau)})'|}{\sqrt[4]{G_k(\tau)}} d\tau, \qquad k = 1,2, \qquad t \geqslant t_0.$$

THEOREM 3.1. *Let the following conditions be satisfied*

1) $G_k(t) > 0, \quad t \geqslant t_0, \quad \lim\limits_{t\to+\infty} \frac{G_k'(t)}{G_k^{3/2}(t)} = \alpha_k,\ |\alpha_k| < 4,\ k = 1,2;$

2) $\mathscr{L}_k(t)$ *and* $Var_{t_0}^{t} \frac{G_k'(t)}{G_k^{3/2}(t)}$ *are bounded* $k = 1,2$.

Then the system (1.1) *is Lyapunov (asymptotically) stable if and only if*

$$\begin{cases} \sup\limits_{t\geqslant t_0} \left[\int\limits_{t_0}^{t} \mathfrak{Re}\, S(\tau) d\tau - \ln|b(t)| - \frac{1}{2}\ln G_1(t) \right] < +\infty, \\ \sup\limits_{t\geqslant t_0} \left[\int\limits_{t_0}^{t} \mathfrak{Re}\, S(\tau) d\tau - \ln|c(t)| - \frac{1}{2}\ln G_2(t) \right] < +\infty. \end{cases} \tag{3.1}$$

$$\left(\begin{cases} \lim\limits_{t\to+\infty} \left[\int\limits_{t_0}^{t} \mathfrak{Re}\, S(\tau) d\tau - \ln|b(t)| - \frac{1}{2}\ln G_1(t) \right] = -\infty, \\ \lim\limits_{t\to+\infty} \left[\int\limits_{t_0}^{t} \mathfrak{Re}\, S(\tau) d\tau - \ln|c(t)| - \frac{1}{2}\ln G_2(t) \right] = -\infty. \end{cases} \right) \tag{3.2}$$

Proof. By virtue of Theorem 2.1 from conditions 1) ,2) it follows that the solutions of Eq. (2.11) and (2.12) are bounded (vanish at $+\infty$) if and only if the inequalities (3.1) (the equalities (3.2)) are satisfied. Then by statement I of Theorem 2.5 the system (1.1) is Lyapunov (asymptotically) stable if and only if the inequalities (3.1) (the equalities (3.2)) are fulfilled. The theorem is proved. $\square$

THEOREM 3.2. *Let the following conditions be satisfied*

3) $G_1(t) > 0,\ t \geqslant t_0, \qquad \lim\limits_{t\to+\infty} \frac{G_1'(t)}{G_1^{3/2}(t)} = \alpha_1,\ |\alpha_1| < 4;$

4) $\mathscr{L}_1(t)$ *and* $Var_{t_0}^t \frac{G_1'(t)}{G_1^{3/2}(t)}$ *are bounded;*

5) $G_2(t) < 0,\ t \geqslant t_0,$ *and is non increasing,* $\frac{G_2'(t)}{|G_2(t)|^{3/2-\varepsilon}}$ *is bounded for some* $\varepsilon > 0$*, or*

5_1) $|G_2(t)| \geqslant \varepsilon > 0,\ \frac{G_2'(t)}{G_2(t)}$ *is bounded and* $\int\limits_{t_0}^{+\infty} \rho_{|G_2|}(\tau) \frac{|G_2'(\tau)|}{|G_2(\tau)|^{3/2}} d\tau < +\infty.$

Then the system (1.1) *is Lyapunov (asymptotically) stable if and only if*

$$\begin{cases} \sup\limits_{t\geqslant t_0} \left[\int\limits_{t_0}^{t} \mathfrak{Re}\, S(\tau) d\tau - \ln|b(t)| - \frac{1}{2}\ln G_1(t) \right] < +\infty, \\ \sup\limits_{t\geqslant t_0} \left[\int\limits_{t_0}^{t} \left(\mathfrak{Re}\, S(\tau) + \sqrt{|G_2(\tau)|} \right) d\tau - \ln|c(t)| - \frac{1}{2}\ln|G_2(t)| \right] < +\infty. \end{cases} \tag{3.3}$$

$$\left(\begin{cases} \lim\limits_{t\to+\infty} \left[\int\limits_{t_0}^{t} \mathfrak{Re}\, S(\tau) d\tau - \ln|b(t)| - \frac{1}{2}\ln G_1(t) \right] = -\infty, \\ \lim\limits_{t\to+\infty} \left[\int\limits_{t_0}^{t} \left(\mathfrak{Re}\, S(\tau) + \sqrt{|G_2(\tau)|} \right) d\tau - \ln|c(t)| - \frac{1}{2}\ln|G_2(t)| \right] = -\infty. \end{cases} \right) \tag{3.4}$$

Proof. By Theorem 2.1 from conditions 3), 4) it follows that the solutions of Eq. (2.11) are bounded (tend to zero for t tending to $+\infty$) if and only if the first of the inequalities (3.3) (the first of the equalities (3.4)) is satisfied. By Theorem 2.3 from conditions 5) or 5_1) it follows that the solutions of Eq. (2.12) are bounded (tend to zero for t tending to $+\infty$) if and only if the second of the inequalities(3.3) (of the equalities (3.4)) is satisfied. Then by Theorem 2.5 the system (1.1) is Lyapunov (asymptotically) stable if and only if the inequalities (3.3) (the equalities (3.4)) are satisfied. The theorem is proved. □

By analogy can be proved

THEOREM 3.3. *Let the following conditions be satisfied*

6) $G_k(t) < 0,\ t \geqslant t_0,\ G_k(t)$ *is non increasing* $k = 1,2$

7) $\frac{G_k'(t)}{|G_k(t)|^{3/2-\varepsilon}}$ *is bounded for some* $\varepsilon > 0,\ k = 1,2$ *or*

7_1) $|G_k(t)| \geqslant \varepsilon > 0,\ t \geqslant t_0,\ \frac{G_k'(t)}{G_k(t)}$ *is bounded and*

$$\int\limits_{t_0}^{+\infty} \rho_{|G_k|}(\tau) \frac{|G_k'(\tau)|}{|G_k(\tau)|^{3/2}} d\tau < +\infty,\ k = 1,2.$$

Then the system (1.1) *is Lyapunov (asymptotically) stable if and only if*

$$\begin{cases} \sup\limits_{t\geqslant t_0}\left[\int\limits_{t_0}^{t}\left(\mathfrak{Re}\, S(\tau)d\tau+\sqrt{|G_1(\tau)|}\right)-\ln|b(t)|-\frac{1}{2}\ln|G_1(t)|\right]<+\infty, \\ \sup\limits_{t\geqslant t_0}\left[\int\limits_{t_0}^{t}\left(\mathfrak{Re}\, S(\tau)+\sqrt{|G_2(\tau)|}\right)d\tau-\ln|c(t)|-\frac{1}{2}\ln|G_2(t)|\right]<+\infty. \end{cases}$$

$$\left(\begin{cases} \lim\limits_{t\to+\infty}\left[\int\limits_{t_0}^{t}\left(\mathfrak{Re}\, S(\tau)+\sqrt{|G_1(\tau)|}\right)d\tau-\ln|b(t)|-\frac{1}{2}\ln|G_1(t)|\right]=-\infty, \\ \lim\limits_{t\to+\infty}\left[\int\limits_{t_0}^{t}\left(\mathfrak{Re}\, S(\tau)+\sqrt{|G_2(\tau)|}\right)d\tau-\ln|c(t)|-\frac{1}{2}\ln|G_2(t)|\right]=-\infty. \end{cases}\right)$$

THEOREM 3.4. *Let the following conditions be satisfied*

8) $a(t)$, $b(t)$ *and* $\frac{1}{b(t)}$ *are bounded;*

9) $G_1(t)>0,\ t\geqslant t_0,\ \lim\limits_{t\to+\infty}\frac{G_1'(t)}{G_1^{3/2}(t)}=\alpha_1,\ |\alpha_1|<4$*;*

10) $\mathscr{L}_1(t)$ *and* $Var_{t_0}^{i}\frac{G_1'(t)}{G_1^{3/2}(t)}$ *are bounded.*

Then the system (1.1) *is Lyapunov (asymptotically) stable if and only if*

$$\begin{cases} \sup\limits_{t\geqslant t_0}\left[\int\limits_{t_0}^{t}\mathfrak{Re}\, S(\tau)d\tau+\ln|b(t)|-\frac{1}{2}\ln G_1(t)+2\ln\left(1+\left|S(t)+\frac{b'(t)}{b(t)}\right|\right)\right]<+\infty, \\ \sup\limits_{t\geqslant t_0}\left[\int\limits_{t_0}^{t}\mathfrak{Re}\, S(\tau)d\tau+\ln|b(t)|+\frac{1}{2}\ln G_1(t)\right]<+\infty. \end{cases} \tag{3.5}$$

$$\left(\begin{cases} \lim\limits_{t\to+\infty}\left[\int\limits_{t_0}^{t}\mathfrak{Re}\, S(\tau)d\tau+\ln|b(t)|-\frac{1}{2}\ln G_1(t)+2\ln\left(1+\left|S(t)+\frac{b'(t)}{b(t)}\right|\right)\right]=-\infty, \\ \lim\limits_{t\to+\infty}\left[\int\limits_{t_0}^{t}\mathfrak{Re}\, S(\tau)d\tau+\ln|b(t)|+\frac{1}{2}\ln G_1(t)\right]=-\infty. \end{cases}\right) \tag{3.6}$$

Proof. By virtue of Theorem 2.2 it follows from the conditions 9), 10) that Eq. (2.11) is Lyapunov (asymptotically) stable if and only if the inequalities (3.5) (the equalities (3.6)) hold. Then by Theorem 2.5 (statement II) from 8) it follows that the system (1.1) is Lyapunov (asymptotically) stable if and only if the inequalities (3.5) (the equalities (3.6)) are satisfied. The theorem is proved. □

By analogy can be proved

THEOREM 3.5. *Let the condition* 8) *of Theorem* 3.4 *and the following conditions be satisfied*

11) $G_1(t) < 0,\ t \geqslant t_0,\ S(t) + \frac{b'(t)}{b(t)}$ *and* $G_1(t)$ *are continuously differentiable on* $[t_0, +\infty)$*;*

12) $G_1(t)$ *is non increasing and for some* $\varepsilon > 0$ *the function* $\frac{G_1'(t)}{|G_1(t)|^{3/2-\varepsilon}}$ *is bounded or*

12_1*)* $-G_1(t) \geqslant \varepsilon > 0$*, the function* $\frac{G_1'(t)}{G_1(t)}$ *is bounded and* $\int\limits_{t_0}^{+\infty} \rho_{|G_1|}(\tau) \frac{|G_1'(\tau)|}{|G_1(\tau)|^{3/2}} d\tau < +\infty$.

Then the system (1.1) *is Lyapunov (asymptotically) stable if and only if*

$$\sup_{t \geqslant t_0} \Bigg[\int\limits_{t_0}^{t} \Big(\mathfrak{Re}\, S(\tau) + 2\sqrt{|G_1(\tau)} \Big) d\tau + \ln|b(t)| + 2\ln\Big[1 + \Big| S(t) + \frac{b'(t)}{b(t)} + 2\sqrt{|G_1(t)|}\Big|\Big] - \frac{1}{2}\ln|G_1(t)| \Bigg] < +\infty$$

$$\Bigg(\lim_{t \to +\infty} \Bigg[\int\limits_{t_0}^{t} \Big(\mathfrak{Re}\, S(\tau) + 2\sqrt{|G_1(\tau)} \Big) d\tau + \ln|b(t)| + 2\ln\Big[1 + \Big| S(t) + \frac{b'(t)}{b(t)} + 2\sqrt{|G_1(t)|}\Big|\Big] - \frac{1}{2}\ln|G_1(t)| \Bigg] = -\infty. \Bigg)$$

REMARK 3.2. On the basis of Corollary 2.1 and Theorem 2.6 one can conclude that the conditions 7) and 7_1) of Theorem 3.3 can be replaced by the following simple ones.

$$-G_k(t) \geqslant \varepsilon > 0,\ t \geqslant t_0,\ \frac{|G_k'(t)|}{|G_k(t)|} \leqslant \frac{M}{(1+t-t_0)}\alpha_k,\ t \geqslant t_0,\ \alpha_k > 0,$$

$$\int\limits_{t_0}^{+\infty} \frac{d\tau}{\sqrt{|G_k(\tau)|}(1+\tau-t_0)^{2\alpha_k}} < +\infty,\ k = 1,2.$$

Similar conclusions are valid with respect to the conditions of Theorem 3.2, Theorem 3.4 and Theorem 3.5.

REMARK 3.3. Let $a_0,\ b_0,\ c_0$ and d_0 be real constants. Consider the linear system

$$\begin{cases} \phi' = a_0\phi + b_0\psi, \\ \psi' = c_0\phi + d_0\psi,\ t \geqslant t_0. \end{cases}$$

According to the Routh-Hurwitz's criterion (see [1], pp. 105, 106) this system is asymptotically stable if and only if

$$a_0 + d_0 < 0 \quad \text{and} \quad a_0 d_0 - b_0 c_0 > 0.$$

Then it is not difficult to verify that (except the trivial cases $G_1(t) = G_2(t) \equiv 0$ and $b(t) = c(t) \equiv 0$) in the two dimensional case the Routh-Hurwitz's criterion is a consequence of the group of Theorem 3.4 and Theorem 3.5 (in these theorems the restrictions on $c(t)$ are not obligatory [see Remark 2.2]).

It should be noted that the obtained results can be used to study the stability of plane oscillation of a feathered rocket about its center of gravity (see [13], pp. 32, 33).

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Georg A. Grigorian
Institute of Mathematics NAS of Armenia
Armenia c. Yerevan, str. M. Bagramian 24/5
e-mail: mathphys2@instmath.sci.am