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GLOBAL EXISTENCE, UNIQUENESS AND ASYMPTOTIC BEHAVIOR FOR A NONLINEAR VISCOELASTIC PROBLEM WITH INTERNAL DAMPING AND LOGARITHMIC SOURCE TERM

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Abstract. This paper is concerned with the existence of global weak solution for a nonlinear viscoelastic problem with internal damping and a logarithmic source term and Dirichlet boundary initial conditions, and with the study of the asymptotic behavior of the problem, involving: a) exponential decay of total energy of solutions for initial data in the set of stability created by the Nehari manifold, b) the exponential growth of the logarithmic source term for negative initial energy. In the existence of global weak solution we employed similar ideas as in the work of S. Cordeiro, J. Ferreira, et al., 2021, where the Faedo-Galerkin method was combined with Aubin-Lions lemmas for the passage to the limit in the nonlinear terms. In the study of the exponential decay of the total energy and in the growth of the logarithmic term of the energy we adapted the perturbed energy methods in a work of Messaoudi & Tatar, 2006 and 2003.

1. Introduction

The existing theory of elasticity accounts for materials that have a capacity to store mechanical energy with no dissipation (of energy). On the other hand, a Newtonian viscous fluid in a nonhydrostatic stress state has a capacity for dissipating energy without storing it. Materials that are outside the scope of these two theories would be those for which some, but not all, of the work done to deform them can be recovered. Such materials possess a capacity for storage and dissipation of mechanical energy. This is the case of 'viscoelastic' materials.

Viscoelastic materials are those for which the behavior combines liquid-like and solid-like characteristics. Viscoelasticity is important in areas such as: biomechanics, power industry or heavy construction, Synthetic polymers, Wood, Human tissue and cartilage, Metals at high temperature, Concrete.

Polymers, for instance, are viscoelastic materials, since they occupy an intermediate position between viscous liquids and elastic solids. The formulation of Boltzmann's superposition principle leads to a memory term involving a relaxation function of exponential type. But, it has been observed that relaxation functions of some viscoelastic

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materials are not necessarily of this type, see [19, 20]. In the present work, we are concerned with the following initial boundary value problem:

$$\begin{cases} |u_t|_{\mathbb{R}}^{\rho} u_{tt} + M(\|u\|^2)(-\Delta u) - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds + u_t \\ \qquad = u|u|_{\mathbb{R}}^{p-2} \ln|u|_{\mathbb{R}}^{k} \quad \text{in} \quad \Omega\times(0,\infty), \\ u = 0 \quad \text{on} \quad \partial\Omega\times[0,\infty), \\ u(x,0) = u_0(x) \quad \text{in} \quad \Omega, \\ u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geqslant 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $p > 2$ and $\rho > 0$ are constants, and $g : \mathbb{R}^+ \to \mathbb{R}^+$ and $M : [0,\infty) \to \mathbb{R}$ are C^1 functions. The notations $\|\cdot\|$, $|\cdot|$, $|\cdot|_{\mathbb{R}}$ and $|\cdot|_{\mathbb{R}^n}$ are defined in Section 2.

From the physical point of view, this is also the type of problems that usually arises in viscoelasticity. It has been considered with a power source term first by Dafermos [11], where the general decay was discussed.

As mentioned in [17], the logarithmic nonlinearity appears in several branches of physics such as inflationary cosmology, nuclear physics, optics, and geophysics. With all this specific underlying meaning in physics, the global-in-time well-posedness of solution to the problem of evolution equation with such logarithmic-type nonlinearity captures lots of attention, see [17] for the references related to each branch listed above.

A problem related to the present work is the nonlinear viscoelastic wave equation

$$|u_t|_{\mathbb{R}}^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \gamma\Delta u_t = b|u|_{\mathbb{R}}^{p-2} u,$$

considered by Messaoudi & Tatar, [20], who used the potential well method to prove existence of global solution. They established results on decay of energy and growth of solution for suitable values of ρ, p and under the assumption of decay of function g. When $\gamma = 0$ this hypothesis on g is dropped in [19] by improving Zuazua's method for the proof of asymptotic decay of energy.

Another problem related to (1) when $\rho = 0$, with a power source term, is

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(\tau)d\tau + au_t = b|u|_{\mathbb{R}}^{p-2} u \quad \text{in} \quad \Omega\times\mathbb{R}^+, \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \\ u_t(x,0) = u_1(x) \quad \text{in } \Omega, \end{cases} \tag{2}$$

which has been extensively studied. When $g = 0$, Zuazua [23] proved that the energy related to the problem (2) decays exponentially depending on the nature of the boundary, while for $g \neq 0$, Cavalcanti et al. [9] studied (2) for a localized damping $a(x)u_t$ and $b < 0$, showing an exponential rate of decay by assuming that the kernel g decays exponentially. It is also possible to prove the exponential decay when $a = 0$ using the ideas introduced by Muñoz Rivera [15] for small initial data. For nonexistence results, Messaoudi [14] considered (2) for $b > 0$ and with a nonlinear mechanical damping of the form $au_t|u_t|^{m-2}$.

For blow-up results related to the problem (2), see Messaoudi [14], who considered $b > 0$ and a nonlinear mechanical damping of the form $au_t|u_t|^{m-2}$. It is also possible to prove finite-time blow-up when $g = 0$ and $a = 0$ and with the source term $bu|u|^{p-2}$ (see [5]). We recommend [20] for a wider exposition of references relating the existence in time for the problem (2) and the terms involving a, b, g and the source.

In [10] a Klein-Gordon equation of Kirchhoff Carrier type was studied, with a strong damping $-\Delta u_t$ and a logarithmic source $u\ln|u|_{\mathbb{R}}$:

$$u_{tt} + M(\|u\|^2)(-\Delta u) + M_1(|u|^2)u - \Delta u_t = u\ln|u|_{\mathbb{R}} \tag{3}$$

proving the global existence of weak solution by using the potential well technique and the exponential decay of total energy by means of Nakao's Lemma. The present work uses ideas similar to [10] to prove the existence of global weak solutions for initial data in the stability set created from the Nehari manifold.

An important class of viscoelastic Kirchhoff equations with a logarithmic term source is that with time varying delay. For instance, we mention the recent work of Nadia Mezour et. al [17] and the references contained therein, where the global and local existence of solutions for the equation

$$\begin{aligned} &|u_t|^{\rho}_{\mathbb{R}}u_{tt} - M(\|\nabla u\|)\Delta u - \Delta u_{tt} + \int_0^t h(t-\tau)\Delta u(s)ds \\ &+ \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau(t))) = \nu u\ln|u|_{\mathbb{R}} \end{aligned}$$

is proved in a suitable Sobolev space using the energy method combined with Faedo-Galerkin procedures.

Our work studies the interaction between the dissipation and the source term and how they interact asymptotically. We prove that, when the initial data are in a stable set, we have global existence. To achieve our goal, we use the potential well method. Moreover, we show that the energy for the problem (1), i.e, the sum of kinetic and potential energies of (1), has an exponential decay for initial data in a subset of a stable set by employing a perturbed energy method. Then we show the exponential growth of the logarithmic term of the energy for a sufficiently large constant ρ and negative initial energy.

This work is structured as follows. Section 2 presents the notation and results underlying the methods used in this paper. In section 3 we describe the set of stability of solutions built by the Nehari manifold by considering a functional energy J, representing the viscoelastic potential energy of the problem (1) including the source and damping terms. Next, in section 4, we state and prove a theorem on global existence of a weak solution for the problem (1). Section 5 establish conditions for the well-posedness of the problem (1). Finally in the sections 6 and 7 we prove results on asymptotic behavior of some terms of the total energy, including exponential decay of total energy and exponential growth of logarithmic term of total energy.

2. Preliminaries and assumptions

DEFINITION 1. Let B be a Banach space and $u:[0,T]\to B$ a measurable function. The vector function spaces $L^p(0,T;B)$, $1\leqslant p\leqslant\infty$, are defined by:

$$L^p(0,T;B)=\left\{u:\left(\int_0^T \|u\|_B^p dt\right)^{1/p}<\infty\right\},\ 1\leqslant p<\infty,$$

and

$$L^\infty(0,T;B)=\{u:\text{ess sup}_{0<t<T}\|u(t)\|_B<\infty\}.$$

DEFINITION 2. If V and W are Banach spaces and $1\leqslant p\leqslant\infty$, then we define

$$W_p(0,T;V,W):=\{f\in L^p(0,T;V)|f'\in L^p(0,T;W)\}.$$

The spaces in the Definition 2 are Banach spaces with the natural norms.

Now we present two well-known compactness results. The compactness is needed to extract a strongly convergent sequence in the set of approximate solutions.

LEMMA 1. (Aubin-Lions lemma, Lions [12], Lemma 1.3, p. 12) *Let $Q=\Omega\times(0,T)$ be an open and bounded set of $\mathbb{R}^n\times\mathbb{R}$. We denote $L^p(0,T;L^p(\Omega))$ by $L^p(Q)$. If g_m, $\forall m\in\mathbb{N}$, and g are functions on $L^p(Q)$, $1<p<\infty$, such that*

$$\|g_m\|_{L^p(Q)}\leqslant C,\ \ g_m\to g \text{ a.e. in } Q$$

then $g_m\rightharpoonup g$ weakly in $L^p(Q)$.

LEMMA 2. (Aubin-Lions lemma, Lions [12], Theorem 5.1, p. 58) *Let B_0, B and B_1 be Banach spaces, B_i, $i=0,1$, reflexive spaces with $B_0\hookrightarrow B$ compactly, $B\hookrightarrow B_1$ continously. Define*

$$W=\{u:u\in L^{p_0}(0,T;B_0);u_t\in L^{p_1}(0,T;B_1)\}$$

where $T>0$ and $1<p_i<\infty$, $i=0,1$. Then $W\subset L^{p_0}(0,T;B)$, equipped with the norm

$$\|w\|=\|u\|_{L^{p_0}(0,T;B_0)}+\|u_t\|_{L^{p_1}(0,T;B_1)},$$

is a Banach space and $W\hookrightarrow L^{p_0}(0,T;B)$ is compact.

For simplicity of notation, hereafter we denote by $|\cdot|$ the Lebesgue space $L^2(\Omega)$ norm, $\|\cdot\|=\int_\Omega|\nabla(\cdot)|^2_{\mathbb{R}^n}dx$ the $H_0^1(\Omega)$ norm, where $|\cdot|_{\mathbb{R}^n}$ is the norm in $\mathbb{R}^n$.

We start by setting several hypotheses for the problem (1).

$(H.1)$ $M\in C^1([0,\infty),\mathbb{R})$ is such that $M(\lambda)\geqslant m_0$ $\forall\lambda\in[0,\infty)$, where $m_0>0$.

$(H.2)$ $g:\mathbb{R}^+\to\mathbb{R}^+$ is a Lebesgue integrable and absolutely continuous function such that

$$1-\int_0^\infty g(s)ds=:l>0.$$

$(H.3)$ There exist positive constants ξ_1 and ξ_2 satisfying

$$-\xi_1 g(t) \leqslant g'(t) \leqslant -\xi_2 g(t) \quad \text{for almost all } t \geqslant 0.$$

We will need the very useful relation

$$\begin{aligned} \int_0^t g(t-\tau)(\nabla u(\tau), \nabla u_t(t))d\tau = &\frac{1}{2}(g' \diamond \nabla u)(t) - \frac{1}{2}(g \diamond \nabla u)'(t) \\ &+ \frac{d}{dt}\left\{\frac{1}{2}\left(\int_0^t g(s)ds\right)|\nabla u(t)|^2\right\} \\ &- \frac{1}{2}g(t)|\nabla u(t)|^2, \end{aligned} \tag{4}$$

which can be checked directly, where

$$(g \diamond y)(t) = \int_0^t g(t-s)|y(t) - y(s)|^2 ds.$$

Let us assume that $p > 2$, $\rho > 0$ and $n = 1$, 2 or 3, and in order to guarantee the Sobolev embedddings, hereafter we assume

$$2 < p \leqslant 3, \;\; 0 < \rho \leqslant 2 \;\text{ if }\; n = 3, \tag{5}$$

$$p > 2, \;\; \rho > 0 \;\text{ if }\; n = 1 \text{ or } 2. \tag{6}$$

Let us denote $\hat{M}(s) = \int_0^s M(\tau)d\tau$. If $u(t), u_t(t) \in H_0^1(\Omega)$, we define the following functionals

$$\mathscr{I}(t) := \hat{M}(\|u\|^2) - \int_0^t g(s)ds\|u\|^2 - \int_\Omega |u|_{\mathbb{R}}^p \ln|u|_{\mathbb{R}}^k dx, \tag{7}$$

$$\begin{aligned} \mathscr{J}(t) := &\frac{1}{2}\left(\hat{M}(\|u\|^2) - \int_0^t g(s)ds\|u\|^2\right) + \frac{1}{2}\|u_t\|^2 \\ &+ \frac{k}{p^2}\int_\Omega |u|_{\mathbb{R}}^p dx + \frac{1}{2}(g \diamond \nabla u)(t) - \frac{1}{p}\int_\Omega |u|_{\mathbb{R}}^p \ln|u|_{\mathbb{R}}^k dx, \end{aligned} \tag{8}$$

$$\mathscr{E}(t) := \frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + \mathscr{J}(t), \tag{9}$$

where $\mathscr{E}(t)$ is called the total energy of (1).

3. Potential well

In this section, we present the potential well corresponding to the logarithmic nonlinearity. It is well known that the energy of a PDE system is, in some sense, split into kinetic and potential energy. Following the idea presented in [21], see also [18], we can construct a stability set as follows. We will prove that there is a valley, or a "well", of depth d created in the potential energy. If this height d is strictly positive, we find that for solutions with initial data in the "good part" of the well, the potential energy of the solution can never escape the well. In general, it is possible for the energy from the source term to cause a blow-up in finite time. However, in the good part of the well it remains bounded. As a result, the total energy of the solution remains finite on any time interval $[0, \infty)$, which provides the global existence of the solution.

3.1. Abstract setting: Nehari manifold

The method of Nehari manifold goes back to Nehari's work [24, 25]. He considered a boundary value problem for a certain nonlinear second-order ordinary differential equation in an interval (a,b) and showed that it has a nontrivial solution, which may be found by constrained minimization of the Euler-Lagrange functional corresponding to the problem. For the basic critical point theory and its applications to nonlinear boundary value problems for elliptic equations, see e.g. [2, 3, 4, 22].

Let E be real Banach space and $J \in C^1(E,\mathbb{R})$ a functional. If $J'(u) = 0$ and $u \neq 0$, then

$$u \in \mathscr{N} := \{u \in E \backslash \{0\} : J'(u)u = 0\}.$$

Thereby $\mathscr{N}$ is a natural constraint for the set of nontrivial solutions. It is called the Nehari manifold though it is not a manifold in general. Assume without loss of generality that $J(0) = 0$ and denote

$$d := \inf_{u \in \mathscr{N}} J(u). \tag{10}$$

The minimum energy d for potential energy along the Nehari manifold will determine the high of the well and will be used to construct a stability set from which the total energy of the system will never escape. However some assumptions on J are required in order to guarantee that $d > 0$. For this purpose we mention next some fundamental results from Ambrossetti and Rabinowitz, which will be useful for proving that $d > 0$.

DEFINITION 3. (Brézis-Coron-Nirenberg, 1980) Let E be a Banach space, $J \in C^1(E,\mathbb{R})$ and $c \in \mathbb{R}$. The function J satisfies $(PS)_c$ condition if any sequence $(u_n) \subset E$ such that

$$J(u_n) \to c, \;\; J'(u_n) \to 0$$

has a convergent subsequence.

LEMMA 3. *Consider*

$$J(u) = \int_\Omega \left(\frac{|\nabla u|^2}{2} + \frac{u^2}{2} - F(u) \right) dx,$$

where $F(u) = \int_0^u f(s)ds$, *with* f *satisfying*

$(f1)$ $f \in C(\mathbb{R})$, *and for some* $2 < p < 2^*$, $c_0 > 0$,

$$|f(s)| \leqslant c_0(|s| + |s|^p),$$

$(f2)$ *there exists* $\alpha > 2$ *such that for every* $s \in \mathbb{R}$,

$$\alpha F(s) \leqslant s f(s).$$

$(f3)$ $f(s) = o(|s|), |s| \to 0.$

Then any sequence $(u_n) \subset E$ *such that*

$$d := \sup_n J(u_n), \ J'(u_n) \to 0,$$

contains a convergent subsequence.

Proof. The proof follows from an adaptation of the proof of Lemma 3.11 in [22]. □

THEOREM 1. (Ambrossetti & Rabinowitz, 1973 [1]) *Let* E *be a Hilbert space,* $J \in C^2(E, \mathbb{R})$, $e \in E$ *and* $r > 0$ *be such that* $\|e\| > r$ *and*

$$\inf_{\|u\|=r} J(u) > J(0) \geqslant J(e).$$

If J *satisfies* $(PS)_c$ *condition, then* c *is a critical value for* J.

3.2. Constructing the stability set

We start by introducing a functional $J : H_0^1(\Omega) \to \mathbb{R}$ by

$$J(u) := -\frac{1+C_p^2}{m_0+l-1}\left[\frac{1}{p}\int_\Omega |u|_{\mathbb{R}}^p \ln|u|_{\mathbb{R}}^k dx - \frac{k}{p^2}\int_\Omega |u|_{\mathbb{R}}^p dx\right] + \frac{1}{2}\int_\Omega |\nabla u|_{\mathbb{R}^n}^2 dx + \frac{1}{2}\int_\Omega |u|_{\mathbb{R}}^2 dx,$$

where C_p stands for the optimal constant in the Poincaré inequality. For all $u \in H_0^1(\Omega)$ and $\lambda > 0$, we have

$$J(\lambda u) = -\frac{\lambda^p(1+C_p^2)}{m_0+l-1}\left[\frac{k}{p}\int_\Omega |u|_{\mathbb{R}}^p dx \ln\lambda + \frac{1}{p}\int_\Omega |u|_{\mathbb{R}}^p \ln|u|_{\mathbb{R}}^k dx - \frac{k}{p^2}\int_\Omega |u|_{\mathbb{R}}^p dx\right] + \frac{\lambda^2}{2}\int_\Omega |\nabla u|_{\mathbb{R}^n}^2 dx + \frac{\lambda^2}{2}\int_\Omega |u|_{\mathbb{R}}^2 dx. \tag{11}$$

The functional $K : H_0^1(\Omega) \to \mathbb{R}$ is defined by $K(u) := \left[\frac{d}{d\lambda}J(\lambda u)\right]_{\lambda=1}$. More precisely,

$$K(u) = -\frac{1+C_p^2}{m_0+l-1}\int_\Omega |u|_{\mathbb{R}}^p \ln|u|_{\mathbb{R}}^k dx + \int_\Omega |\nabla u|_{\mathbb{R}^n}^2 dx + \int_\Omega |u|_{\mathbb{R}}^2 dx.$$

Defining $f \in C^1(\mathbb{R})$ by

$$f(s) := \begin{cases} \frac{(1+C_p^2)|s|^{p-2}s\ln|s|^k}{(m_0+l-1)}, & s \neq 0 \\ 0, & s = 0 \end{cases} \tag{12}$$

and denoting $F(u) := \int_0^u f(s)ds$, we see that

$$J(u) = \int_\Omega \left[\frac{|\nabla u|^2_{\mathbb{R}^n}}{2} + \frac{|u|^2_{\mathbb{R}}}{2} - F(u) \right] dx.$$

Associated with J, we have the well-known Nehari manifold

$$\mathcal{N} = \left\{ u \in H_0^1(\Omega) \setminus \{0\} : K(u) = 0 \right\}.$$

LEMMA 4. *The following statements hold:*

(i) Assume $m_0 + l - 1 > 0$. For any $u \in H_0^1(\Omega) \setminus \{0\}$, if $g(\lambda) := J(\lambda u)$, then there exists $\lambda(u)$ such that $g'(\lambda) > 0$ if $\lambda < \lambda(u)$ and $g'(\lambda) < 0$ if $\lambda > \lambda(u)$. In particular, $\lambda(u)$ is the unique $\lambda > 0$ such that $\lambda u \in \mathcal{N}$ and the maximum of $J(\lambda u)$ for $\lambda \geqslant 0$ is achieved at $\lambda = \lambda(u)$.

(ii) There exists $\delta > 0$ such that $\lambda(u) \geqslant \delta$ for all $u \in S$, where $S = \{u \in H_0^1(\Omega) : \|u\| = 1\}$. Consequently $\mathcal{N} \cap B(0,\delta) = \emptyset$, which yields that $\mathcal{N}$ is a closed subset of $H_0^1(\Omega)$.

Proof. (i) Let $u \in H_0^1(\Omega)$ be fixed. Taking the derivative with relation to λ in (11), we get

$$\frac{g'(\lambda)}{\lambda} = -a\lambda^{p-2} \ln \lambda^k + b\lambda^{p-2} + c =: h(\lambda)$$

where $c := \|u\|^2$, $a := \frac{1+C_p^2}{m_0+l-1} \int_\Omega |u|^p_{\mathbb{R}} dx$ and $b := \frac{1+C_p^2}{m_0+l-1} \int_\Omega |u|^p_{\mathbb{R}} \ln |u|^k_{\mathbb{R}} dx$. Note that $a,\ c > 0$ since $u \neq 0$ and $m_0 + l - 1 > 0$. Also we can see that h is positive near to zero, $\lim_{\lambda \to 0} h(\lambda) = -\infty$, and that there exists a unique critical point λ_* for h, namely $\lambda_* = \ln^{-1} \frac{(p-2)b-a}{(p-2)a}$. These facts suffice to prove (i).

(ii) We suppose by contradiction that there exists a sequence $(u_n) \in S$ such that $0 < \lambda_n := \lambda(u_n) < 1$ and $\lambda_n \to 0$. Denoting $\mu := \frac{1+C_p^2}{m_0+l-1}$, it follows from the equation

$$g'(\lambda_n) = \left[\frac{d}{d\lambda} J(\lambda u_n) \right]_{\lambda = \lambda_n} = 0$$

that

$$\|u_n\|^2 = \frac{\mu \int_\Omega |\lambda_n u_n|^p_{\mathbb{R}} \ln |\lambda_n u_n|^k_{\mathbb{R}} dx}{\lambda_n^2}.$$

Since $u_n \in S$, it follows that

$$1 \leqslant \frac{\mu k \int_\Omega |\lambda_n u_n|^{p+1}_{\mathbb{R}} dx}{\lambda_n^2} \leqslant \mu k C_*^{p+1} \lambda_n^{p-1} \|u_n\|^{p+1} \leqslant \mu k C_*^{p+1} \lambda_n^{p-1},$$

where C_* is the embedding constant for $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$. Hence,

$$1 \leqslant \mu k C_*^{p+1} \lambda_n^{p-1} \to 0,$$

which is a contradiction. Thus the proof of (ii) is complete. Consequently, if $u \in \mathscr{N}$, we have

$$\mathscr{N} \ni u = \frac{u}{\|u\|}\|u\| \Rightarrow \|u\| = \lambda\left(\frac{u}{\|u\|}\right) \geqslant \delta. \quad \square$$

As a consequence of Lemma 4 (i), we can define, as in the Mountain Pass Theorem due to Ambrossetti and Rabinowitz,

$$d = \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \sup_{\lambda > 0} J(\lambda u), \tag{13}$$

which is equivalent to the definition given in (10). Furthermore, from Lemma 4 (i), if d is attained, then $d > 0$. Whether d is attained or not, by the continuous embedding $H_0^1(\Omega) \subset L^p(\Omega)$ and the inequality $\int_\Omega F(u)dx \leqslant \frac{1+C_p^2}{m_0+l-1}(|u|^2 + \|u\|_p^p)$, it is easy to check that if $C_p^2(C_p^2+1) < m_0 + l - 1$, then $d > 0$. However this imposes a restriction on the domain through the Poincaré constant. Theorem 1 makes such restriction unnecessary by showing that d is indeed attained and it is a critical value of J. The critical point where d is attained is called *ground state*. In short:

THEOREM 2. *If* $0 < m_0 + l - 1$, *then* d *is a critical value of* J. *In particular,* $d > 0$.

Proof. Observe that Lemma 4 (ii) implies the existence of $r > 0$ such that $B(0,r) \cap \mathscr{N} = \emptyset$. Consequently from Lemma 4 (i), J is strictly increasing on each ray starting at zero and ending on $\partial B(0,r)$. Thus, $\inf_{\|u\|=r} J(u) > J(0)$. The equation (11) shows that for any $u \neq 0$, $\lim_{\lambda \to \infty} J(\lambda u) = -\infty$. Thus, there exists $e \in H_0^1(\Omega)$ such that $J(e) < J(0) = 0$, that is,

$$\inf_{\|u\|=r} J(u) > J(0) > J(e).$$

Since J is $(PS)_d$ (Lemma 3), it follows from Theorem 1 that d is a critical value for J. $\square$

We now introduce

$$\mathscr{W} = \{u \in H_0^1(\Omega) : J(u) < d\}$$

and partition it into two parts, as follows:

$$\mathscr{W}_1 = \{u \in \mathscr{W} : K(u) > 0\} \cup \{0\},$$
$$\mathscr{W}_2 = \{u \in \mathscr{W} : K(u) < 0\}.$$

LEMMA 5. $\mathscr{W}_1$ *is open.*

Proof. We know from Lemma 4 (ii) that $\mathscr{N} \cap B(0,\delta) = \emptyset$. Hence for all $0 \neq u \in B(0,\delta)$, it follows that $K(u) > 0$, since every ray starting from zero does not intersect the Nehari manifold inside the ball (see Lemma 4 (i)). Further, $0 \in A := \{u \in H_0^1(\Omega) : J(u) < d\} \cap B(0,\delta) \subset \mathscr{W}_1$, where A is open. Therefore, $\mathscr{W}_1$ an open set in $H_0^1(\Omega)$. $\square$

We refer to $\mathscr{W}_1$ as the "good" part and $\mathscr{W}_2$ as the "bad" part of the well. Then we define by $\mathscr{W}_1$ the stability set for the problem (1).

4. Existence of a global weak solution

DEFINITION 4. Let $T > 0$. A function $u \in C^1([0,T], H_0^1(\Omega))$ is called a weak solution of (1) on $[0,T]$ if for any $\omega \in H_0^1(\Omega)$ and $t \in [0,T]$

$$\begin{cases} \frac{d}{dt}(|u_t(t)|^\rho u_t(t), \omega) + M(\|u(t)\|^2)(\nabla u(t), \nabla\omega) + (\nabla u_{tt}(t), \nabla\omega) \\ -\int_0^t g(t-s)(\nabla u(s), \nabla\omega)ds + (u_t, \omega) = (u(t)|u(t)|^{p-2}\ln|u(t)|^k, \omega) \\ u(0) = u_0 \quad , \quad u_t(0) = u_1. \end{cases} \tag{14}$$

LEMMA 6. *If u is a weak solution of* (1) *on* $[0,T]$*, then* $\mathscr{E}'(t) \leqslant 0$*,* $\forall t \in [0,T]$.

Proof. Setting $\omega = u_t$ in the equation in (14), it follows that

$$\begin{aligned} &(|u_t(t)|^\rho u_{tt}(t), u_t(t)) + M(\|u(t)\|^2)(\nabla u(t), \nabla u_t(t)) + (\nabla u_{tt}(t), \nabla u_t(t)) \\ &-\int_0^t g(t-s)(\nabla u(s), \nabla u_t(t))ds + (u_t(t), u_t(t)) \\ =\ &(u(t)|u(t)|^{p-2}\ln|u(t)|^k, u_t(t)). \end{aligned}$$

Noting that $\rho/[2(\rho+1)] + 1/[2(\rho+1)] + 1/2 = 1$ from the generalized Hölder inequality, the nonlinear term $\int_\Omega |u_t|_{\mathbb{R}}^\rho u_{tt} u_t dx$ in the above definition makes sense.

Thus we get

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + \frac{1}{2}\hat{M}(\|u\|^2) + \frac{1}{2}|\nabla u_t(t)|^2 + \frac{k}{p^2}\int_\Omega |u|_{\mathbb{R}}^p dx - \frac{1}{p}\int_\Omega |u|_{\mathbb{R}}^p \ln|u|_{\mathbb{R}}^k dx\right\} \\ &= -|u_t(t)|^2 + \int_0^t g(t-s)(\nabla u(s), \nabla u_t(t))ds. \end{aligned} \tag{15}$$

Combining (15) with (4) and then using the hypotheses $(H.2)$ and $(H.3)$, we obtain

$$\mathscr{E}'(t) = -|u_t(t)|^2 + \frac{1}{2}(g' \diamond \nabla u)(t) - \frac{1}{2}g(t)|\nabla u(t)|^2 \leqslant 0. \quad \square \tag{16}$$

LEMMA 7. *Suppose that* $0 < m_0 + l - 1 \leqslant 1$ *and that the hypothesis* $(H.1)$–$(H.3)$ *and the conditions over* p *and* ρ *hold. Assume there exists a solution* $u : [0,T] \to H_0^1(\Omega)$ *for the problem* (1) *with initial data* $u_0 \in \mathscr{W}_1$ *and* $u_1 \in H_0^1(\Omega)$ *such that* $0 < \mathscr{E}(0) < \frac{(m_0+l-1)d}{1+C_p^2}$.

Then $K(t) \geqslant 0$ *for all* $0 \leqslant t \leqslant T$*. Consequently* $\mathscr{I}(t) \geqslant 0$ *and*

$$\begin{aligned} \mathscr{E}(0) \geqslant\ &\frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + \frac{p-2}{2p}(m_0+l-1)\|u\|^2 \\ &+ \frac{1}{2}\|u_t\|^2 + \frac{k}{p^2}\int_\Omega |u|_{\mathbb{R}}^p dx + \frac{1}{2}(g \diamond \nabla u)(t) \end{aligned} \tag{17}$$

for all $t \in [0,T]$.

Proof. First, we will show that $\mathscr{I}(t) \geqslant 0$ for all $0 < t \leqslant T$. We notice that

$$\begin{aligned}\frac{m_0+l-1}{1+C_p^2}J(u)(t) &= -\frac{1}{p}\int_\Omega |u|_{\mathbb{R}}^p \ln|u|^k dx + \frac{k}{p^2}\int_\Omega |u|_{\mathbb{R}}^p dx \\ &\quad + \frac{m_0+l-1}{2(1+C_p^2)}\int_\Omega |\nabla u|_{\mathbb{R}^n}^2 dx + \frac{m_0+l-1}{2(1+C_p^2)}\int_\Omega |u|_{\mathbb{R}}^2 dx \\ &\leqslant -\frac{1}{p}\int_\Omega |u|_{\mathbb{R}}^p \ln|u|_{\mathbb{R}}^k dx + \frac{k}{p^2}\int_\Omega |u|_{\mathbb{R}}^p dx + \frac{m_0+l-1}{2}\int_\Omega |\nabla u|_{\mathbb{R}^n}^2 dx \\ &\leqslant \mathscr{E}(t).\end{aligned}$$

It follows from Lemma 6 that

$$J(u)(t) \leqslant \frac{1+C_p^2}{m_0+l-1}\mathscr{E}(t) \leqslant \frac{1+C_p^2}{m_0+l-1}\mathscr{E}(0) < d.$$

Claim: $u(t) \in \mathscr{W}_1$, $\forall t \in [0,T]$

Indeed, the proof is similar to [21]. Since $\mathscr{W}_1$ is an open set and $u : [0,T] \to H_0^1(\Omega)$ is continuous, we suppose by contradiction that there exists the first $t_1 \in (0,T]$ such that $u(t_1) \in \partial\mathscr{W}_1$. Then we have the following possibilities:

$$\text{(i) } J(u(t_1)) = d, \tag{18}$$

$$\text{(ii) } u(t_1) \neq 0 \quad \text{and} \quad K(u(t_1)) = 0. \tag{19}$$

We have just seen that $J(u(t)) < d$ for all $t \in (0,T]$. Thus, (i) never occurs. Now, supposing that $K(u(t_1)) = 0$, $u(t_1) \neq 0$, it follows that $u(t_1) \in \mathscr{N}$ and from Lemma 4, item (i), that $\lambda = \lambda(u(t_1)) = 1$ is the unique $\lambda > 0$ for which $\lambda u(t_1) \in \mathscr{N}$, and the maximum of $J(\lambda u(t_1))$ is achieved when $\lambda = \lambda(u(t_1))$. Hence, we have

$$\sup_{\lambda \geqslant 0} J(\lambda u(t_1)) = J(\lambda u(t_1))|_{\lambda=1} = J(u(t_1)) < d, \tag{20}$$

which contradicts with definition of d. Thus (ii) does not occur either. This proves our claim. Therefore, since $\frac{m_0+l-1}{1+C_p^2}K(u)(t) \leqslant \mathscr{I}(t)$ and $K(u)(t) \geqslant 0$, for all $t \in [0,T]$, we have also proved that $\mathscr{I}(t) \geqslant 0$, for all $t \in [0,T]$.

Finally, note that the following inequalities hold for all $0 < t \leqslant T$:

$$\begin{aligned}\mathscr{E}(0) \geqslant \mathscr{E}(t) &= \frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + \mathscr{J}(t) \\ &= \frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + \left(\frac{1}{2} - \frac{1}{p}\right)\left(\hat{M}(\|u\|^2) - \int_0^t g(s)ds\|u\|^2\right) \\ &\quad + \frac{1}{2}\|u_t\|^2 + \frac{k}{p^2}\int_\Omega |u|_{\mathbb{R}}^p dx + \frac{1}{2}(g \diamond \nabla u)(t) + \frac{1}{p}\mathscr{I}(t) \\ &\geqslant \frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + \frac{p-2}{2p}\left(\hat{M}(\|u\|^2) - \int_0^t g(s)ds\|u\|^2\right) \\ &\quad + \frac{1}{2}\|u_t\|^2 + \frac{k}{p^2}\int_\Omega |u|_{\mathbb{R}}^p dx + \frac{1}{2}(g \diamond \nabla u)(t)\end{aligned}$$

which imply the bound (17). □

THEOREM 3. *Suppose that* $0 < m_0 + l - 1$ *and that the hypotheses* $(H.1)$–$(H.3)$ *and the conditions* (5) *and* (6) *over* p *and* ρ *hold. Further, assume that the initial data*

$$u_0 \in \mathscr{W}_1, \quad u_1 \in H_0^1(\Omega) \tag{21}$$

are such that

$$0 < \mathscr{E}(0) < \frac{(m_0 + l - 1)d}{1 + C_p^2}. \tag{22}$$

Then there exists a function $u : [0,T] \to H_0^1(\Omega) \cap H^2(\Omega)$ *in the class*

$$u, u_t \in C(0,T; H_0^1(\Omega))$$
$$u_{tt} \in L^2(0,T; H_0^1(\Omega))$$

such that for all $\omega \in H_0^1(\Omega)$ *we have*

$$\frac{d}{dt}\frac{1}{\rho+1}(|u_t|^\rho u_t, \omega) + M(\|u\|^2)(\nabla u, \nabla \omega) + (\nabla u_{tt}, \nabla \omega)$$
$$-\int_0^t g(t-s)(\nabla u(s), \nabla \omega)ds + (u_t, \omega) = (u|u|_{\mathbb{R}}^{p-2} \ln |u|_{\mathbb{R}}^k, \omega)$$

In particular the function u *is a solution for the problem* (1).

Proof. Let $(\omega_\nu)_{\nu \in \mathbb{N}} \subset H_0^1(\Omega) \cap H^2(\Omega)$ be a basis of $L^2(\Omega)$ from the eigenvectors of the operator $-\Delta$. It is known that $(\omega_\nu)_{\nu \in \mathbb{N}}$ is a complete and orthonormal system of $H_0^1(\Omega)$. Let $V_m = \text{span}[\omega_1, \cdots, \omega_m]$ be the space generated by the first m eigenvector of the system $(\omega_\nu)_{\nu \in \mathbb{N}}$ and $u^m(t) = \sum_{j=1}^m g_{jm}(t)\omega_j$ be a solution in the interval $[0, t_m)$ of the approximate problem

$$\begin{aligned}
&(|u_t^m(t)|^\rho u_{tt}^m, \omega) + M(\|u^m(t)\|^2)(\nabla u^m(t), \nabla \omega) + (\nabla u_{tt}^m(t), \nabla \omega) \\
&\quad - \int_0^t g(t-s)(\nabla u^m(s), \nabla \omega)ds + (u_t^m, \omega) \\
&= (u^m |u^m|^{p-2} \ln |u^m|^k, \omega) \quad \forall \omega \in V_m
\end{aligned} \tag{23}$$

$$u^m(0) = u_{0m} \to u_0 \text{ strongly in } H_0^1(\Omega) \tag{24}$$

$$u_t^m(0) = u_{1m} \to u_1 \text{ strongly in } H_0^1(\Omega). \tag{25}$$

The solutions (u^m) exist by Carathéodory's Theorem.

First a priori estimate: Setting $\omega = u_t^m$ in the equation (14), we see that u^m satisfies (15) in the interval $[0, t_m]$. By continuity of $\mathscr{E}$ and the fact that $\mathscr{W}_1$ is open, we can suppose that $0 < \mathscr{E}(u_{0m}) < \frac{m_0 + l - 1}{1 + C_p^2} d$ and $u_{0m} \in \mathscr{W}_1$ for all m. Hence, from Lemma 7 we infer that

$$\frac{p-2}{2p}(m_0 + l - 1)\|u^m(t)\|^2 + \frac{1}{2}\|u_t^m(t)\|^2 \leqslant \mathscr{E}(u_{0m}, u_{1m}) \tag{26}$$

for all $t \in [0,t_m]$. By continuity of $\mathscr{E}$ once again, there exists $L_1 > 0$ independent of $t \in [0,t_m]$ and m such that

$$\|u^m(t)\|^2 + \|u_t^m(t)\|^2 < L_1. \tag{27}$$

Hence, we can extend the approximate solution $u^m(t)$ to the interval $[0,T]$. It follows, in particular, from (27) that

$$(u^m), (u_t^m) \text{ are bounded in } L^\infty(0,T;H_0^1(\Omega)).$$

Second a priori estimate: Taking $\omega = u_{tt}^m$ in (23), we have

$$\begin{aligned}
&\int_\Omega |u_t^m(t)|_{\mathbb{R}}^\rho |u_{tt}^m|_{\mathbb{R}}^2 dx + \|u_{tt}^m\|^2 \\
&= -(u_t^m, u_{tt}^m) - M(\|u^m(t)\|^2)(\nabla u^m, \nabla u_{tt}^m) \\
&\quad + \int_0^t g(t-s)(\nabla u^m(s), \nabla u_{tt}^m) ds + (u^m|u^m|_{\mathbb{R}}^{p-2} \ln |u^m|_{\mathbb{R}}^k, u_{tt}^m) \\
&\quad + \frac{C_p^2}{4\eta}\|u_t^m\|^2 + C_p^2\eta\|u_{tt}\|^2 + \overline{m}\left(\frac{1}{4\eta}\|u^m\|^2 + \eta\|u_{tt}^m\|^2\right) \\
&\quad + \|u_{tt}^m\| \int_0^t |g(t-s)|\|u^m(s)\| ds + \left(\int_\Omega \left|u^m|u^m|_{\mathbb{R}}^{p-2} \ln |u^m|_{\mathbb{R}}^k\right|_{\mathbb{R}}^2 dx\right)^{1/2} |u_{tt}^m| \\
&\leqslant \frac{C_p^2}{4\eta}\|u_t^m\|^2 + C_p^2\eta\|u_{tt}^m\|^2 + \overline{m}\left(\frac{1}{4\eta}\|u^m\|^2 + \eta\|u_{tt}^m\|^2\right) \\
&\quad + \eta\|u_{tt}\|^2 + \frac{1}{4\eta}\int_0^t |g(t-s)|\|u^m(s)\|^2 ds \\
&\quad + \frac{1}{4\eta}\left(\int_\Omega \left|u^m|u^m|_{\mathbb{R}}^{p-2} \ln |u^m|_{\mathbb{R}}^k\right|_{\mathbb{R}}^2 dx\right) + C_p^2\eta\|u_{tt}^m\|^2.
\end{aligned}$$

The first a priori estimate allows us to find a constant $C > 0$ such that

$$\begin{aligned}
\|u_{tt}\|^2 &\leqslant (2C_p^2 + \overline{m} + 1)\eta\|u_{tt}^m\|^2 \\
&\qquad + \frac{1}{4\eta}\left(\int_\Omega \left|u^m|u^m|_{\mathbb{R}}^{p-2} \ln |u^m|_{\mathbb{R}}^k\right|^2 dx\right) + C.
\end{aligned}$$

The constant η can be chosen small enough for that $\eta(2C_p^2 + \overline{m} + 1) \leqslant \frac{1}{2}$. Notice that from the elementary inequality

$$\left|\xi|\xi|_{\mathbb{R}}^{p-2} \ln \xi\right|_{\mathbb{R}} \leqslant c_0(|\xi|_{\mathbb{R}} + |\xi|_{\mathbb{R}}^p),$$

it follows that

$$\int_\Omega \left|u^m|u^m|_{\mathbb{R}}^{p-2} \ln |u^m|_{\mathbb{R}}^k\right|^2 = k^2 c_0 \int_\Omega (|u^m|_{\mathbb{R}} + |u^m|_{\mathbb{R}}^p)^2 dx.$$

Since the assumptions on p assure the embeddings $H_0^1(\Omega) \hookrightarrow L^{2p}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^{p+1}(\Omega)$, there exists a positive constant $\tilde{C}$ independently on m and t such that

$$\int_\Omega |u^m|u^m|_{\mathbb{R}}^{p-2} \ln |u^m|_{\mathbb{R}}^k|^2 \leqslant \tilde{\tilde{C}}. \tag{28}$$

Therefore, there is a positive constant L_2 independent of m and t such that

$$\|u_{tt}^m\|^2 \leqslant L_2.$$

This is the second a priori estimate, from which we have

$$(u_{tt}^m) \text{ is bounded in } L^\infty(0,T;H_0^1(\Omega)). \tag{29}$$

Third a priori estimate: We set $\omega = -\Delta u_t^m$ in (23). We have

$$\begin{aligned}
&M(\|u^m\|^2)\frac{d}{dt}|\Delta u^m|^2 + \frac{d}{dt}|\Delta u^m|^2 \\
&\leqslant \frac{1}{2}\left(\|u_t^m\|_{2(\rho+1)}\|u_{tt}^m\|_{2(\rho+1)}\right)^2 + \frac{1}{2}|\Delta u_t^m|^2 \\
&\quad + \frac{1}{2}\|g\|_{L^\infty(0,\infty)}\int_0^t |\Delta u^m(s)|^2 ds + \frac{1}{2}\|g\|_{L^1(0,\infty)}|\Delta u_t^m|^2 + \|u_t^m\|^2 \\
&\quad + \frac{1}{2}|u^m|u^m|^{p-2}\ln|u^m|^k|^2 + \frac{1}{2}|\Delta u_t^m|^2.
\end{aligned}$$

Due to the first and second a priori estimates and (28), we get

$$\begin{aligned}
&\frac{d}{dt}\left(M(\|u^m\|^2)|\Delta u^m|^2 + |\Delta u_t^m|^2\right) \\
&\leqslant \frac{d}{dt}M(\|u^m\|^2)|\Delta u^m|^2\left(\frac{1}{2}\|g\|_{L^1(0,\infty)} + 1\right)|\Delta u_t^m|^2 \\
&\quad + \frac{1}{2}\|g\|_{L^\infty(0,\infty)}\int_0^t |\Delta u^m(s)|^2 ds + C.
\end{aligned} \tag{30}$$

Since $\{\|u^m(t)\|\}$ is uniformly bounded for $t \in [0,T]$ and $m \in \mathbb{N}$, and $M \in C^1[0,\infty)$, there exists $\tilde{M} > 0$ such that $\frac{d}{dt}(M(\|u\|^2)) \leqslant \tilde{M}$. Integrating the inequality (30) from 0 to t we obtain

$$\begin{aligned}
&M(\|u^m\|^2)|\Delta u^m|^2 + |\Delta u_t^m|^2 \\
&\leqslant \tilde{M}\int_0^t |\Delta u^m(s)|^2 ds + \left(\frac{1}{2}\|g\|_{L^1(0,\infty)} + 1\right)\int_0^t |\Delta u_t^m(s)|^2 ds \\
&\quad + \frac{1}{2}\|g\|_{L^\infty(0,\infty)} T \int_0^t |\Delta u^m(s)|^2 ds + C \\
&= C_1\int_0^t |\Delta u^m(s)|^2 ds + C_2\int_0^t |\Delta u_t^m(s)|^2 ds + C \\
&\leqslant C_3\int_0^t (m_0|\Delta u^m(s)|^2 + |\Delta u_t^m(s)|^2) ds + C.
\end{aligned}$$

Hence,

$$m_0|\Delta u^m|^2+|\Delta u_t^m|^2\leqslant C_3\int_0^t(m_0|\Delta u^m(s)|^2+|\Delta u_t^m(s)|^2)ds+C.$$

It follows from Gronwall inequality that there is a constant $L_3>0$ independent of m and t such that

$$m_0|\Delta u^m|^2+|\Delta u_t^m|^2\leqslant L_3.$$

In particular,

$$|\Delta u^m|^2 \text{ is bounded in } L^\infty(0,T;L^2(\Omega)).$$

Combining the above boundedness with the first a priori estimate, we obtain

$$(u^m) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \tag{31}$$

$$(u_t^m) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)). \tag{32}$$

Passage to the limit: From the estimates (31), (32) and (29), there exists a subsequence of (u^m), also denoted by (u^m), such that

$$u^m \overset{*}{\rightharpoonup} u \text{ weakly star in } L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \tag{33}$$

$$u_t^m \rightharpoonup u_t \text{ weakly in } L^2(0,T;H_0^1(\Omega)), \tag{34}$$

$$u_{tt}^m \rightharpoonup u_{tt} \text{ weakly in } L^2(0,T;H_0^1(\Omega)). \tag{35}$$

Note that from (31), (32) we have

$$(u^m) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \tag{36}$$

$$(u_t^m) \text{ is bounded in } L^2(0,T;L^2(\Omega)). \tag{37}$$

Thus, putting $B_0=H_0^1(\Omega)\cap H^2(\Omega)$, $B=H_0^1(\Omega)$, $B_1=L^2(\Omega)$, and

$$W=\{u:u\in L^2(0,T,B_0);u_t\in L^2(0,T,B_1)\}$$

equipped with the norm

$$\|w\|=\|u\|_{L^2(0,T,B_0)}+\|u_t\|_{L^2(0,T,B_1)}$$

results from (36) and (37) that

$$(u^m) \text{ is bounded in } W.$$

Then, by Aubin-Lions Lemma (Lemma 2) we extract a subsequence of (u^m) which we continue to denote by (u^m), such that

$$u^m\to u \text{ strongly in } L^2(0,T;H_0^1(\Omega)). \tag{38}$$

In particular

$$u^m\to u \text{ a.e in } \Omega\times(0,T),$$
$$\|u^m\|\to\|u\| \text{ a.e in } (0,T).$$

Since M is continuous, it follows from the Dominated Convergence Theorem that

$$M(\|u^m\|^2) \to M(\|u\|^2) \text{ strongly in } L^2(0,T). \tag{39}$$

Therefore

$$(M(\|u^m\|^2)\nabla u^m, \nabla w) \to (M(\|u\|^2)\nabla u, \nabla w) \text{ strongly in } L^2(0,T). \tag{40}$$

Clearly the convergence in (40) is also valid in $D'(0,T)$ for every $w \in H_0^1(\Omega)$. Now, since that $f(s) = s|s|_{\mathbb{R}}^{p-2} \ln |s|_{\mathbb{R}}^k$ is continuous, we have the convergence

$$u^m|u^m|_{\mathbb{R}}^{p-2} \ln |u^m|_{\mathbb{R}}^k \to u|u|_{\mathbb{R}}^{p-2} \ln |u|_{\mathbb{R}}^k \text{ a.e. in } \Omega \times (0,T). \tag{41}$$

It follows from (28) and (41) and Lemma 1 that

$$u^m|u^m|_{\mathbb{R}}^{p-2} \ln |u^m|_{\mathbb{R}}^k \rightharpoonup u|u|_{\mathbb{R}}^{p-2} \ln |u|_{\mathbb{R}}^k \text{ weakly in } L^2(0,T;L^2(\Omega)). \tag{42}$$

Hence, as before, for every $w \in H_0^1(\Omega)$,

$$(u^m|u^m|_{\mathbb{R}}^{p-2} \ln |u^m|_{\mathbb{R}}^k, w) \to (u|u|_{\mathbb{R}}^{p-2} \ln |u|_{\mathbb{R}}^k, w) \text{ in } D'(0,T). \tag{43}$$

Let us make the similar analysis for the nonlinear term that induces the sequence $|u_t^m|^\rho u_t^m$.

We know that

$$(u_t^m) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)), \tag{44}$$

$$(u_{tt}^m) \text{ is bounded in } L^2(0,T;L^2(\Omega)) \tag{45}$$

Thus, from (44), (45) and Lemma 2 we have, up to a subsequence,

$$u_t^m \to u_t \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{46}$$

Therefore

$$|u_t^m|_{\mathbb{R}}^\rho u_t^m \to |u_t|_{\mathbb{R}}^\rho u_t \text{ a.e. in } \Omega \times (0,T). \tag{47}$$

Note that

$$\begin{aligned} \||u_t^m|_{\mathbb{R}}^\rho u_t^m\|_{L^2(0,T;L^2(\Omega))}^2 &= \int_0^T \|u_t^m(s)\|_{2(\rho+1)}^{2(\rho+1)} ds \\ &\leqslant C \int_0^T \|u_t^m\|^{2(\rho+1)}(s) ds \leqslant CTL_1^{\rho+1}. \end{aligned} \tag{48}$$

Combining (47), (48) and Aubin-Lions Lemma (Lemma 1), we deduce

$$|u_t^m|_{\mathbb{R}}^\rho u_t^m \rightharpoonup |u_t|_{\mathbb{R}}^\rho u_t \text{ weakly in } L^2(0,T;L^2(\Omega)). \tag{49}$$

Convergence (49) implies

$$(|u_t^m|_{\mathbb{R}}^\rho u_t^m, w) \to (|u_t|_{\mathbb{R}}^\rho u_t, w) \text{ in } \mathscr{D}'(0,T). \tag{50}$$

Multiplying (23) by $\theta \in \mathscr{D}(0;T)$ and integrating the obtained result over $(0;T)$, we obtain

$$\begin{aligned} &-\frac{1}{\rho+1}\int_0^T (|u_t^m(t)|_{\mathbb{R}}^{\rho} u_t^m, \omega)\theta'(t)dt + \int_0^T M(\|u^m(t)\|^2)(\nabla u^m(t), \nabla\omega)\theta(t)dt \\ &+\int_0^T (\nabla u_{tt}^m(t), \nabla\omega)\theta(t)dt - \int_0^T\int_0^\tau g(\tau-s)(\nabla u^m(s), \nabla\omega)\theta(t)dsd\tau \\ &+\int_0^T (u_t^m, \omega)\theta(t)dt = \int_0^T (u^m|u^m|_{\mathbb{R}}^{p-2}\ln|u^m|_{\mathbb{R}}^k, \omega)\theta(t)dt \quad \forall\omega\in V_m \end{aligned} \tag{51}$$

The convergences (35), (36), (37), (40), (43) and (50) are sufficient to pass to the limit in (51) in order to obtain, for all $w \in \bigcup_{m\in\mathbb{N}} V_m$,

$$\begin{aligned} &-\frac{1}{\rho+1}\int_0^T (|u_t(t)|_{\mathbb{R}}^{\rho} u_t, \omega)\theta'(t)dt + \int_0^T M(\|u(t)\|^2)(\nabla u(t), \nabla\omega)\theta(t)dt \\ &+\int_0^T (\nabla u_{tt}(t), \nabla\omega)\theta(t)dt - \int_0^T\int_0^\tau g(\tau-s)(\nabla u(s), \nabla\omega)\theta(\tau)dsd\tau \end{aligned} \tag{52}$$

$$+\int_0^T (u_t, \omega)\theta(t)dt = \int_0^T (u|u|_{\mathbb{R}}^{p-2}\ln|u|_{\mathbb{R}}^k, \omega)\theta(t)dt. \tag{53}$$

This means that

$$\begin{aligned} &\frac{d}{dt}\frac{1}{\rho+1}(|u_t|_{\mathbb{R}}^{\rho} u_t, \omega) + M(\|u\|^2)(\nabla u, \nabla\omega) + (\nabla u_{tt}, \nabla\omega) \\ &-\int_0^t g(t-s)(\nabla u(s), \nabla\omega)ds + (u_t, \omega) = (u|u|_{\mathbb{R}}^{p-2}\ln|u|_{\mathbb{R}}^k, \omega) \end{aligned} \tag{54}$$

holds in $\mathscr{D}'(0,T)$.

As $\bigcup_{m\in\mathbb{N}} V_m$ is dense in $H_0^1(\Omega)$, (54) is valid for all ω in $H_0^1(\Omega)$.

Verification of initial data: From the convergences (38) and (46), we infer that

$$u^m \to u \text{ in } C([0,T], H_0^1(\Omega)).$$

Hence

$$u^m(0) \to u(0) \text{ in } H_0^1(\Omega).$$

Also

$$u^m(0) \to u_0 \text{ in } H_0^1(\Omega).$$

Hence $u(x,0) = u_0(x)$. Furthermore, by the compact embedding $L^\infty(0,T,H_0^1(\Omega)) \subset L^2(0,T,L^2(\Omega))$ and the bound (29), we have, up to a subsequence, that

$$u_{tt}^m \to u_{tt} \text{ strongly in } L^2(0,T,L^2(\Omega)). \tag{55}$$

Therefore, the convergences (32) and (55) imply

$$u_t^m \to u_t \text{ strongly in } W_2(0,T;L^2(\Omega),L^2(\Omega)).$$

Thus, it follows that

$$u_t^m \to u \text{ strongly in } C([0,T];L^2(\Omega)).$$

Hence

$$u_t^m(0) \to u_t(0) \text{ in } L^2(\Omega).$$

Since

$$u_t^m(0) \to u_1 \text{ in } H_0^1(\Omega).$$

it follows that $u_t(x,0) = u_1(x)$. $\square$

5. Uniqueness of global weak solution

In the previous section, we proved that if initial data are in the stability set $\mathscr{W}_1$, the problem (1) possesses a global weak solution. In this section we prove the uniqueness of the weak solutions for $\rho \geqslant 1$ and initial data in a neighborhood of $\mathscr{W}_1$ containing zero.

THEOREM 4. *Assume that all the hypotheses of the Theorem 3 hold and that*

- $\rho \geqslant 1$, *if* $n = 1$ *or* 2 *and* $1 \leqslant \rho \leqslant 2$, *if* $n = 3$,
- $p > 2$, *if* $n = 1$ *or* 2 *and* $\frac{11}{5} \leqslant p \leqslant 3$, *if* $n = 3$,
- $(2\mathscr{E}(0))^{\rho/2} C_*^{\rho+1} C_p < 1$,

where C_* *and* C_p *are constants from the embedding* $H_0^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega)$ *and the Poicanré inequality. Then the solution for the problem* (1) *is unique.*

Proof. Let $u, \tilde{u}$ solutions for the problem (1) with initial data $u(0) = u_0 = \tilde{u}(0)$ and $u_t(0) = u_1 = \tilde{u}_t(0)$. Denoting $U = u - \tilde{u}$, we see that U, u and $\tilde{u}$ satisfy the equation

$$\begin{aligned} &(M(\|U\|^2)\nabla U, \nabla \omega) - (\int_0^t g(t-s)\nabla U(s)ds, \nabla \omega) + (\nabla U_{tt}, \nabla \omega) + (U_t, \omega) \\ &= (M(\|\tilde{u}\|^2)\nabla \tilde{u}, \nabla \omega) - (M(\|u\|^2)\nabla u, \nabla \omega) + (M(\|U\|^2)\nabla U, \nabla \omega) \\ &\quad + (u|u|_{\mathbb{R}}^{p-2} \ln |u|_{\mathbb{R}}^k - \tilde{u}|\tilde{u}|_{\mathbb{R}}^{p-2} \ln |\tilde{u}|_{\mathbb{R}}^k + |\tilde{u}|_{\mathbb{R}}^{\rho} \tilde{u}_{tt} - |u_t|_{\mathbb{R}}^{\rho} u_{tt}, \omega), \quad \forall \omega \in H_0^1(\Omega). \end{aligned} \tag{56}$$

Setting $\omega = U_t$ in (56), we obtain

$$\frac{1}{2}\frac{d}{dt}\left[|\nabla U_t|^2 + \hat{M}(|\nabla U|^2) - \int_0^t g(s)ds |\nabla U|^2 + g \diamond \nabla U \right] \leqslant G(t), \tag{57}$$

where

$$G(t) = (M(\|\tilde{u}\|^2)\nabla\tilde{u} - M(\|u\|^2)\nabla u + M(\|U\|^2)\nabla U, \nabla U_t)$$
$$+ (u|u|_{\mathbb{R}}^{p-2}\ln|u|_{\mathbb{R}}^k - \tilde{u}|\tilde{u}|_{\mathbb{R}}^{p-2}\ln|\tilde{u}|_{\mathbb{R}}^k + |\tilde{u}|_{\mathbb{R}}^{\rho}\tilde{u}_{tt} - |u_t|_{\mathbb{R}}^{\rho}u_{tt}, U_t). \quad (58)$$

Now, we need to estimate $G(t)$.

Claim: there exists a constant $C_1 > 0$ such that

$$(M(\|\tilde{u}\|^2)\nabla\tilde{u} - M(\|u\|^2)\nabla u + M(\|U\|^2)\nabla U, \nabla U_t) \leqslant C_1\|U\|\|U_t\|, \quad (59)$$

for all $t \in [0,T]$. Indeed: by the Mean Value Theorem, there exists $\mu \in (0,1)$ such that

$$M(\|u\|^2) - M(\|\tilde{u}\|^2) = M'(\|u\|^2 + \mu(\|\tilde{u}\|^2 - \|u\|^2))(\|u\|^2 - \|\tilde{u}\|^2).$$

We have

$$\begin{aligned}
&|M(\|\tilde{u}\|^2)\nabla\tilde{u} - M(\|u\|^2)\nabla u| \\
&\leqslant |M(\|\tilde{u}\|^2)\nabla\tilde{u} - M(\|\tilde{u}\|^2)\nabla u| + |M(\|u\|^2)\nabla u - M(\|\tilde{u}\|^2)\nabla u| \\
&\leqslant M(\|\tilde{u}\|^2)\|u-\tilde{u}\| + |M'(\|u\|^2 + \mu(\|\tilde{u}\|^2 - \|u\|^2))|_{\mathbb{R}}|\|u\| - \|\tilde{u}\||_{\mathbb{R}}|\nabla u| \\
&\leqslant \tilde{m}\|U\|
\end{aligned} \quad (60)$$

where $\tilde{m}$ is obtained from the fact that $u, \tilde{u} \in L^\infty(0,T;H_0^1(\Omega))$ and that $M \in C^1(0,\infty)$. Using Hölder inequality, we obtain (59).

Claim: there exist a constant $C_2 > 0$ such that

$$\int_\Omega \Big(u|u|^{p-2}\ln|u|^k - \tilde{u}|\tilde{u}|^{p-2}\ln|\tilde{u}|^k\Big)U_t\,dx \leqslant C_2\|U\|\|U_t\|. \quad (61)$$

Indeed, since $\dfrac{d}{d\xi}\Big(\xi|\xi|^{p-2}\ln|\xi|^k\Big) = k(p-1)|\xi|^{p-2}\ln|\xi| + k|\xi|^{p-2}$, from the elementary inequality

$$\Big|\xi|\xi|_{\mathbb{R}}^{p-2}\ln\xi\Big|_{\mathbb{R}} \leqslant c_0(|\xi|_{\mathbb{R}} + |\xi|_{\mathbb{R}}^p),$$

we have $\Big|\dfrac{d}{d\xi}\Big(\xi|\xi|^{p-2}\ln|\xi|^k\Big)\Big| \leqslant C(|\xi| + |\xi|^{p-1} + |\xi|^{p-2})$. By the Mean Value Theorem, there exists $\theta \in (0,1)$ such that, if we denote $\bar{u} := u + \theta(u - \tilde{u})$, then

$$\begin{aligned}
&\int_\Omega \Big(u|u|^{p-2}\ln|u|^k - \tilde{u}|\tilde{u}|^{p-2}\ln|\tilde{u}|^k\Big)U_t \\
&\leqslant C\int_\Omega \Big(|\bar{u}|_{\mathbb{R}} + |\bar{u}|_{\mathbb{R}}^{p-1} + |\bar{u}|_{\mathbb{R}}^{p-2}\Big)UU_t\,dx \\
&\leqslant C\Big(\int_\Omega (|\bar{u}|_{\mathbb{R}} + |\bar{u}|_{\mathbb{R}}^{p-1} + |\bar{u}|_{\mathbb{R}}^{p-2}\Big)^s dx\Big)^{\frac{1}{s}}\Big(\int_\Omega |U|^r dx\Big)^{\frac{1}{r}}\Big(\int_\Omega |U_t|^q dx\Big)^{\frac{1}{q}} \\
&\leqslant \tilde{C}\Big(\|\bar{u}\|_s + \|\bar{u}\|_{s(p-1)}^{p-1} + \|\bar{u}\|_{s(p-2)}^{p-2}\Big)\|U\|_r\|U_t\|_q,
\end{aligned} \quad (62)$$

where $\frac{1}{s}+\frac{1}{r}+\frac{1}{q}=1$. In order to have suitable embeddings for our estimate, we set s, r and q as follows: $s=\frac{p-1}{p-2}$ and $r=q=2(p-1)$ if $n=1$ or 2, and $s=r=q=3$ if $n=3$. Hence,

$$\int_\Omega \Big(u|v|_{\mathbb{R}}^{p-2}\ln|u|_{\mathbb{R}}^k-\tilde{u}|\tilde{u}|_{\mathbb{R}}^{p-2}\ln|\tilde{u}|_{\mathbb{R}}^k\Big)U_t dx\leqslant \tilde{\tilde{C}}\Big(\|\tilde{u}\|+\|\tilde{u}\|^{p-1}+\|\tilde{u}\|^{p-2}\Big)\|U\|\|U_t\|. \tag{63}$$

Since $u,\tilde{u}\in L^\infty(0,T;H_0^1(\Omega))$, there exists $C_2>0$ satisfying (61).

For the next nonlinear term of $G(t)$, we use the following identity:

$$|\tilde{u}_t|^\rho\tilde{u}_{tt}-|u_t|^\rho u_{tt}=-\frac{1}{2}\left(|\tilde{u}_t|^\rho+|u_t|^\rho\right)U_{tt}+\frac{1}{2}\left(|\tilde{u}_t|^\rho-|u_t|^\rho\right)(\tilde{u_{tt}}+u_{tt}). \tag{64}$$

Multiplying (64) by U_t and integrating over Ω, we obtain

$$\begin{aligned}\int_\Omega \Big(|\tilde{u}_t|^\rho\tilde{u}_{tt}-|u_t|^\rho u_{tt}\Big)U_t dx\leqslant{}& \frac{1}{2}\Big(\|u_t\|_{2(\rho+1)}^\rho+\|\tilde{u}_t\|_{2(\rho+1)}^\rho\Big)\|U_{tt}\|_{2(\rho+1)}|U_t|\\ &+\frac{1}{2}\int_\Omega(|\tilde{u}_t|_{\mathbb{R}}^\rho-|u_t|_{\mathbb{R}}^\rho)(u_{tt}+\tilde{u}_{tt})U_t dx.\end{aligned} \tag{65}$$

Recalling that $u,\tilde{u}\in L^\infty(0,T;H_0^1(\Omega))$ and using the embedding $H_0^1(\Omega)\hookrightarrow L^{2(\rho+1)}(\Omega)$, results for the first term of the right side of the inequality (65) that there exists a constant $C_3>0$ such that

$$\Big(\|u_t\|_{2(\rho+1)}^\rho+\|\tilde{u}_t\|_{2(\rho+1)}^\rho\Big)\|U_{tt}\|_{2(\rho+1)}|U_t|\leqslant C_3\|U_{tt}\|\|U_t\|,\quad t\in[0,T]. \tag{66}$$

The second term of the right side of the inequality (65) can be estimated as follows: By Mean Value Theorem there exists a function $\lambda:(0,T)\to(0,1)$ such that

$$\Big||v_t|_{\mathbb{R}}^\rho-|\tilde{v}_t|_{\mathbb{R}}^\rho\Big|\leqslant\rho|v_t+\lambda(v_t-\tilde{v})_t|_{\mathbb{R}}^{\rho-1}|v_t-\tilde{v}_t|_{\mathbb{R}}.$$

Using generalized Hölder inequality with

$$\frac{1}{\frac{r}{\rho-1}}+\frac{1}{\frac{3r}{r-(\rho-1)}}+\frac{1}{\frac{3r}{r-(\rho-1)}}+\frac{1}{\frac{3r}{r-(\rho-1)}}=1,$$

where $r\geqslant\rho$, if $n=1$ or 2, or $r=6$ if $n=3$, and the embeddings $H_0^1(\Omega)\hookrightarrow L^r(\Omega)$ and $H_0^1(\Omega)\hookrightarrow L^{\frac{3r}{r-(\rho-1)}}(\Omega)$, we obtain

$$\begin{aligned}&\frac{1}{2}\int_\Omega(|u_t|_{\mathbb{R}}^\rho-|\tilde{u}_t|_{\mathbb{R}}^\rho)(u_{tt}+\tilde{u}_{tt})U_t dx\\ &\leqslant\frac{\rho}{2}\|u_t+\lambda(u_t-\tilde{u}_t)\|_{L^r(\Omega)}^{\rho-1}\|u_t-\tilde{u}_t\|_{L^{\frac{3r}{r-(\rho-1)}}(\Omega)}\|u_{tt}+\tilde{u}_{tt}\|_{L^{\frac{3r}{r-(\rho-1)}}(\Omega)}\|U_t\|_{L^{\frac{3r}{r-(\rho-1)}}(\Omega)}.\end{aligned} \tag{67}$$

Since $u_t, \tilde{u}_t, u_{tt}, \tilde{u}_{tt} \in L^\infty(0,T;H_0^1(\Omega))$, there exists $C_4 > 0$ such that

$$\int_\Omega \frac{1}{2}(|\tilde{u}_t|_{\mathbb{R}}^\rho - |u_t|_{\mathbb{R}}^\rho)(u_{tt} + \tilde{u}_{tt})U_t \leqslant C_4\|U_t\|^2. \tag{68}$$

Therefore, we get the following estimate for G:

$$G(t) \leqslant C_1\|U\|\|U_t\| + C_2\|U\|\|U_t\| + C_3\|U_{tt}\|\|U_t\| + C_4\|U_t\|^2, \tag{69}$$

and by the Young inequality, we obtain

$$G(t) \leqslant \left(\frac{C_1}{2} + \frac{C_2}{2}\right)\|U\|^2 + \left(\frac{C_1}{2} + \frac{C_2}{2} + \frac{C_3}{2} + C_4\right)\|U_t\|^2 + \frac{C_3}{2}\|U_{tt}\|^2 \tag{70}$$

for all $t \in [0,T]$. Now, set $\omega = U_{tt}$ in (56). We have

$$(M(\|U\|^2)\nabla U, \nabla U_{tt}) - (\int_0^t g(t-s)\nabla U(s)ds, \nabla U_{tt}) + \|U_{tt}\|^2 + (U_t, U_{tt}) = H(t), \tag{71}$$

where

$$\begin{aligned} H(t) = {} & (M(\|\tilde{u}\|^2)\nabla\tilde{u} - M(\|u\|^2)\nabla u + M(\|U\|^2)\nabla U, \nabla U_{tt}) \\ & + (u|u|_{\mathbb{R}}^{p-2}\ln|u|_{\mathbb{R}}^k - \tilde{u}|\tilde{u}|_{\mathbb{R}}^{p-2}\ln|\tilde{u}|_{\mathbb{R}}^k + |\tilde{u}|_{\mathbb{R}}^\rho\tilde{u}_{tt} - |u_t|_{\mathbb{R}}^\rho u_{tt}, U_{tt}). \end{aligned} \tag{72}$$

Hence,

$$\|U_{tt}\|^2 = -(M(\|U\|^2)\nabla U, \nabla U_{tt}) + (\int_0^t g(t-s)\nabla U(s), \nabla U_{tt}) + (U_t, U_{tt}) + H(t). \tag{73}$$

For the first term of the right side of the inequality (73), we use the Young inquality. Thus,

$$(M(\|U\|^2)\nabla U, \nabla U_{tt}) \leqslant \tilde{m}\left(\frac{1}{4\eta}\|U\|^2 + \eta\|U_{tt}\|^2\right) \tag{74}$$

where $\tilde{m} = \sup_{t\in[0,T]} M(\|U\|^2)$.

For the second term, we have

$$\begin{aligned} & (\int_0^t g(t-s)\nabla U(s), \nabla U_{tt}) \\ & = \int_0^t g(t-s)(\nabla U(s) - \nabla U, \nabla U_{tt})ds + \int_0^t g(t-s)(\nabla U, \nabla U_{tt}) \\ & \leqslant \frac{1}{4\eta}(g\diamond\nabla U)(t) + \eta\int_0^t g(t-s)ds|\nabla U_{tt}|^2 + \int_0^t g(t-s)\left(\frac{|\nabla U|^2}{4\eta} + \eta|\nabla U_{tt}|^2\right) \\ & \leqslant \frac{1}{4\eta}(g\diamond\nabla U)(t) + (1-l)\left(2\eta\|U_{tt}\|^2 + \frac{1}{4\eta}\|U\|^2\right). \end{aligned} \tag{75}$$

For the third term, we use the Poincaré and Young inequalities:

$$(U_t, U_{tt}) \leqslant \frac{C_p^2}{4\eta}\|U_t\|^2 + \eta C_p^2 \|U_{tt}\|^2. \tag{76}$$

To estimate $H(t)$, we can proceed in the same way as to obtain the estimate for $G(t)$. Then, it follows that there exists positive constants C_5 and C_6 such that

$$(M(\|\tilde{u}\|^2)\nabla\tilde{u} - M(\|u\|^2)\nabla u + M(\|U\|^2)\nabla U, \nabla U_{tt}) \leqslant C_5\|U\|\|U_{tt}\| \tag{77}$$

and

$$(u|u|_{\mathbb{R}}^{p-2}\ln|u|_{\mathbb{R}}^{k} - \tilde{u}|\tilde{u}|_{\mathbb{R}}^{p-2}\ln|\tilde{u}|_{\mathbb{R}}^{k}, U_{tt}) \leqslant C_6\|U\|U_{tt}\|. \tag{78}$$

Furthermore,

$$\begin{aligned}
&\int_\Omega \left(|\tilde{u}_t|^\rho \tilde{u}_{tt} + |u_t|^\rho u_{tt}\right) U_{tt} \\
&\leqslant \frac{1}{2}\left(\|u_t\|_{2(\rho+1)}^\rho + \|\tilde{u}_t\|_{2(\rho+1)}^\rho\right)\|U_{tt}\|_{2(\rho+1)}|U_{tt}| \\
&\quad + \frac{\rho}{2}\|u_t + \lambda(u_t - \tilde{u}_t)\|_{L^r(\Omega)}^{\rho-1}\|u_t - \tilde{u}_t\|_{L^{\frac{3r}{r-(\rho-1)}}(\Omega)}\|u_{tt} + \tilde{u}_{tt}\|_{L^{\frac{3r}{r-(\rho-1)}}(\Omega)}\|U_{tt}\|_{L^{\frac{3r}{r-(\rho-1)}}(\Omega)} \\
&\leqslant \frac{1}{2}C_*^{\rho+1}C_p\left(\|u_t\|^\rho + \|\tilde{u}_t\|^\rho\right)\|U_{tt}\|^2 + C_7\|U_t\|\|U_{tt}\| \\
&\leqslant C_*^{\rho+1}C_p(2\mathscr{E}(0))^{\rho/2}\|U_{tt}\|^2 + C_7\|U_t\|\|U_{tt}\|.
\end{aligned} \tag{79}$$

Thus, we obtain the following estimate for $H(t)$:

$$H(t) \leqslant (C_5 + C_6)\|U\|\|U_{tt}\| + C_*^{\rho+1}C_p(2\mathscr{E}(0))^{\rho/2}\|U_{tt}\|^2 + C_7\|U_t\|\|U_{tt}\|. \tag{80}$$

From the Young inequality, it follows

$$\begin{aligned}
H(t) \leqslant\ & (C_5 + C_6)\left(\frac{1}{4\eta}\|U\|^2 + \eta\|U_{tt}\|^2\right) \\
& + C_*^{\rho+1}C_p(2\mathscr{E}(0))^{\rho/2}\|U_{tt}\|^2 + C_7\left(\frac{1}{4\eta}\|U_t\|^2 + \eta\|U_{tt}\|^2\right).
\end{aligned} \tag{81}$$

From the estimates (74), (75), (76) and (81), the inequality (73) takes the form:

$$\begin{aligned}\|U_{tt}\|^2 \leqslant \tilde{m}\left(\frac{1}{4\eta}\|U\|^2+\eta\|U_{tt}\|^2\right)+\frac{1}{4\eta}(g\diamond\nabla U)(t)\\ +(1-l)\left(2\eta\|U_{tt}\|^2+\frac{1}{4\eta}\|U\|^2\right)+(C_5+C_6)\left(\frac{1}{4\eta}\|U\|^2+\eta\|U_{tt}\|^2\right)\\ +C_*^{\rho+1}C_p(2\mathscr{E}(0))^{\rho/2}\|U_{tt}\|^2+C_7\left(\frac{1}{4\eta}\|U_t\|^2+\eta\|U_{tt}\|^2\right)\\ +\frac{C_p^2}{4\eta}\|U_t\|^2+\eta C_p^2\|U_{tt}\|^2. \qquad (82)\end{aligned}$$

Hence,

$$\begin{aligned}\left[1-C_*^{\rho+1}C_p(2\mathscr{E}(0))^{\rho/2}-\eta(\tilde{m}+2(1-l)+(C_5+C_6)+C_p^2)\right]\|U_{tt}\|^2\\ \leqslant\frac{1}{4\eta}(\tilde{m}+1-l+C_5+C_6)\|U\|^2+\frac{1}{4\eta}(g\diamond\nabla U)(t)+\frac{1}{4\eta}(C_7+C_p^2)\|U_t\|^2. \qquad (83)\end{aligned}$$

Since by hypothesis we assumed $1-C_*^{\rho+1}C_p(2\mathscr{E}(0))^{\rho/2}>0$, we can take $\eta>0$ sufficiently small such that

$$1-C_*^{\rho+1}C_p(2\mathscr{E}(0))^{\rho/2}-\eta(\tilde{m}+2(1-l)+(C_5+C_6)+C_p^2)>0.$$

Combining (83) with (70), we deduce that there exist positive constants C_8, C_9 e C_{10} such that

$$G(t)\leqslant C_8\|U\|^2+C_9\|U_t\|^2+C_{10}(g\diamond\nabla U)(t),\quad \forall t\in[0,T]. \qquad (84)$$

Finally, integrating (57) from 0 to $t\in[0,T]$ and employing Gronwall inequality, we obtain

$$\|U(t)\|^2+\|U_t(t)\|^2+(g\diamond\nabla U)(t)=0,\quad \forall t\in[0,T]. \qquad (85)$$

therefore $u(t)=\tilde{u}(t)$ for all $t\in[0,T]$. This prove the uniqueness of weak solution. □

REMARK 1. Throughout this section, the reader will realize the recurrent mention of Mean Value Theorem (MVT), for which the use demanded diferentiability of real functions involving the nonlinear terms, namely, $\xi|\xi|_{\mathbb{R}}^{p-2}\ln|\xi|_{\mathbb{R}}$, $|\xi|_{\mathbb{R}}^{\rho}$ and $M(\xi)$. Consequently, the derivative in the formula of MVT depends on objects which depends on t. However, as a function of t, such derivatives remains bounded due to their arguments depend, on norm in $H_0^1(\Omega)$, of functions the vary in a ball of $L^\infty(0,T,H_0^1(\Omega))$. This shows the Lipschitz character of the nonlinearities, but the same not happens for $|\cdot|_{\mathbb{R}}^{\rho}$ near zero when $0<\rho<1$. We recommend [26] for a study of well-posedness of a kind of equation with $|u_t|_{\mathbb{R}}^{\rho}u_{tt}$ and $0<\rho<1$.

6. Exponential decay

According to what we have just proved, if the initial data are in the stable set $\mathscr{W}_1$, we guarantee the existence of a global weak solution and the associated energy $\mathscr{E}(t)$ is always non-negative and non-increasing. We shall prove that in fact if a global weak solution exists with small positive initial energy, then the total energy has an exponential decay property. For this purpose we will assume a fourth hypothesis over the problem (1):

$(H.4)$ $M(\tau)\tau \geqslant \hat{M}(\tau),\ \tau \geqslant 0$

THEOREM 5. *Considering* $(H.4)$, *the hypotheses of Theorem 3, and in addition* $I(0) > 0$. *Then there exists* $\zeta > 0$ *such that if* $0 < \mathscr{E}(0) < \zeta$, *then there exist positive constants* C_0 *and* μ *such that*

$$\mathscr{E}(t) \leqslant C_0 e^{-\mu t} \quad (t > 0). \tag{86}$$

Proof. Let us define the functional

$$\mathscr{F}(t) := \mathscr{E}(t) + \frac{\varepsilon}{\rho+1}\int_\Omega |u_t|_{\mathbb{R}}^{\rho} u_t u dx + \varepsilon \int_\Omega \nabla u \nabla u_t dx - \frac{\varepsilon}{2}\int_\Omega |u|_{\mathbb{R}}^2 dx + \theta\chi(t) \tag{87}$$

where

$$\begin{aligned}\chi(t) = &-\int_\Omega \nabla u_t \int_0^t g(t-s)(\nabla u(t) - \nabla u(s)) ds dx \\ &-\int_\Omega \frac{|u_t|_{\mathbb{R}}^{\rho} u_t}{\rho+1}\int_0^t g(t-s)(u(t)-u(s)) ds dx\end{aligned}$$

and ε and θ are small positive constants so that there exist positive constants α_1 and α_2 satisfying

$$\alpha_1 \mathscr{F}(t) \leqslant \mathscr{E}(t) \leqslant \alpha_2 \mathscr{F}(t). \tag{88}$$

Such choices for ε and θ are possible due to (17). Indeed, we have

$$\begin{aligned}\int_\Omega |u_t|_{\mathbb{R}}^{\rho} u_t u dx &\leqslant \frac{1}{2}\int_\Omega |u_t|_{\mathbb{R}}^{2(\rho+1)} dx + \frac{1}{2}\int_\Omega |u|_{\mathbb{R}}^2 dx \\ &\leqslant |\Omega|^{\rho/(\rho+1)} C_p^{2(\rho+1)} (2\mathscr{E}(0))^{\rho}\int_\Omega |\nabla u_t|_{\mathbb{R}^n}^2 dx + \frac{C_p^2}{2}\int_\Omega |\nabla u|_{\mathbb{R}^n}^2 dx\end{aligned}$$

also

$$\begin{aligned}&\int_\Omega \nabla u_t \int_0^t g(t-s)(\nabla u(t) - \nabla u(s)) ds dx \\ &\leqslant \frac{1}{2}\int_\Omega |\nabla u_t|_{\mathbb{R}^n}^2 dx + \frac{1}{2}\|g\|_{L_1}|\Omega|(g \diamond \nabla u)(t)\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega |u_t|_{\mathbb{R}}^{\rho} u_t \int_0^t g(t-s)(u(t)-u(s))dsdx\\&\leqslant \frac{1}{2}\int_\Omega |u_t|_{\mathbb{R}}^{2(\rho+1)}dx+\frac{1}{2}\|g\|_{L^1}C_p^2(g\diamond\nabla u)(t)\\&\leqslant |\Omega|^{\rho/(\rho+1)}C_s^{2(\rho+1)}\int_\Omega |\nabla u_t|_{\mathbb{R}^n}^2 dx+\frac{1}{2}\|g\|_{L^1}C_p^2(g\diamond\nabla u)(t)\end{aligned}$$

where C_s and C_p are the embedding $H_0^1(\Omega)\hookrightarrow L^{2(\rho+1)}(\Omega)$ and the Poincaré constants, respectively. By mean of (17) there exists a constant $C>0$ such that

$$|\mathscr{F}(t)-\mathscr{E}(t)|\leqslant C\varepsilon\mathscr{E}(t).$$

Denoting

$$F(t):=\mathscr{E}(t)+\frac{\varepsilon}{\rho+1}\int_\Omega |u_t|_{\mathbb{R}}^{\rho}u_t u dx+\varepsilon\int_\Omega \nabla u\nabla u_t dx-\frac{\varepsilon}{2}\int_\Omega |u|_{\mathbb{R}}^2 dx \tag{89}$$

and differentiating (89) and next adding and subtracting $\varepsilon\int_0^t g(t-s)ds\|u\|^2$ to its expression, we obtain

$$\begin{aligned}F'(t)=\mathscr{E}'(t)-\varepsilon M(\|u\|^2)(\|u\|^2)+\varepsilon\int_0^t g(t-s)(\nabla u(s)-\nabla u(t),\nabla u(t))ds\\+\varepsilon\int_0^t g(t-s)ds\|u\|^2+\varepsilon\int_\Omega |u|_{\mathbb{R}}^p\ln|u|_{\mathbb{R}}^k dx+\varepsilon\int_\Omega |\nabla u_t|_{\mathbb{R}^n}^2 dx\\+\varepsilon\frac{1}{\rho+1}\int_\Omega |u_t|_{\mathbb{R}}^{\rho+2}dx.\end{aligned}\tag{90}$$

Now we pick $\alpha\in(0,1)$ and estimate $\int_\Omega |u|_{\mathbb{R}}^p\ln|u|_{\mathbb{R}}^k dx$ from the formulae (9):

$$\begin{aligned}&\int_\Omega |u(t)|_{\mathbb{R}}^p\ln|u(t)|_{\mathbb{R}}^k dx\\&=\alpha\int_\Omega |u(t)|_{\mathbb{R}}^p\ln|u(t)|_{\mathbb{R}}^k dx+(1-\alpha)\int_\Omega |u(t)|_{\mathbb{R}}^p\ln|u(t)|_{\mathbb{R}}^k dx\\&\leqslant\alpha\Big\{\frac{p}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2}+\frac{p}{2}\Big[\hat{M}(\|u\|^2)-\int_0^t g(s)ds\|u\|^2\Big]+\frac{p}{2}\|u_t\|^2\\&\quad+\frac{k}{p}\int_\Omega |u(t)|_{\mathbb{R}}^p dx+\frac{p}{2}(g\diamond\nabla u)(t)-p\mathscr{E}(t)\Big\}\\&\quad+(1-\alpha)\int_\Omega |u(t)|_{\mathbb{R}}^p\ln|u(t)|_{\mathbb{R}}^k dx.\end{aligned}\tag{91}$$

Exploit Young's inequality, it derives, for any $\delta>0$,

$$\int_0^t g(t-s)(\nabla u(s)-\nabla u(t),\nabla u(t))ds\leqslant\frac{1}{4\delta}(g\diamond\nabla u)(t)+\delta\int_0^t g(t-s)ds\|u\|^2. \tag{92}$$

The logarithmic term $\int_\Omega |u|^p_{\mathbb{R}} \ln |u|^k_{\mathbb{R}} dx$ can be estimated as follows:

$$\begin{aligned}\int_\Omega |u|^p_{\mathbb{R}} \ln |u|^k_{\mathbb{R}} dx &\leqslant k\int_\Omega |u|^{p+1}_{\mathbb{R}} dx \leqslant kC_*^{p+1}\|u\|^{p+1} \\ &= \frac{kC_*^{p+1}}{l+m_0-1}\|u\|^{p-1}(l+m_0-1)\|u\|^2 \leqslant \beta(l+m_0-1)\|u\|^2 \\ &\leqslant \beta\left(\hat{M}(\|u\|^2) - \int_0^t g(s)ds\|u\|^2\right).\end{aligned}$$

where

$$\beta := \frac{kC_*^{p+1}}{l+m_0-1}\left[\frac{2p}{(p-2)(l+m_0-1)}\mathscr{E}(0)\right]^{(p-1)/2}. \tag{93}$$

Shortly

$$\int_\Omega |u|^p_{\mathbb{R}} \ln |u|^k_{\mathbb{R}} dx \leqslant \beta\left(\hat{M}(\|u\|^2) - \int_0^t g(s)ds\|u\|^2\right). \tag{94}$$

Combining (91), (92), (94) and noting that $\mathscr{E}'(t) \leqslant \frac{1}{2}(g' \diamond \nabla u)(t)$, it follows

$$\begin{aligned}F'(t) \leqslant{}& \frac{1}{2}(g' \diamond \nabla u)(t) - \varepsilon M(\|u\|^2)(\|u\|^2) + \varepsilon\left\{\frac{1}{4\delta}(g\diamond\nabla u)(t) + \delta\int_0^t g(s)ds\|u\|^2\right\} \\ &+ \varepsilon\int_0^t g(t-s)ds\|u\|^2 + \varepsilon\alpha\left\{\frac{p}{\rho+2}\|u_t\|^{\rho+2}_{\rho+2} + \frac{p}{2}\left[\hat{M}(\|u\|^2) - \int_0^t g(s)ds\|u\|^2\right]\right. \\ &\left. + \frac{p}{2}\|u_t\|^2 + \frac{k}{p}\int_\Omega |u(t)|^p_{\mathbb{R}} dx + \frac{p}{2}(g\diamond\nabla u)(t) - p\mathscr{E}(t)\right\} \\ &+ \varepsilon(1-\alpha)\beta\left(\hat{M}(\|u\|^2) - \int_0^t g(s)ds\|u\|^2\right) \\ &+ \varepsilon\int_\Omega |\nabla u_t|^2_{\mathbb{R}^n} dx + \varepsilon\frac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2}_{\mathbb{R}} dx.\end{aligned}$$

Thus

$$\begin{aligned}F'(t) \leqslant{}& \frac{1}{2}(g' \diamond \nabla u)(t) + \varepsilon\left(\frac{\alpha p}{2}+1\right)\|u_t\|^2 + \varepsilon\left(\frac{\alpha p}{2} + \frac{1}{4\delta}\right)(g\diamond\nabla u)(t) \\ &+ \varepsilon\left(\frac{1}{\rho+1} + \frac{\alpha p}{\rho+2}\right)\|u_t(t)\|^{\rho+2}_{\rho+2} + \varepsilon\left(-M(\|u\|^2) + \int_0^t g(s)ds\right)\|u\|^2 \\ &+ \varepsilon\frac{\alpha p}{2}\left(\hat{M}(\|u\|^2) - \int_0^t g(s)ds\|u\|^2\right) \\ &+ \varepsilon(1-\alpha)\beta\left(\hat{M}(\|u\|^2) - \int_0^t g(s)ds\|u\|^2\right) \\ &+ \varepsilon\delta\int_0^t g(s)ds\|u(t)\|^2 + \frac{\varepsilon\alpha k}{p}\int_\Omega |u(t)|^p_{\mathbb{R}} dx - \varepsilon\alpha p\mathscr{E}(t).\end{aligned} \tag{95}$$

Note that by embeddings $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^{\rho+2}(\Omega)$ we obtain

$$\begin{aligned}\int_\Omega |u|_{\mathbb{R}}^p dx &\leqslant |\Omega|^{1/(p+1)} C_*^p \|u\|^p = |\Omega|^{1/(p+1)} C_*^p \|u\|^2 (\|u\|^2)^{(p-2)/2} \\ &\leqslant |\Omega|^{1/(p+1)} C_*^p \frac{2p}{(p-2)(m_0+l-1)} \mathscr{E}(t) \left(\frac{2p}{(p-2)(m_0+l-1)} \mathscr{E}(0) \right)^{(p-2)/2} \\ &= |\Omega|^{1/(p+1)} C_*^p \left(\frac{2p}{(p-2)(m_0+l-1)} \right)^{p/2} \mathscr{E}(0)^{(p-2)/2} \mathscr{E}(t) \end{aligned} \tag{96}$$

and

$$\|u_t\|_{\rho+2}^{\rho+2} \leqslant C_{**}^{\rho+2} \|u_t\|^{\rho+2} = C_{**}^{\rho+2} \|u_t\|^2 (\|u_t\|^2)^{2\rho} \leqslant 2C_{**}^{\rho+2} (2\mathscr{E}(0))^{2\rho} \mathscr{E}(t). \tag{97}$$

The hypotheses $(H3)$ and $(H4)$ and the estimate (96) and (97) lead us to

$$\begin{aligned} F'(t) \leqslant & \left[\frac{1}{2} - \frac{\varepsilon}{\xi_2} \left(\frac{\alpha p}{2} + \frac{1}{4\delta} \right) \right] (g' \diamond \nabla u)(t) + \varepsilon \left(\frac{\alpha p}{2} + 1 \right) \|u_t\|^2 \\ &+ \varepsilon \left[-1 + \left(\frac{\alpha p}{2} + (1-\alpha)\beta \right) + \frac{\delta(1-l)}{m_0+l-1} \right] \left(\hat{M}(\|u\|^2) - \int_0^t g(s)ds \|u\|^2 \right) \\ &+ \varepsilon \Bigg[\left(\frac{1}{\rho+1} + \frac{\alpha p}{\rho+2} \right) 2C_{**}^{\rho+2} (2\mathscr{E}(0))^{2\rho} \\ &+ \frac{\alpha k}{p} |\Omega|^{1/(p+1)} C_*^p \left(\frac{2p}{(p-2)(m_0+l-1)} \right)^{p/2} \mathscr{E}(0)^{(p-2)/2} - \alpha p \Bigg] \mathscr{E}(t). \end{aligned} \tag{98}$$

Next steps are devoted to deal with estimates of derivative of χ. We have

$$\begin{aligned} \chi'(t) = & M(\|u\|^2) \int_\Omega \nabla u(t) \int_0^t g(t-s)(\nabla u(t) - \nabla u(s)) ds \\ &- \int_\Omega \left(\int_0^t g(t-s) \nabla u(s) ds \right) \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s)) ds \right) dx \\ &+ \int_\Omega u_t \int_0^t g(t-s)(u(t)-u(s)) ds dx \\ &- \int_\Omega u|u|_{\mathbb{R}}^{p-2} \ln|u|^k \int_0^t g(t-s)(u(t)-u(s)) ds dx \\ &- \frac{1}{\rho+1} \int_\Omega |u_t|_{\mathbb{R}}^\rho u_t \int_0^t g'(t-s)(u(t)-u(s)) ds dx \\ &- \frac{1}{\rho+1} \left(\int_0^t g(s) ds \right) \int_\Omega |u_t|_{\mathbb{R}}^{\rho+2} dx \\ &- \int_\Omega \nabla u_t(t) \int_0^t g'(t-s)(\nabla u(t) - \nabla u(s)) ds dx - \left(\int_0^t g(s) ds \right) \int_\Omega |\nabla u_t|_{\mathbb{R}^n}^2 dx. \end{aligned} \tag{99}$$

Next we estimate some of the terms of (99). The estimate for the first term is obtained by using Young inequality followed by Hölder inequality in the second integrand:

$$M(\|u\|^2)\int_\Omega \nabla u(t)\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds$$
$$\leqslant \overline{m}\delta\int_\Omega |\nabla u|^2_{\mathbb{R}^n}dx+\frac{\overline{m}}{4\delta}\left(\int_0^t g(s)ds\right)(g\diamond\nabla u)(t) \tag{100}$$

where $\overline{m}=\max_{\tau>0}M(\tau)$.

The second term is handled as follows:

$$\int_\Omega\left(\int_0^t g(t-s)\nabla u(s)ds\right)\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx$$
$$\leqslant \delta\int_\Omega\left|\int_0^t g(t-s)\nabla u(s)ds\right|^2dx+\frac{1}{4\delta}\int_\Omega\left|\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right|^2dx$$
$$=\delta\int_\Omega\left|\int_0^t g(t-s)\nabla u(s)ds\right|^2dx+\frac{1}{4\delta}\left(\int_0^t g(s)ds\right)(g\diamond\nabla u)(t). \tag{101}$$

The first expression of (101) leads us to

$$\int_\Omega\left|\int_0^t g(t-s)\nabla u(s)ds\right|^2dx$$
$$\leqslant\left(\int_0^t g(s)ds\right)\int_\Omega\int_0^t g(t-s)(|\nabla u(t)-\nabla u(s)|+|\nabla u(t)|)^2dsdx$$
$$=\left(\int_0^t g(s)ds\right)\left(\int_\Omega\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2dsdx\right.$$
$$\left.+2\int_\Omega\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)||\nabla u(t)|dsdx+\int_\Omega\int_0^t g(t-s)|\nabla u(t)|^2dsdx\right)$$
$$\leqslant\left(\int_0^t g(s)ds\right)\left(2(g\diamond\nabla u)(t)+2\left(\int_0^t g(s)ds\right)\int_\Omega|\nabla u(t)|^2_{\mathbb{R}^n}dx\right). \tag{102}$$

Connecting (102) to (101), it follows

$$\int_\Omega\left(\int_0^t g(t-s)\nabla u(s)ds\right)\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx$$
$$\leqslant\left(2\delta+\frac{1}{4\delta}\right)\left(\int_0^t g(s)ds\right)(g\diamond\nabla u)(t)+2\delta(1-l)^2\int_\Omega|\nabla u(t)|^2_{\mathbb{R}^n}dx. \tag{103}$$

The estimate of third expression of (99) is obtained by mean of Young and Poincaré inequalities:

$$\int_\Omega u_t\int_0^t g(t-s)(u(t)-u(s))dsdx\leqslant\delta\int_\Omega|u|^2_{\mathbb{R}}dx+\frac{C_p^2}{4\delta}\left(\int_0^t g(s)ds\right)(g\diamond\nabla u)(t)$$
$$\leqslant\delta C_p^2\int_\Omega|\nabla u|^2_{\mathbb{R}^n}dx+\frac{C_p^2}{4\delta}\left(\int_0^t g(s)ds\right)(g\diamond\nabla u)(t). \tag{104}$$

The fourth term requires the elemental logarithmic identity:

$$\left|\xi|\xi|^{p-2}\ln|\xi|\right| \leqslant c_0(|\xi|+|\xi|^p) \quad (\forall \xi \in \mathbb{R}).$$

Hence

$$\begin{aligned}
&-\int_\Omega u|u|_{\mathbb{R}}^{p-2}\ln|u|^k \int_0^t g(t-s)(u(t)-u(s))dsdx \\
&\leqslant kc_0 \int_\Omega (|u|_{\mathbb{R}}+|u|_{\mathbb{R}}^p)\int_0^t g(t-s)|u(t)-u(s)|dsdx \\
&\leqslant \delta kc_0 \int_\Omega (|u|_{\mathbb{R}}+|u|_{\mathbb{R}}^p)^2 dx + \frac{k}{4\delta}\left(\int_0^t g(t-s)|u(t)-u(s)|ds\right)^2 dx \\
&\leqslant \delta kc_0 C_p^2\|u\|^2 + 2\delta kc_0 C_*^{p+1}\|u\|^{p+1} + \delta kc_0 C_r^{2p}\|u\|^{2p} \\
&\quad + \frac{kc_0C_p^2}{4\delta}\left(\int_0^t g(s)ds\right)(g\diamond\nabla u)(t)
\end{aligned} \tag{105}$$

where C_r is a constant obtained from the embedding $H_0^1(\Omega)\hookrightarrow L^{2p}(\Omega)$. Note that we obtain directly from (17) the following inequality:

$$\|u\|^2 \leqslant \frac{2p}{(p-2)(m_0+l-1)}\mathscr{E}(0).$$

Thus the fourth term of (99) is estimated as

$$\begin{aligned}
&-\int_\Omega u|u|_{\mathbb{R}}^{p-2}\ln|u|_{\mathbb{R}}^k \int_0^t g(t-s)(u(t)-u(s))dsdx \\
&\leqslant \left[\delta kC_p^2 + 2\delta kC_*^{p+1}\left(\frac{2p}{(p-2)(m_0+l-1)}\mathscr{E}(0)\right)^{(p-1)/2}\right. \\
&\quad \left. + \delta kC_r^{2p}\left(\frac{2p}{(p-2)(m_0+l-1)}\mathscr{E}(0)\right)^{p-1}\right]\|u\|^2 \\
&\quad + \frac{kC_p^2}{4\delta}\left(\int_0^t g(s)ds\right)(g\diamond\nabla u)(t).
\end{aligned} \tag{106}$$

For the fifth term we own that $g'<0$ and the $H_0^1(\Omega)\hookrightarrow L^{2(\rho+1)}(\Omega)$ embedding in order to obtain

$$\begin{aligned}
&-\frac{1}{\rho+1}\int_\Omega |u_t|_{\mathbb{R}}^\rho u_t \int_0^t g'(t-s)(u(t)-u(s))dsdx \\
&\leqslant \frac{\delta C_s^{2(\rho+1)}}{\rho+1}(2\mathscr{E}(0))^\rho \int_\Omega |\nabla u_t|_{\mathbb{R}^n}^2 dx + \frac{g(0)C_p^2}{4\delta(\rho+1)}(-g\diamond\nabla u)(t).
\end{aligned} \tag{107}$$

Finally the seventh term is estimated as follows:

$$\begin{aligned}
&-\int_\Omega \nabla u_t(t)\int_0^t g'(t-s)(\nabla u(t)-\nabla u(s))dsdx \\
&\leqslant \delta\int_\Omega |\nabla u_t|_{\mathbb{R}^n}^2 dx + \frac{g(0)}{4\delta}(-g'\diamond\nabla u)(t).
\end{aligned} \tag{108}$$

The estimates (100), (103), (104), (106), (107), (108) and (98) induce the following estimate for the derivative of $\mathscr{F}$:

$$\mathscr{F}'(t) \leqslant A\left(\hat{M}(\|u\|^2) - \int_0^t g(s)ds\|u\|^2\right) + B(g' \diamond \nabla u)(t) + C\|u_t\|^2 - D\mathscr{E}(t) \tag{109}$$

where A, B and D are constants depending on α, δ, ε and θ, and C depends further on t. For the instance

$$\begin{aligned} A = \varepsilon&\left[-1 + \left(\frac{\alpha p}{2} + (1-\alpha)\beta\right) + \frac{\delta(1-l)}{m_0+l-1}\right] \\ &+ \frac{\theta\delta}{m_0+l-1}\left[\overline{m} + 2(1-l)^2 + C_p^2 + kc_0C_p^2\right. \\ &+ 2kc_0C_*^{p+1}\left(\frac{2p}{(p-2)(m_0+l-1)}\mathscr{E}(0)\right)^{(p-1)/2} \\ &\left.+ kc_0C_r^{2p}\left(\frac{2p}{(p-2)(m_0+l-1)}\mathscr{E}(0)\right)^{p-1}\right] \end{aligned} \tag{110}$$

$$\begin{aligned} B = &\left[\frac{1}{2} - \frac{\varepsilon}{\xi_2}\left(\frac{\alpha p}{2} + \frac{1}{4\delta}\right)\right] \\ &- \theta\left[\frac{\overline{m}}{4\delta\xi_2}(1-l) + \left(2\delta + \frac{1}{4\delta}\right)\frac{(1-l)}{\xi_2} + \frac{C_p^2}{4\delta\xi_2}(1-l)\right. \\ &\left.+ \frac{kc_0C_p^2}{4\delta}\frac{(1-l)}{\xi_2} + \frac{g(0)C_p^2}{4\delta(\rho+1)} + \frac{g(0)}{4\delta}\right] \end{aligned} \tag{111}$$

$$C = \left[\varepsilon\left(\frac{\alpha p}{2} + 1\right) - \theta\left(\int_0^t g(s)ds\right)\right] + \theta\delta\left[\frac{C_s^{2(\rho+1)}}{\rho+1}(2\mathscr{E}(0))^\rho + 1\right] \tag{112}$$

$$\begin{aligned} D = \varepsilon&\left[\alpha p - \left(\frac{1}{\rho+1} + \frac{\alpha p}{\rho+2}\right)2C_{**}^{\rho+2}(2\mathscr{E}(0))^{2\rho}\right. \\ &\left.- \frac{\alpha k}{p}|\Omega|^{1/(p+1)}C_*^p\left(\frac{2p}{(p-2)(m_0+l-1)}\mathscr{E}(0)\right)^{p/2}\mathscr{E}(0)^{(p-2)/2}\right]. \end{aligned} \tag{113}$$

The constants α, δ, ε and θ will be chosen next in order to obtain $A < 0$, $B, D > 0$ and $C(t) < 0$ for all t far from zero.

We start by imposing that $\mathscr{E}(0)$ is small enough for that β defined in (93) is less than 1, and assume that $\alpha := \frac{1-\beta}{p-2\beta}$. Hence α belongs to the interval $(0,1)$. In order to obtain $D > 0$ we suppose that $\mathscr{E}(0)$ is even smaller satisfying

$$\begin{aligned} &\left(\frac{p-2\beta}{(1-\beta)(\rho+1)} + \frac{p}{\rho+2}\right)2C_{**}^{\rho+2}(2\mathscr{E}(0))^{2\rho} \\ &+ \frac{k}{p}|\Omega|^{1/(p+1)}C_*^p\left(\frac{2p}{(p-2)(m_0+l-1)}\mathscr{E}(0)\right)^{p/2}\mathscr{E}(0)^{(p-2)/2} < \alpha p. \end{aligned} \tag{114}$$

Let $t_0>0$ and denote $a:=\int_0^{t_0} g(s)ds$. We then put θ depending on ε in this way: $\theta:=\frac{\alpha p+2}{a}\varepsilon$. This implies

$$\varepsilon\left(\frac{\alpha p}{2}+1\right)-\theta\left(\int_0^t g(s)ds\right)\leqslant\varepsilon\left(\frac{\alpha p}{2}+1\right)-\theta a<0$$

which allows us to find $\delta_0>0$ such that if $\delta<\delta_0$, then $C(t)<0$ for all $t>t_0$. The choice of α is also enough for that

$$-1+\left(\frac{\alpha p}{2}+(1-\alpha)\beta\right)<0$$

holds. Thus we can find δ even smaller and sufficient to make $A<0$. Finally we set ε so small so that the constant B becomes positive. The inequality for derivative of $\mathscr{F}$ in (109) yields to the following shortened one

$$\mathscr{F}'(t)\leqslant -D\mathscr{E}(t),\quad (t>t_0).$$

Recurring to (88) twice, the proof of theorem is completed. □

7. Exponential growth

In this section we shall prove that the logarithmic term of the total energy is unbounded when the initial data are large enough in some sense. That is, it will be proved that the term $\int_\Omega |u|_{\mathbb{R}}^p \ln|u|_{\mathbb{R}}^k dx$ grows unboundedly as an exponential function. This will be established here in spite of the strong exponential decreasingness of the relaxation function $g(t)$.

THEOREM 6. *Assuming that* $0<\rho\leqslant p-2$ *then the solution of problem* (1) *grows up exponentially in the sense that there exist positive constants* c,b *and* λ *such that*

$$\int_\Omega |u|_{\mathbb{R}}^p \ln|u|_{\mathbb{R}}^k dx+c\geqslant be^{\lambda t} \tag{115}$$

provided that $l>\frac{4}{\rho+2}$ *and that initial conditions satisfy* $\mathscr{E}(0)<0$.

Proof. Let us define the functional

$$\mathscr{F}(t)=\mathscr{E}(t)-\varepsilon\Psi(t) \tag{116}$$

where $\Psi(t)$ is defined by

$$\Psi(t)=\frac{1}{\rho+1}\int_\Omega |u_t|_{\mathbb{R}}^\rho u_t u dx+\int_\Omega \nabla u\nabla u_t dx \tag{117}$$

and ε is to be defined later.

Firstly observe that we can find positive constants a_i, $i=1,\cdots,5$, such that $a_i = a_i(\rho,p,\varepsilon,l,|\Omega|)$ such that

$$\begin{aligned}\mathscr{F}(t) \geqslant\ & a_1\int_\Omega |u_t|_{\mathbb{R}}^{\rho+2}dx + a_2\int_\Omega |\nabla u_t(t)|_{\mathbb{R}^n}^2 dx + \frac{1}{2}(g\diamond\nabla u)(t)\\ & + a_3\int_\Omega |u|_{\mathbb{R}}^p dx + a_4\int_\Omega |\nabla u|_{\mathbb{R}^n}^2 dx - a_5\int_\Omega |u|_{\mathbb{R}}^p \ln|u|_{\mathbb{R}}^k dx - a_6. \end{aligned}\tag{118}$$

Indeed, by our assumption $\rho \leqslant p-2$, then $\frac{p(\rho+1)}{p-1} \leqslant \rho+2$. Therefore

$$\int_\Omega |u_t|_{\mathbb{R}}^\rho u_t u \leqslant \frac{1}{p}\int_\Omega |u|_{\mathbb{R}}^p dx + \frac{(p-1)|\Omega|}{p} + \frac{p-1}{p}\int_\Omega |u_t|_{\mathbb{R}}^{\rho+2}dx. \tag{119}$$

Thus

$$\begin{aligned}\mathscr{F}(t) \geqslant\ & \left(\frac{1}{\rho+2} - \frac{\varepsilon(p-1)}{p(\rho+1)}\right)\int_\Omega |u_t|_{\mathbb{R}}^{\rho+2}dx + \frac{1}{2}(1-\varepsilon)\int_\Omega |\nabla u_t|_{\mathbb{R}^n}^2 dx\\ & + \left(\frac{k}{p^2} - \frac{\varepsilon}{p(\rho+1)}\right)\int_\Omega |u|_{\mathbb{R}}^p dx + \frac{1}{2}\left(l+m_0-1-\varepsilon\right)\int_\Omega |\nabla u|_{\mathbb{R}^n}^2 dx\\ & + \frac{1}{2}(g\diamond\nabla u)(t) - \frac{1}{p}\int_\Omega |u|_{\mathbb{R}}^p \ln|u|_{\mathbb{R}}^k dx - \frac{\varepsilon(p-1)|\Omega|}{p(\rho+1)}. \end{aligned}\tag{120}$$

Choosing $\varepsilon > 0$ small enough we have (118) verified.
Secondly, differentiating $\mathscr{F}(t)$ with respect to t yields

$$\begin{aligned}\mathscr{F}'(t) =\ & -\int_\Omega |u_t|_{\mathbb{R}}^2 dx + \frac{1}{2}(g'\diamond\nabla u)(t) - \frac{1}{2}g(t)\int_\Omega |\nabla u(t)|_{\mathbb{R}^n}^2 dx\\ & -\varepsilon\left\{M(\|u\|^2)\int_\Omega |\nabla u|_{\mathbb{R}^n}^2 dx + \int_0^t g(t-s)(\nabla u(s),\nabla u(t))ds\right.\\ & \left. -\int_\Omega u_t u dx + \int_\Omega |u|_{\mathbb{R}}^p \ln|u|_{\mathbb{R}}^k + \frac{1}{\rho+1}\int_\Omega |u_t|_{\mathbb{R}}^{\rho+2}dx + \int_\Omega |\nabla u_t|_{\mathbb{R}^n}^2 dx\right\}. \end{aligned}\tag{121}$$

The following estimate is obtained easily from Young inequality:

$$\begin{aligned}\varepsilon\int_0^t g(t-s)(\nabla u(s),\nabla u(t))ds &\geqslant \frac{\varepsilon}{2}\int_0^t g(s)ds\int_\Omega |\nabla u|_{\mathbb{R}^n}^2 dx - \frac{\varepsilon}{2}(g\diamond\nabla u)\\ &\geqslant -\frac{\varepsilon}{2}(g\diamond\nabla u). \end{aligned}\tag{122}$$

We will also use the estimate

$$\int_\Omega u_t u dx \leqslant C_p\delta\int_\Omega |\nabla u|_{\mathbb{R}^n}^2 dx + \frac{1}{4\delta}\int_\Omega |u_t|_{\mathbb{R}}^2 dx \quad (\delta > 0) \tag{123}$$

where C_p is obtained from the Sobolev embbeding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$.

Now we add and subtract $\gamma\varepsilon\mathscr{F}(t)$ in the right side of the inequality (121) and obtain

$$\begin{aligned}
\mathscr{F}'(t) \leqslant \gamma\varepsilon\mathscr{F}(t) - \gamma\varepsilon\Big\{ &\left(\frac{1}{\rho+2} - \frac{\varepsilon(p-1)}{p(\rho+1)}\right)\int_\Omega |u_t|_{\mathbb{R}}^{\rho+2}dx + \frac{1}{2}(1-\varepsilon)\int_\Omega |\nabla u_t|_{\mathbb{R}^n}^2 dx \\
&+ \left(\frac{k}{p^2} - \frac{\varepsilon}{p(\rho+1)}\right)\int_\Omega |u|_{\mathbb{R}}^p dx + \frac{1}{2}(l+m_0-1-\varepsilon)\int_\Omega |\nabla u|_{\mathbb{R}^n}^2 dx \\
&+ \frac{1}{2}(g\diamond\nabla u)(t) - \frac{1}{p}\int_\Omega |u|_{\mathbb{R}}^p \ln|u|_{\mathbb{R}}^k dx - \frac{\varepsilon(p-1)|\Omega|}{p(\rho+1)}\Big\} \\
&- \int_\Omega |u_t|_{\mathbb{R}}^2 dx + \frac{1}{2}(g'\diamond\nabla u)(t) - \frac{1}{2}g(t)\int_\Omega |\nabla u(t)|_{\mathbb{R}^n}^2 dx \\
&- \varepsilon M(\|u^2\|)\int_\Omega |\nabla u|_{\mathbb{R}^n}^2 dx + \frac{\varepsilon}{2}(g\diamond\nabla u)(t) + \varepsilon\int_\Omega u_t u dx \\
&- \varepsilon\int_\Omega |u|_{\mathbb{R}}^p \ln|u|_{\mathbb{R}}^k dx - \varepsilon\frac{1}{\rho+1}\int_\Omega |u_t|_{\mathbb{R}}^{\rho+2}dx - \varepsilon\int_\Omega |\nabla u_t|_{\mathbb{R}^n}^2 dx. \qquad (124)
\end{aligned}$$

Knowing that $M(\tau) \geqslant m_0 > 1-l$ for all $\tau > 0$, also that $g'(t) \leqslant -\xi_2 g(t)$ and using the estimates (120) and (123) in the inequality (124) yields

$$\begin{aligned}
\mathscr{F}'(t) \leqslant \gamma\varepsilon\mathscr{F}(t) &- \varepsilon\left[\frac{\gamma}{\rho+2} + \frac{\gamma\varepsilon(p-1)}{p(\rho+1)}\right]\int_\Omega |u_t|_{\mathbb{R}}^{\rho+2}dx \\
&- \varepsilon\left[\frac{\gamma}{2}(l+m_0-1-\varepsilon) + m_0 - C_p\delta\right]\int_\Omega |\nabla u|_{\mathbb{R}^n}^2 dx \\
&- \left[\frac{\gamma\varepsilon(1-\varepsilon)}{2} + \varepsilon\right]\int_\Omega |\nabla u_t|_{\mathbb{R}^n}^2 dx - \left[1 - \frac{\varepsilon}{4\delta}\right]\int_\Omega |u_t|_{\mathbb{R}}^2 \\
&- \varepsilon\left[1 - \frac{\gamma}{p}\right]\int_\Omega |u|_{\mathbb{R}}^p \ln|u|_{\mathbb{R}}^k dx - \frac{1}{2}\left[\gamma\varepsilon + \xi_2 - \varepsilon\right](g\diamond\nabla u)(t) \\
&- \varepsilon\left[\frac{k\gamma}{p^2} - \frac{\gamma\varepsilon^2}{p^2(\rho+1)}\right]\int_\Omega |u|_{\mathbb{R}}^p dx - \frac{1}{2}g(t)\int_\Omega |\nabla u|_{\mathbb{R}^n}^2 + \frac{\gamma\varepsilon^2(p-1)|\Omega|}{p(\rho+1)}. \qquad (125)
\end{aligned}$$

Note that taking $\gamma = p$, the logarithmic term vanishes. Then we can find positive values for ε and δ such that the terms in the brackets are all positive. Thus

$$\mathscr{F}'(t) \leqslant \gamma\varepsilon\mathscr{F}(t) + \Lambda, \quad (t>0) \qquad (126)$$

where

$$\Lambda := \frac{\varepsilon^2(p-1)|\Omega|}{\rho+1}. \qquad (127)$$

Integrating (126), yields, for $M(t) = -\mathscr{F}(t)$

$$M(t) \geqslant \left(M(0) - \frac{\Lambda}{\gamma\varepsilon}\right)e^{\gamma\varepsilon t} + \frac{\Lambda}{\gamma\varepsilon} \geqslant \left(M(0) - \frac{\Lambda}{\gamma\varepsilon}\right)e^{\gamma\varepsilon t}. \qquad (128)$$

The initial data are assumed to satisfy $M(0)-\frac{\Lambda}{\gamma\varepsilon}:=b>0$, or

$$M(0)-\frac{\Lambda}{\gamma\varepsilon}>0$$

or even

$$-\mathscr{E}(0)+\varepsilon\Psi(0)-\frac{\Lambda}{\gamma\varepsilon}>0. \tag{129}$$

Since $\mathscr{E}(0)<0$, we can choose ε even smaller in such way that (129) is guaranteed.

The inequalities (128) and (118) imply

$$a_5\int_\Omega |u|_{\mathbb{R}}^p \ln|u|_{\mathbb{R}}^k dx + a_6 \geqslant -\mathscr{F}(t) = M(t) \geqslant be^{\gamma\varepsilon t}. \quad \square \tag{130}$$

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