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PERIODIC AND SUBHARMONIC SOLUTIONS FOR A CLASS OF SUPERQUADRATIC FIRST ORDER HAMILTONIAN SYSTEMS

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Abstract. In this paper, we investigate the existence of periodic and subharmonic solutions for the first order Hamiltonian systems. By virtue of auxiliary functions, we obtain some kinds of new superquadratic growth conditions. Using the minimax methods in critical point theory, several new existence and multiplicity theorems are established.

1. Introduction and main results

In this paper, we are concerned with the following non-autonomous first order Hamiltonian systems

$$-J\dot{u}-B(t)u=\nabla H(t,u), \tag{1.1}$$

where $B(t)$ is a symmetric $2N\times 2N$-matrix, continuous and T-periodic in t, $T>0$, $H\in C^1(\mathbb{R}\times\mathbb{R}^{2N},\mathbb{R})$ is a T-periodic function in t, $J=\begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ is the standard $2N\times 2N$ symplectic matrix.

In the celebrated paper [8], making use of critical point theory, Rabinowitz has established the existence of periodic solutions of the non-autonomous Hamiltonian systems with a classical superquadratic condition, namely,

(S) there exist $\mu>2$ and $L_1>0$ such that

$$0<\mu H(t,z)\leqslant(\nabla H(t,z),z),\quad \forall(t,z)\in\mathbb{R}\times\mathbb{R}^{2N} \text{ with } |z|\geqslant L_1,$$

where $(\cdot,\cdot)$ denotes the Euclid's inner product and $|\cdot|$ denotes the corresponding Euclid's norm. For results on the existence of periodic and subharmonic solutions of Hamiltonian systems under condition (S), we refer the reader to the book of Mawhin and Willem [6], where an extensive literature is given. After that, generalized superquadratic conditions covering condition (S) were raised in many literature, such as [1, 2, 3, 4, 5, 7, 10, 11, 12, 13, 14, 15, 16, 17]. Particularly, for periodic solution, in [3], Li, Ou and Tang have proved the following theorem with the aid of the local linking theorem.

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THEOREM A. (see [3]) *Suppose that* $H \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ *satisfies the following conditions:*

(H_0) $H(t,z)$ *is a* T*-periodic function in* t*;*

(H_1) $\lim_{|z|\to+\infty} \frac{H(t,z)}{|z|^2} = +\infty$ *uniformly for all* $t \in [0,T]$*;*

(H_2) $\lim_{|z|\to 0} \frac{H(t,z)}{|z|^2} = 0$ *uniformly for all* $t \in [0,T]$*;*

(H_3) *there exist* $\lambda > 2$ *and* $d_1 > 0$ *such that*

$$|\nabla H(t,z)| \leqslant d_1(1+|z|^{\lambda-1}) \quad \text{for all } (t,z) \in [0,T] \times \mathbb{R}^{2N};$$

(H_4) *there exist* $\beta > \lambda - 2$ *such that*

$$\liminf_{|z|\to+\infty} \frac{(\nabla H(t,z),z) - 2H(t,z)}{|z|^\beta} > 0, \qquad \text{uniformly for all } t \in [0,T].$$

If 0 *is an eigenvalue of* $-J(d/dt) - B(t)$ *(with periodic boundary conditions), assume also the condition:*

(H_5) *there exists* $\delta > 0$ *such that*

(i) $H(t,z) \geqslant 0, \quad \forall |z| \leqslant \delta,\ \forall t \in [0,T]$*, or*
(ii) $H(t,z) \leqslant 0, \quad \forall |z| \leqslant \delta,\ \forall t \in [0,T]$.

Then problem (1.1) *has at least one non-trivial* T*-periodic solution.*

Instead hypothesis (H_5) with the following condition:

(H_6) $H(t,z) \geqslant 0, \quad \forall (t,z) \in [0,T] \times \mathbb{R}^{2N},$

they also have considered infinitely many subharmonic solutions of problem (1.1) by virtue of generalized mountain pass theorem. Specially, they have obtained the following results.

THEOREM B. (see [3]) *Suppose that* $H \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ *satisfies* (H_0)–(H_2)*,* (H_6) *and the following conditions:*

(H_7) *there exist* $L_2 > 0$ *and* $d_2 > 0$ *such that*

$$(\nabla H(t,z),z) - 2H(t,z) \geqslant d_2|z|^2, \quad \forall |z| \geqslant L_2;$$

(H_8) *there exist* $L_3 > 0$*,* $d_3 > 0$ *and* $\sigma > 1$ *such that*

$$(\nabla H(t,z),z) - 2H(t,z) \geqslant d_3 \frac{|\nabla H(t,z)|^\sigma}{|z|^\sigma}, \quad \forall |z| \geqslant L_3.$$

Then there exist infinitely many subharmonic solutions of problem (1.1).

THEOREM C. (see [3]) *Suppose that* $H \in C^1(\mathbb{R}\times\mathbb{R}^{2N},\mathbb{R})$ *satisfies* (H_0)–(H_4) *and* (H_6). *Then there exist infinitely many subharmonic solutions of problem* (1.1).

Motivated by the results of [5, 13, 16, 17, 3, 15, 4], in present paper, introducing several auxiliary functions, we will present some existence and multiplicity theorems under new superquadratic growth conditions. For periodic solutions, we have the following theorems.

THEOREM 1. *Suppose that* $H \in C^1(\mathbb{R}\times\mathbb{R}^{2N},\mathbb{R})$ *satisfies* (H_0)–(H_2) *and the following conditions:*

(H_9) *there exist* $d_4 > 0$, $d_5 > 0$, $\lambda > 2$ *and* $h \in C(\mathbb{R}^+,\mathbb{R}^+)$ *with* $h(s)$ *non-decreasing for all* $s \in \mathbb{R}^+$, *such that*

$$|\nabla H(t,z)| \leqslant d_4[1+h(|z|)] \quad \textit{for all } (t,z) \in [0,T]\times\mathbb{R}^{2N},$$

where h *satisfies the following condition:*

$$h(s) \leqslant d_5(1+s^{\lambda-1}) \quad \textit{for all } s \in \mathbb{R}^+;$$

(H_{10}) *there exist* $\sigma_1 > 1$, $M_1 > 0$, $\theta_1 \in C(\mathbb{R}^+,\mathbb{R}^+)$ *with* $\lim_{s\to+\infty}\frac{h(s)}{\theta_1(s)s} = 0$, *where* $h(s)$ *is defined in* (H_9), *such that*

$$(\nabla H(t,z),z) - 2H(t,z) \geqslant \theta_1^{\sigma_1}(|z|), \quad \forall |z| \geqslant M_1 \textit{ and for all } t \in [0,T].$$

Also assume (H_5) *holds if* 0 *is an eigenvalue of* $-J(d/dt)-B(t)$ *(with periodic boundary conditions). Then problem* (1.1) *has at least one non-trivial* T*-periodic solution.*

REMARK 1. (1) We claim that there exists $d_6 > 0$ such that $h(s) \geqslant d_6 s$ for s large enough. To see this, by (H_2), one has $H(t,0) = 0$. Then it follows from (H_9) that there exists $\alpha \in (0,1)$, such that

$$\begin{aligned} |H(t,z)| = |H(t,z) - H(t,0)| &\leqslant |\nabla H(t,\alpha z)||z| \\ &\leqslant d_4\,[1+h(|\alpha z|)]\,|z| \\ &\leqslant d_4 h(|z|)|z| + d_4|z| \quad \text{for all } (t,z) \in [0,T]\times\mathbb{R}^{2N}. \end{aligned}$$

Moreover, by (H_1), we infer that there exist $d_7 > 0$ and $L_4 > 0$ such that

$$d_7|z|^2 \leqslant H(t,z) \leqslant d_4 h(|z|)|z| + d_4|z|, \quad \forall |z| \geqslant L_4 \text{ and for all } t \in [0,T],$$

which implies that the conclusion holds.

(2) We confirm that $\theta_1(s) \to +\infty$ as $s \to +\infty$. Indeed, taking account of $h(s) \geqslant d_6 s$ for s large enough and $\lim_{s\to+\infty}\frac{h(s)}{\theta(s)s} = 0$, then we can find that the assertion holds.

(3) We observe that (H_3) and (H_4) are special cases of (H_9) and (H_{10}) when $\beta/\sigma_1 > \lambda - 2$. In fact, we only need to put $d_4 = d_1, h(s) = s^{\lambda-1}$ and $\theta_1(s) = s^{\beta/\sigma_1}$, it

is not difficult to verify that $h(s) \leqslant d_5(1+s^{\lambda-1})$, $\lim_{s\to+\infty} \frac{h(s)}{\theta_1(s)s} = 0$, then (H_9), (H_{10}) become (H_3) and (H_4). So, from this sense, Theorem 1 partly generalizes Theorem A.

(4) From (H_{10}), the above discussions of (1) and (2), there exist $L_5 > M_1 > 0$ such that

$$\frac{|\nabla H(t,z)|}{|z|} \geqslant \frac{(\nabla H(t,z),z)}{|z|^2} \geqslant \frac{2H(t,z)}{|z|^2}, \quad \forall |z| \geqslant L_5 \text{ and for all } t \in [0,T],$$

which, by (H_1), leads that

$$\lim_{|z|\to+\infty} \frac{|\nabla H(t,z)|}{|z|} = +\infty \quad \text{uniformly for all } t \in [0,T].$$

Hence, there exists $d_8 > 0$ such that

$$\frac{|z|}{|\nabla H(t,z)|} \leqslant d_8 \quad \text{for all } (t,z) \in [0,T] \times \mathbb{R}^{2N}. \tag{1.2}$$

(5) There exists function $H(t,z)$ satisfying Theorem 1 and not satisfying the corresponding results in [8, 5, 3, 15]. For example, let

$$H(t,z) := g(t)k(z), \quad \forall (t,z) \in \mathbb{R} \times \mathbb{R}^{2N},$$

where $g(t) \in C(\mathbb{R},\mathbb{R}^+)$, $g(t+T) = g(t)$, $T > 0$, $\inf_{t\in[0,T]} g(t) > 0$ and

$$k(z) := |z|^2 \ln(1+|z|^2) + \sin|z|^2 - \ln^2(1+|z|^2) - |z|^2, \quad \forall z \in \mathbb{R}^{2N}.$$

Obviously, $H(t,z)$ does not satisfy the condition (S). At the same time, $\forall \nu > 0$, we have

$$\liminf_{|z|\to+\infty} \frac{(\nabla H(t,z),z) - 2H(t,z)}{|z|^{\nu}} = 0$$

uniformly for all $t \in [0,T]$, which means that (H_4) and (H_7) do not hold. Therefore, the results of Theorem A, Theorem B and Theorem C cannot be applied. Nevertheless, select $h(s) = s\ln(1+s^2)$, $\theta_1(s) = \ln^{3/2}(1+s^2)$, $\sigma_1 = 10/9$, it is easy to check that $H(t,z)$ satisfies all conditions of Theorem 1. Then, by Theorem 1, problem (1.1) has at least one non-trivial T-periodic solution.

THEOREM 2. *Suppose that $H \in C^1(\mathbb{R}\times\mathbb{R}^{2N},\mathbb{R})$ satisfies (H_0)–(H_2) and the following condition:*

(H_{11}) *there exist $M_2 > 0$, $\sigma_2 > 1$, $\theta_2 \in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\lim_{s\to+\infty} \theta_2(s) = +\infty$, such that*

$$(\nabla H(t,z),z) - 2H(t,z) \geqslant \left(\theta_2(|z|)\frac{|\nabla H(t,z)|}{|z|}\right)^{\sigma_2}, \quad \forall |z| \geqslant M_2 \text{ and for all } t \in [0,T].$$

Also assume (H_5) holds if 0 is an eigenvalue of $-J(d/dt) - B(t)$ (with periodic boundary conditions). Then problem (1.1) has at least one non-trivial T-periodic solution.

REMARK 2. We see that assumptions (H_3) and (H_4) could imply (H_{11}). As a matter of fact, by (H_3), there exists $d_9 > 0$ and $L_8 > 0$ such that

$$|\nabla H(t,z)| \leqslant d_9 |z|^{\lambda-1}, \quad \forall |z| \geqslant L_8 \text{ and for all } t \in [0,T]. \tag{1.3}$$

Let $\varepsilon_1 := \frac{1}{2}(\beta - \lambda + 2)$. Note that we have $\beta > \lambda - 2$, hence $\varepsilon_1 > 0$. Let $\theta_2(s) = \ln(1+s), s \in \mathbb{R}^+, \sigma_2 = \frac{\beta-\varepsilon_1}{\lambda-2} > 1$, from (H_4) and (1.3), we conclude that there exist $M_2 > L_8 > 0$ and $d_{10} > 0$ such that

$$\begin{aligned}
(\nabla H(t,z),z) - 2H(t,z)) \geqslant d_{10}|z|^\beta &\geqslant d_9^{\sigma_2} \frac{\ln^{\sigma_2}(1+|z|)}{|z|^{\varepsilon_1}} |z|^\beta \\
&= d_9^{\sigma_2} \left(\ln(1+|z|)|z|^{\lambda-2} \right)^{\sigma_2} \\
&\geqslant \left(\theta_2(|z|) \frac{|\nabla H(t,z)|}{|z|} \right)^{\sigma_2}, \quad \forall |z| \geqslant M_2 \text{ and for all } t \in [0,T],
\end{aligned}$$

which implies (H_{11}) is true. Hence, Theorem 2 greatly improves Theorem A.

Next, turn our attentions to the subharmonic solutions, we have

THEOREM 3. *Suppose that $H \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ satisfies (H_0)–(H_2), (H_6), (H_9) and (H_{10}). Then there exist infinitely many subharmonic solutions of problem* (1.1).

REMARK 3. By Remark 1 (3), then Theorem 3 partially generalizes Theorem C.

THEOREM 4. *Suppose that $H \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ satisfies (H_0)–(H_2), (H_6) and (H_{11}). Then there exist infinitely many subharmonic solutions of problem* (1.1).

REMARK 4. Clearly, (H_{11}) is stronger than condition (H_8), however, Theorem 4 is a novel contribution that does not rely on assumption (H_7) in Theorem B. As such, it can be regarded as a complementary extension of Theorem B. Furthermore, from Remark 2, we know Theorem 4 completely extends Theorem C.

Finally, we would like to point out that the idea of introducing auxiliary functions and obtaining new superquadratic conditions used here is essentially due to [16, 17], where the authors deal with the existence of periodic solutions to certain second order Hamiltonian systems. However, we cannot apply the methods of [16, 17] directly, because problem (1.1) is strongly indefinite, this causes some new difficulties, we need some crucial modifications for our proofs.

The paper is organized as follows. In Section 2, we set up the basic framework in which we study the variational problem associated to (1.1), and we also collect some elementary facts that will be used later. In Section 3, applying local linking theorem and generalized mountain pass theorem, we discuss the periodic and subharmonic solutions of problem (1.1) under different types of potential conditions.

2. Preliminaries

First of all, let $S_T := \mathbb{R}/(T\mathbb{Z})$, $E := W^{1/2,2}(S_T, \mathbb{R}^{2N})$ be the Sobolev space of T-periodic $\mathbb{R}^{2N}$-valued functions with the inner product $(\cdot,\cdot)_E$ and $\|\cdot\|_E$ defined by

$$(u,v)_E := Ta_0\overline{b}_0 + T\sum_{k\neq 0}|k|a_k\overline{b}_k \quad \text{and} \quad \|u\|_E := \left(T|a_0|^2 + T\sum_{k\neq 0}|k||a_k|^2\right)^{1/2}$$

for $u, v \in E$, where

$$u(t) = \sum_{k\in\mathbb{Z}} \exp\left(i\frac{2k\pi t}{T}\right) a_k,\ a_k \in \mathbb{C}^{2N}, a_{-k} = \overline{a}_k$$

and

$$v(t) = \sum_{k\in\mathbb{Z}} \exp\left(i\frac{2k\pi t}{T}\right) b_k,\ b_k \in \mathbb{C}^{2N}, b_{-k} = \overline{b}_k.$$

Define two self-adjoint operators A, $B \in L(E)$ by extending the bilinear forms

$$(Au,v) = \int_0^T (-J\dot{u}, v)dt, \quad (Bu,v) = \int_0^T (B(t)u,v)dt, \quad \forall u,v \in E.$$

Let E^+, E^- and E^0 be the positive, negative and null eigenspace of the linear operator $A-B$, respectively. Then we can consider the splitting $E = E^- \oplus E^0 \oplus E^+$ and an equivalent inner product in E, denoted by $(\cdot,\cdot)$, for $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+ \in E = E^- \oplus E^0 \oplus E^+$, defined by

$$(u,v) = ((A-B)u^+, v^+)_E - ((A-B)u^-, v^-)_E + (u^0, v^0)_E.$$

Then, we have

$$\int_0^T (-J\dot{u} - B(t)u, u)dt = ((A-B)u,u)_E = \|u^+\|^2 - \|u^-\|^2.$$

As we know, space E has the following important embedding property.

LEMMA 1. *E is compactly embedding in $L^\gamma(S_T, \mathbb{R}^{2N})$ for $\gamma \in [1,+\infty)$ and there exists $\tau_\gamma > 0$ such that*

$$\|u\|_{L^\gamma} \leqslant \tau_\gamma \|u\|, \quad \forall u \in E, \tag{2.1}$$

where $\|\cdot\|_{L^\gamma}$ denotes the usual norm on L^γ for all $1 \leqslant \gamma < +\infty$.

Now we define a functional φ on E by

$$\begin{aligned}\varphi(u) :&= \frac{1}{2}\int_0^T (-J\dot{u} - B(t)u, u)dt - \int_0^T H(t,u)dt \\ &= \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \int_0^T H(t,u)dt. \end{aligned} \tag{2.2}$$

It is well known that T-periodic solutions of problem (1.1) correspond to the critical points of φ, and

$$\begin{aligned}(\varphi'(u),v) &= \int_0^T (-J\dot{u}-B(t)u,v)dt - \int_0^T (\nabla H(t,u),v)dt \\ &= (u^+-u^-,v) - \int_0^T (\nabla H(t,u),v)dt \qquad (2.3)\end{aligned}$$

for any $u,v \in E$.

3. Proofs of main results

In the following, we will denote various positive constants as C_i, $i=1,2\cdots$. To prove Theorem 1, we shall use the following local linking theorem (Theorem 2.2 in [11]). Let X be a real Banach space with $X = X^1 \oplus X^2$ and $X_0^j \subset X_1^j \subset \cdots \subset X^j$ such that $X^j = \overline{\bigcup_{n\in\mathbb{N}} X_n^j}$, $j=1,2$. For every multi-index $\alpha = (\alpha_1,\alpha_2) \in \mathbb{N}^2$. Let $X_\alpha = X_\alpha^1 \oplus X_\alpha^2$. We say $\alpha \leqslant \beta \Leftrightarrow \alpha_1 \leqslant \beta_1, \alpha_2 \leqslant \beta_2$. A sequence $(\alpha_n) \in \mathbb{N}^2$ is admissible if, for every $\alpha \in \mathbb{N}^2$ there is $m \in \mathbb{N}$ such that $n \geqslant m \Rightarrow \alpha_n \geqslant \alpha$.

DEFINITION 1. We say that $\varphi \in C^1(X,\mathbb{R})$ satisfies the $(C)^*$ condition if every sequence (u_{α_n}) such that (α_n) is admissible and satisfying

$$u_{\alpha_n} \in X_{\alpha_n}, \quad \sup_{n\in\mathbb{N}} \varphi(u_{\alpha_n}) < +\infty, \quad (1+\|u_{\alpha_n}\|)\varphi'(u_{\alpha_n}) \to 0$$

contains a subsequence which converges to a critical point of φ.

LEMMA 2. (Luan and Mao [11]) *Suppose that $\varphi \in C^1(X,\mathbb{R})$ satisfies the following assumptions:*

(a) $X^1 \neq \{0\}$ and φ has a local linking at 0, that is, for some $r>0$,

$$\varphi(u) \geqslant 0, \quad \forall u \in X^1 \text{ with } \|u\| \leqslant r,$$

$$\varphi(u) \leqslant 0, \quad \forall u \in X^2 \text{ with } \|u\| \leqslant r;$$

(b) φ satisfies $(C)^*$ condition;

(c) φ maps bounded sets into bounded sets;

(d) For every $m \in \mathbb{N}$, $\varphi(u) \to -\infty$ as $\|u\| \to +\infty$ on $X_m^1 \oplus X^2$.

Then φ has at least one non-zero critical point.

The following lemmas give the basic compactness assumptions needed to use minimax methods.

LEMMA 3. *Assume that (H_9) and (H_{10}) hold. Then φ satisfies $(C)^*$ condition.*

Proof. Set $X=E, X^1=E^+\oplus E^0$ and $X^2=E^-$. Choose Hilbert basis $\{e_n\}_{n\geqslant 1}$ for X^1 and $\{e_n\}_{n\leqslant -1}$ for X^2, define

$$X_n^1 := \operatorname{span}\{e_1,\cdots,e_n\}, \quad n\in\mathbb{N},$$
$$X_n^2 := \operatorname{span}\{e_{-1},\cdots,e_{-n}\}, \quad n\in\mathbb{N},$$
$$X^j := \overline{\bigcup_{n\in\mathbb{N}} X_n^j}, \quad j=1,2.$$

Let (u_{α_n}) be a sequence in E such that (α_n) is admissible and satisfying

$$u_{\alpha_n}\in X_{\alpha_n}, \quad \sup_{n\in\mathbb{N}}\varphi(u_{\alpha_n})<+\infty, \quad (1+\|u_{\alpha_n}\|)\varphi'(u_{\alpha_n})\to 0. \tag{3.1}$$

By a standard argument, we only need to prove that (u_{α_n}) is a bounded sequence in X. Otherwise, going if necessary to a subsequence, we can assume that $\|u_{\alpha_n}\|\to+\infty$ as $n\to+\infty$.

Since $\lim_{s\to+\infty}\frac{h(s)}{\theta_1(s)s}=0$, then, $\forall\varepsilon_2>0$, we can easily get that there exists $M_3\geqslant M_1>0$ such that

$$\frac{h(s)}{\theta_1(s)}\leqslant\varepsilon_2 s, \quad \forall s\geqslant M_3. \tag{3.2}$$

Denote $\Omega_{1n}:=\{t\in[0,T]\,|\,|u_{\alpha_n}|>M_3\}$ and $\Omega_{2n}:=\{t\in[0,T]\,|\,|u_{\alpha_n}|\leqslant M_3\}$. Using (2.2), (2.3), (3.1) and (H_{10}), for $n\in\mathbb{N}$, we deduce that

$$\begin{aligned}
C_1 &\geqslant 2\varphi(u_{\alpha_n})-(\varphi'(u_{\alpha_n}),u_{\alpha_n})\\
&=\int_0^T[(\nabla H(t,u_{\alpha_n}),u_{\alpha_n})-2H(t,u_{\alpha_n})]dt\\
&=\int_{\Omega_{1n}}[(\nabla H(t,u_{\alpha_n}),u_{\alpha_n})-2H(t,u_{\alpha_n})]dt+\int_{\Omega_{2n}}[(\nabla H(t,u_{\alpha_n}),u_{\alpha_n})-2H(t,u_{\alpha_n})]dt\\
&\geqslant\int_{\Omega_{1n}}\theta_1^{\sigma_1}(|u_{\alpha_n}|)dt-C_0,
\end{aligned} \tag{3.3}$$

which implies that

$$\int_{\Omega_{1n}}\theta_1^{\sigma_1}(|u_{\alpha_n}|)dt\leqslant C_2. \tag{3.4}$$

Applying Hölder's inequality, (2.1), (3.2) and (3.4), it follows that

$$\begin{aligned}
\int_{\Omega_{1n}}h(|u_{\alpha_n}|)|u_{\alpha_n}^+|dt &= \int_{\Omega_{1n}}\theta_1(|u_{\alpha_n}|)\frac{h(|u_{\alpha_n}|)}{\theta_1(|u_{\alpha_n}|)}|u_{\alpha_n}^+|dt\\
&\leqslant\left(\int_{\Omega_{1n}}\theta_1^{\sigma_1}(|u_{\alpha_n}|)dt\right)^{\frac{1}{\sigma_1}}\left[\int_{\Omega_{1n}}\left(\frac{h(|u_{\alpha_n}|)}{\theta_1(|u_{\alpha_n}|)}|u_{\alpha_n}^+|\right)^{\frac{\sigma_1}{\sigma_1-1}}dt\right]^{\frac{\sigma_1-1}{\sigma_1}}\\
&\leqslant C_2^{\frac{1}{\sigma_1}}\left[\int_{\Omega_{1n}}\left(\frac{h(|u_{\alpha_n}|)}{\theta_1(|u_{\alpha_n}|)}\right)^{\frac{2\sigma_1}{\sigma_1-1}}dt\right]^{\frac{\sigma_1-1}{2\sigma_1}}\left(\int_{\Omega_{1n}}|u_{\alpha_n}^+|^{\frac{2\sigma_1}{\sigma_1-1}}dt\right)^{\frac{\sigma_1-1}{2\sigma_1}}
\end{aligned}$$

$$\begin{aligned} &\leqslant \varepsilon_2 C_2^{\frac{1}{\sigma_1}} \|u_{\alpha_n}\|_{L^{\frac{2\sigma_1}{\sigma_1-1}}} \|u_{\alpha_n}^+\|_{L^{\frac{2\sigma_1}{\sigma_1-1}}} \\ &\leqslant \varepsilon_2 C_2^{\frac{1}{\sigma_1}} \tau^2_{\frac{2\sigma_1}{\sigma_1-1}} \|u_{\alpha_n}\| \|u_{\alpha_n}^+\| \end{aligned} \tag{3.5}$$

for all $n \in \mathbb{N}$. Obviously, we have

$$\int_{\Omega_{2n}} h(|u_{\alpha_n}|)|u_{\alpha_n}^+|dt \leqslant C_3 \|u_{\alpha_n}^+\|_{L^1} \leqslant C_3 \tau_1 \|u_{\alpha_n}^+\| \tag{3.6}$$

for all $n \in \mathbb{N}$. As a consequence, for all $n \in \mathbb{N}$, (2.1), (2.3), (3.1), (3.5), (3.6) and (H_9) give that

$$\begin{aligned} (\varphi'(u_{\alpha_n}), u_{\alpha_n}^+) &= \|u_{\alpha_n}^+\|^2 - \int_0^T (\nabla H(t, u_{\alpha_n}), u_{\alpha_n}^+) dt \\ &\geqslant \|u_{\alpha_n}^+\|^2 - \int_0^T |\nabla H(t, u_{\alpha_n})||u_{\alpha_n}^+| dt \\ &\geqslant \|u_{\alpha_n}^+\|^2 - d_4 \int_0^T h(|u_{\alpha_n}|)|u_{\alpha_n}^+| dt - d_4 \int_0^T |u_{\alpha_n}^+| dt \\ &\geqslant \|u_{\alpha_n}^+\|^2 - \varepsilon_2 d_4 C_2^{\frac{1}{\sigma_1}} \tau^2_{\frac{2\sigma_1}{\sigma_1-1}} \|u_{\alpha_n}\| \|u_{\alpha_n}^+\| - d_4 C_3 \tau_1 \|u_{\alpha_n}^+\| - d_4 \tau_1 \|u_{\alpha_n}^+\|, \end{aligned}$$

which implies that

$$\frac{\|u_{\alpha_n}^+\|}{\|u_{\alpha_n}\|} \to 0 \quad \text{as } n \to +\infty. \tag{3.7}$$

Similarly for $u_{\alpha_n}^-$, one has

$$\frac{\|u_{\alpha_n}^-\|}{\|u_{\alpha_n}\|} \to 0 \quad \text{as } n \to +\infty. \tag{3.8}$$

On the other hand, from (3.2) and Remark 1 (1), for $\varepsilon_2 > 0$ mentioned above and $n \in \mathbb{N}$, we conclude

$$\begin{aligned} \int_{\Omega_{1n}} |u_{\alpha_n}|^2 dt &= \int_{\Omega_{1n}} \theta_1(|u_{\alpha_n}|) \frac{|u_{\alpha_n}|^2}{\theta_1(|u_{\alpha_n}|)} dt \\ &\leqslant \left(\int_{\Omega_{1n}} \theta_1^{\sigma_1}(|u_{\alpha_n}|) dt \right)^{\frac{1}{\sigma_1}} \left[\int_{\Omega_{1n}} \left(\frac{|u_{\alpha_n}|^2}{\theta_1(|u_{\alpha_n}|)} \right)^{\frac{\sigma_1}{\sigma_1-1}} dt \right]^{\frac{\sigma_1-1}{\sigma_1}} \\ &\leqslant \frac{\varepsilon_2}{d_6} C_2^{\frac{1}{\sigma_1}} \|u_{\alpha_n}\|^2_{L^{\frac{2\sigma_1}{\sigma_1-1}}} \\ &\leqslant \frac{\varepsilon_2}{d_6} C_2^{\frac{1}{\sigma_1}} \tau^2_{\frac{2\sigma_1}{\sigma_1-1}} \|u_{\alpha_n}\|^2. \end{aligned} \tag{3.9}$$

Moreover, we know, for $n \in \mathbb{N}$,

$$\int_{\Omega_{2n}} |u_{\alpha_n}|^2 dt \leqslant C_4. \tag{3.10}$$

Then, noting $\dim E^0 < +\infty$, for n large enough, (3.9) and (3.10) ensure that

$$\begin{aligned} C_5 \|u^0_{\alpha_n}\|^2 &\leqslant \int_0^T |u^0_{\alpha_n}|^2 dt \\ &\leqslant \int_0^T |u_{\alpha_n}|^2 dt \\ &\leqslant \frac{\varepsilon_2}{d_6} C_2^{\frac{1}{\sigma_1}} \tau^2_{\frac{2\sigma_1}{\sigma_1 - 1}} \|u_{\alpha_n}\|^2 + C_4, \end{aligned}$$

which implies that

$$\frac{\|u^0_{\alpha_n}\|}{\|u_{\alpha_n}\|} \to 0 \quad \text{as } n \to +\infty. \tag{3.11}$$

Consequently, together with (3.7), (3.8) and (3.11), we obtain

$$1 = \frac{\|u_{\alpha_n}\|}{\|u_{\alpha_n}\|} \leqslant \frac{\|u^0_{\alpha_n}\| + \|u^-_{\alpha_n}\| + \|u^+_{\alpha n}\|}{\|u_{\alpha_n}\|} \to 0 \quad \text{as } n \to +\infty,$$

a contradiction. Therefore, (u_{α_n}) is bounded. Then we conclude that the $(C)^*$ condition is satisfied. □

LEMMA 4. *Assume that (H_{11}) holds. Then φ satisfies $(C)^*$ condition.*

Proof. Let (u_{α_n}) be a sequence in E such that (α_n) is admissible and satisfying (3.1). Now we will prove that (u_{α_n}) is bounded. Otherwise, without loss of generality, we may assume that $\|u_{\alpha_n}\| \to +\infty$ as $n \to +\infty$. In view of $\lim_{s \to +\infty} \theta_2(s) = +\infty$, then $\forall \varepsilon_3 > 0$, there exists $M_4 > 0$ such that

$$\frac{1}{\theta_2(s)} < \varepsilon_3, \quad \forall s \geqslant M_4. \tag{3.12}$$

Let $\Omega^*_{1n} := \{t \in [0,T] | |u_n(t)| > M_4\}$, $\Omega^*_{2n} := \{t \in [0,T] | |u_n(t)| \leqslant M_4\}$. Using the similar way of (3.3), by (H_{11}), one has

$$\int_{\Omega^*_{1n}} \left(\theta_2(|u_{\alpha_n}|) \frac{|\nabla H(t, u_{\alpha_n})|}{|u_{\alpha_n}|} \right)^{\sigma_2} dt \leqslant C_6, \tag{3.13}$$

By (3.12), (3.13), (2.1) and Hölder's inequality, for $n \in \mathbb{N}$, we derive that

$$\begin{aligned}
&\int_{\Omega_{1n}^*} |\nabla H(t,u_{\alpha_n})||u_{\alpha_n}^+|dt \\
&= \int_{\Omega_{1n}^*} \theta_2(|u_{\alpha_n}|)\frac{|\nabla H(t,u_{\alpha_n})|}{|u_{\alpha_n}|}\frac{|u_{\alpha_n}|}{\theta_2(|u_{\alpha_n}|)}|u_{\alpha_n}^+|dt \\
&\leqslant \left[\int_{\Omega_{1n}^*} \left(\theta_2(|u_{\alpha_n}|)\frac{|\nabla H(t,u_{\alpha_n})|}{|u_{\alpha_n}|}\right)^{\sigma_2} dt\right]^{\frac{1}{\sigma_2}} \left[\int_{\Omega_{1n}^*} \left(\frac{|u_{\alpha_n}|}{\theta_2(|u_{\alpha_n}|)}|u_{\alpha_n}^+|\right)^{\frac{\sigma_2}{\sigma_2-1}} dt\right]^{\frac{\sigma_2-1}{\sigma_2}} \\
&\leqslant C_5^{\frac{1}{\sigma_2}}\varepsilon_3\left(\int_{\Omega_{1n}^*} |u_{\alpha_n}|^{\frac{2\sigma_2}{\sigma_2-1}}dt\right)^{\frac{\sigma_2-1}{2\sigma_2}}\left(\int_{\Omega_{1n}^*} |u_{\alpha_n}^+|^{\frac{2\sigma_2}{\sigma_2-1}}dt\right)^{\frac{\sigma_2-1}{2\sigma_2}} \\
&\leqslant C_5^{\frac{1}{\sigma_2}}\varepsilon_3\tau_{\frac{2\sigma_2}{\sigma_2-1}}^2\|u_{\alpha_n}\|\|u_{\alpha_n}^+\|. \qquad (3.14)
\end{aligned}$$

In addition, we obtain

$$\int_{\Omega_{2n}^*} |\nabla H(t,u_{\alpha_n})||u_{\alpha_n}^+|dt \leqslant C_7\|u_{\alpha_n}^+\|_{L^1} \leqslant C_7\tau_1\|u_{\alpha_n}^+\| \qquad (3.15)$$

for $n \in \mathbb{N}$. It follows (3.14), (3.15) and (2.3), for all $n \in \mathbb{N}$, that

$$\begin{aligned}
(\varphi'(u_{\alpha_n}),u_{\alpha_n}^+) &= \|u_{\alpha_n}^+\|^2 - \int_0^T (\nabla H(t,u_{\alpha_n}),u_{\alpha_n}^+)dt \\
&\geqslant \|u_{\alpha_n}^+\|^2 - \int_0^T |\nabla H(t,u_{\alpha_n}||u_{\alpha_n}^+|dt \\
&\geqslant \|u_{\alpha_n}^+\|^2 - C_5^{\frac{1}{\sigma_2}}\varepsilon_3\tau_{\frac{2\sigma_2}{\sigma_2-1}}^2\|u_{\alpha_n}\|\|u_{\alpha_n}^+\| - C_7\tau_1\|u_{\alpha_n}^+\|, \qquad (3.16)
\end{aligned}$$

which implies that

$$\frac{\|u_{\alpha_n}^+\|}{\|u_{\alpha_n}\|} \to 0 \quad \text{as } n \to +\infty. \qquad (3.17)$$

Similarly for $u_{\alpha_n}^-$, one has

$$\frac{\|u_{\alpha_n}^-\|}{\|u_{\alpha_n}\|} \to 0 \quad \text{as } n \to +\infty. \qquad (3.18)$$

Furthermore, by (1.2), (2.1), (3.12), (3.14) and Hölder's inequality, for $n \in \mathbb{N}$, we know

$$\begin{aligned}
\int_{\Omega_{1n}^*} |u_{\alpha_n}|^2dt &= \int_{\Omega_{1n}^*} \theta_2(|u_{\alpha_n}|)\frac{|\nabla H(t,u_{\alpha_n})|}{|u_{\alpha_n}|}\frac{1}{\theta_2(|u_{\alpha_n}|)}\frac{|u_{\alpha_n}|}{|\nabla H(t,u_{\alpha_n})|}|u_{\alpha_n}|^2dt \\
&\leqslant \varepsilon_3 d_8\left[\int_{\Omega_{1n}^*} \left(\theta_2(|u_{\alpha_n}|)\frac{|\nabla H(t,u_{\alpha_n})|}{|u_{\alpha_n}|}\right)^{\sigma_2} dt\right]^{\frac{1}{\sigma_2}}\left(\int_{\Omega_{1n}^*} |u_{\alpha_n}|^{\frac{2\sigma_2}{\sigma_2-1}}dt\right)^{\frac{\sigma_2-1}{\sigma_2}} \\
&\leqslant \varepsilon_3 d_8 C_6^{\frac{1}{\sigma_2}}\tau_{\frac{2\sigma_2}{\sigma_2-1}}^2\|u_{\alpha_n}\|^2. \qquad (3.19)
\end{aligned}$$

Moreover, for $n \in \mathbb{N}$, one has

$$\int_{\Omega_{2n}^*} |u_{\alpha_n}|^2 dt \leqslant C_8. \tag{3.20}$$

Combing (3.19) with (3.20), for $n \in \mathbb{N}$, we have

$$C_5 \|u_{\alpha_n}^0\|^2 \leqslant \int_0^T |u_{\alpha_n}^0|^2 dt \leqslant \int_0^T |u_{\alpha_n}|^2 dt \leqslant \varepsilon_3 d_8 C_6^{\frac{1}{\sigma_2}} \tau^2_{\frac{2\sigma_2}{\sigma_2 - 1}} \|u_{\alpha_n}\|^2 + C_8,$$

which implies that

$$\frac{\|u_{\alpha_n}^0\|}{\|u_{\alpha_n}\|} \to 0 \quad \text{as } n \to +\infty. \tag{3.21}$$

So, together with (3.17), (3.18) and (3.21), one has

$$1 = \frac{\|u_{\alpha_n}\|}{\|u_{\alpha_n}\|} \leqslant \frac{\|u_{\alpha_n}^0\| + \|u_{\alpha_n}^-\| + \|u_{\alpha n}^+\|}{\|u_{\alpha_n}\|} \to 0 \quad \text{as } n \to +\infty,$$

a contradiction. Therefore, (u_{α_n}) is bounded. Then $(C)^*$ condition holds. □

Now, we can prove our main results.

Proof of Theorem 1. From (H_9), we can obtain there exists $d_{13} > 0$ such that

$$|\nabla H(t,z)| \leqslant d_{13}(1 + |z|^{\lambda - 1}), \quad \forall (t,z) \in [0,T] \times \mathbb{R}^{2N}.$$

Then, using the same arguments of Theorem 1 in [3], we know φ satisfies conditions (a), (c) and (d) of Lemma 2. What's more, by Lemma 3, we see that condition (b) of Lemma 2 is also satisfied. Thus, from Lemma 2, problem (1.1) has at least one non-trivial T-periodic solution. □

Proof of Theorem 2. Noting that (H_{11}) implies (H_8), then, using Lemma 4 and the proof of Theorem 1, we infer that problem (1.1) has at least one non-trivial T-periodic solution. □

Proof of Theorem 3. For a given $k \in \mathbb{N}$, making the change of variables $\xi = k^{-1}t$, thus, if $u(t)$ is a kT-periodic solution of problem (1.1), $\eta(\xi) = u(k\xi)$ satisfies

$$-J\dot{\eta}(\xi) - kB(k\xi)\eta = k\nabla H(k\xi, \eta). \tag{3.22}$$

Hence, finding a kT-periodic solution of problem (1.1) is equivalent to finding a T-periodic solution of problem (3.22). Certainly, $kH(k\xi, \eta)$ satisfies the conditions of our Theorem 1, there is a solution $\eta_k(\xi)$ of problem (3.22), which is a critical point of

$$\varphi_k(\eta) = \frac{1}{2}\int_0^T (-J\dot{\eta} - kB(k\xi)\eta, \eta) d\xi - k\int_0^T H(k\xi, \eta) d\xi.$$

Using Lemma 3, the arguments of [3] and the generalized mountain pass theorem in [9], we can easily get problem (3.22) has infinitely many T-periodic solution, that is, problem (1.1) has infinitely many subharmonic solutions. □

Proof of Theorem 4. From Lemma 4 and the proof of Theorem 3, we see that there exist infinitely many subharmonic solutions of problem (1.1). □

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