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FUNDAMENTAL SOLUTIONS: A BRIEF REVIEW

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Abstract. We review briefly the fundamental solutions to some of the most important partial differential operators. These are very crucial in analysis and partial differential equations (PDEs). Among several applications, these are used, for instance, in studying regularity and growth of solutions.

1. Introduction

Fundamental solutions play an important role in formulation and solving of many local and non-local boundary value problems. The goal of this note is to discuss the fundamental solutions to some of the most important partial differential operators. Let us recall the definition of the fundamental solution to a linear operator L.

DEFINITION 1. Let L be a linear differential operator. We call a distribution E, a fundamental solution of L if

$$LE = \delta \tag{1.1}$$

in the sense of distribution. Here, δ is the Dirac *delta function*, also called the Dirac distribution.

One may see the book *La théorie des distributions* by Schwartz [52] for the details, where the author developed the theory of distributions in order to provide tools for solving PDEs. Let us go over some of the historical appearances of the topic. Before the first edition of the first part of the book, even the question of existence of a fundamental solution was not well posed. Even a generally adopted definition of a fundamental solution was not there before Schwartz [52]. In 1948, Schwartz posed the question of

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the existence of a fundamental solution to the not identically vanishing constant coefficient linear partial differential operator. The question was independently answered by Malgrange [49] and Ehrenpreis [25]. Following that, several researchers contributed to the finding of explicit general formulae for fundamental solutions. We refer to the book [50] for the details on solutions to linear partial differential operators. If u is a solution of the homogeneous equation $Lu = 0$ then $E+u$ is also a fundamental solution of (1.1). Using this definition, the additive and multiplicative constants can be computed. For example, the fundamental solutions of Laplace operator are upto some additive and multiplicative constants. We ignore these constants for time being.

This note is organized as follows. In Section 1, we review the fundamental solutions to the well-known operators. In Section 2, we derive the fundamental solution to fully nonlinear parabolic partial differential equations. In Section 3 and 4, we review fundamental solutions in the Heisenberg group and a Grushin-type space, respectively.

Let us consider the *Laplace* operator $-\Delta$ on $\mathbb{R}^N$, $N \geqslant 2$. The fundamental solution $\Phi(x)$ of $-\Delta$ is given by

$$\Phi(x) := \begin{cases} |x|^{2-N} & \text{for } N > 2, \\ -\log|x| & \text{for } N = 2, \end{cases} \tag{1.2}$$

for $x \in \mathbb{R}^N \setminus \{0\}$. More precisely, the fundamental solution of the Laplace operator is the function

$$\Phi(x) := \begin{cases} \frac{1}{N(N-2)\alpha(N)}|x|^{2-N} & \text{for } N > 2, \\ -\frac{1}{2\pi}\log|x| & \text{for } N = 2, \end{cases}$$

for $x \in \mathbb{R}^N \setminus \{0\}$. Here, $\alpha(N)$ is the volume of the unit ball in $\mathbb{R}^N$. Let us consider p-*Laplace* operator, defined as

$$-\Delta_p u := -div(|\nabla u|^{p-2}\nabla u) \text{ in } \mathbb{R}^N, \tag{1.3}$$

where $1 \leqslant p < \infty$. (1.3) is the Laplace operator when $p = 2$. (1.3) is called degenerate elliptic when $p > 2$ and for $1 < p < 2$, it is called singular at points where $\nabla u = 0$. The fundamental solution $\Phi_p(x)$ of p-Laplacian is given by

$$\Phi_p(x) := \begin{cases} |x|^{\frac{p-N}{p-1}}, & \text{if } p \neq N, \\ -\log|x|, & \text{if } p = N, \end{cases} \tag{1.4}$$

$x \in \mathbb{R}^N \setminus \{0\}$. When $p = \infty$, (1.3) is called *infinity Laplacian*, which is defined as follows:

$$-\Delta_\infty u := -\langle D^2u \cdot \nabla u, \nabla u\rangle = -\sum_{i,j=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}. \tag{1.5}$$

The cone functions

$$\Phi_\infty(x) := a|x| + b \tag{1.6}$$

are solutions of (1.5) in $\mathbb{R}^N \setminus \{0\}$, where a and b are constants. These are called the fundamental solutions of (1.5), see [20, 48].

Next, let us consider the *Hardy-Sobolev* operator:

$$-\Delta u - \frac{\lambda u}{|x|^2} \text{ in } \mathbb{R}^N \setminus \{0\}, \tag{1.7}$$

where

$$-\infty < \lambda \leqslant \frac{(N-2)^2}{4}.$$

When $-\infty < \lambda < \frac{(N-2)^2}{4}$, the fundamental solutions to (1.7) are given by

$$\Psi_\lambda^+(x) = |x|^{2-N+p} \text{ and } \Psi_\lambda^-(x) = |x|^{-p}, \;\; x \in \mathbb{R}^N \setminus \{0\}, \tag{1.8}$$

where $p := \frac{N-2}{2} - \sqrt{\frac{(N-2)^2}{4} - \lambda}$. When $\lambda = \frac{(N-2)^2}{4}$, the fundamental solutions are given by

$$\Psi^+(x) = |x|^{-\frac{(N-2)}{2}} \log\left(\frac{1}{|x|}\right) \text{ and } \Psi^-(x) = |x|^{-\frac{(N-2)}{2}}, \;\; x \in \mathbb{R}^N \setminus \{0\},$$

see [18, 41] for the details.

For a given λ, Λ satisfying $0 < \lambda \leqslant \Lambda < \infty$, *Pucci's extremal* operators are defined as follows:

$$\mathscr{M}^+_{\lambda,\Lambda}(M) := \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i,$$

and

$$\mathscr{M}^-_{\lambda,\Lambda}(M) := \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i, \tag{1.9}$$

where e_i's are the eigenvalues of M and $M \in S_N$. Here, S_N denotes the set of all $N \times N$ real symmetric matrices. In case when $\lambda = \Lambda = 1$, it is easy to see that

$$\mathscr{M}^+_{\lambda,\Lambda}(D^2u) = \mathscr{M}^-_{\lambda,\Lambda}(D^2u) = \Delta u.$$

Consider a function

$$\xi_\alpha(x) := \begin{cases} |x|^{-\alpha}, & \text{if } \alpha > 0, \\ -\log|x|, & \text{if } \alpha = 0, \\ -|x|^{-\alpha}, & \text{if } \alpha < 0 \end{cases}$$

for $x \in \mathbb{R}^N \setminus \{0\}$. A direct computation shows that the fundamental solutions of the Pucci extremal operators $\mathscr{M}^-_{\lambda,\Lambda}$ and $\mathscr{M}^+_{\lambda,\Lambda}$, are given by

$$\xi_{\frac{\lambda(N-1)}{\Lambda}-1} \text{ and } \xi_{\frac{\Lambda(N-1)}{\lambda}-1}, \tag{1.10}$$

respectively. We refer to [2, 21] for the details. By the well-known theory of elliptic partial differential equations, if Φ is a fundamental solution of a linear operator L then $L\Phi$ is interpreted as the Dirac mass at the origin. It is important to mention that this is not true for fully nonlinear operators as was seen by Labutin [47]. Labutin showed that if $\lambda \neq \Lambda$, then $\mathscr{M}^{+}_{\lambda,\Lambda}(\xi_{\frac{\Lambda(N-1)}{\lambda}-1})$ is not the Dirac mass at the origin but it vanishes near the origin in a reasonable weak sense.

Next, we consider partial trace operators, $\mathscr{P}^{\pm}_{k}$. Let S_N be as mentioned above. For $A \in S_N$, *partial trace operators* (truncated Laplacian) are defined as follows:

$$\mathscr{P}^{-}_{k}(A) := \sum_{i=1}^{k} e_i(A) \quad \text{and} \quad \mathscr{P}^{+}_{k}(A) := \sum_{i=N-k+1}^{N} e_i(A), \tag{1.11}$$

for $1 \leqslant k \leqslant N$. Here, $\{e_i\}$ is the ordered eigenvalues of A, i.e., $e_1(A) \leqslant e_2(A) \ldots \leqslant e_N(A)$. The fundamental solution of $\mathscr{P}^{+}_{k}$, i.e., the classical radial solution, $\Phi(x)$ of equation

$$\mathscr{P}^{+}_{k}(D^2 u) = 0 \text{ in } \mathbb{R}^N \setminus \{0\}, \tag{1.12}$$

is given by

$$\Phi(x) = \begin{cases} -c_1|x| + c_2 & \text{if } k = 1, \\ -c_1 \log|x| + c_2 & \text{if } k = 2, \\ c_1|x|^{2-k} + c_2 & \text{if } k > 2, \end{cases} \tag{1.13}$$

where $c_1 \geqslant 0$ and $c_2 \in \mathbb{R}$ are constants. We mention that the radial solutions of (1.12) are concave and increasing only if $k = N$, i.e., in case of Laplacian.

The fundamental solution of the *fractional Laplacian* $(-\Delta)^s$ in $\mathbb{R}^N \setminus \{0\}$ is given by

$$\Phi(x) := \begin{cases} a(N,s)|x|^{-N+2s}, & \text{if } N \neq 2s, \\ -\frac{1}{\pi}\log(|x|) & \text{if } N = 2s, \end{cases} \tag{1.14}$$

where $a(N,s)$ is a dimensional constant and $(-\Delta)^s$ stands for the fractional Laplacian and is defined as follows:

$$(-\Delta)^s u(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy = C_{N,s} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy, \ u \in S,$$

where P.V. stands for *in the principal value sense*, S is the Schwartz space of rapidly decreasing functions, $s \in (0,1)$ is fixed and

$$C_{N,s} := \frac{1}{\int_{\mathbb{R}^N} \frac{1-\cos(\xi_1)}{|\xi|^{N+2s}} d\xi}, \tag{1.15}$$

which is a normalization constant, see [13]. Here, ξ_1 is the first coordinate of $\xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{R}^N$.

Let $\mathscr{K}_0$ be a family of symmetric kernels, $K\colon \mathbb{R}^N \setminus \{0\} \longrightarrow \mathbb{R}$ satisfying

$$\lambda \frac{C(N,s)}{|x|^{N+2s}} \leqslant K(x) \leqslant \Lambda \frac{C(N,s)}{|x|^{N+2s}}, \quad x \neq 0, \tag{1.16}$$

where $c(N,s)$ is a positive constant. Consider *a class of nonlinear integral operators* defined as follows:

$$\mathscr{M}_{\mathscr{I}}{}^{+}u(x) = \sup_{\mathscr{I}\in\mathscr{L}_0} \mathscr{I}u(x) \quad \text{and} \quad \mathscr{M}_{\mathscr{I}}{}^{-}u(x) = \inf_{\mathscr{I}\in\mathscr{L}_0} \mathscr{I}u(x), \tag{1.17}$$

where $\mathscr{L}_0$ denotes a class of all linear operators $\mathscr{I}$ of the following form:

$$\mathscr{I}u(x) = \int_{\mathbb{R}^N} \Big(u(x+y) + u(x-y) - 2u(x)\Big) K(y)dy \ \text{ for } x \in \mathbb{R}^N. \tag{1.18}$$

Here, $K\colon \mathbb{R}^N \to \mathbb{R}$ denotes a kernel which belongs to the family $\mathscr{K}_0$ and u is such that the integrand in (1.18) is integrable in $\mathbb{R}^N \setminus B(0,\varepsilon)$ for $\varepsilon > 0$ and u is of class $C^{1,1}(x)$ in the sense of Caffarelli and Silvestre [15]. In particular, $\mathscr{I}u$ is well defined at x, when $u \in C^{1,1}(x)$ and is bounded and continuous. The fundamental solutions of $\mathscr{M}_{\mathscr{I}}{}^{+}$ in $\mathbb{R}^N \setminus \{0\}$ is given by

$$\Phi_{N^+}(|x|) := \begin{cases} |x|^{-N^+ + 2s} & \text{if } N^+ > 2s, \\ -\log(|x|) & \text{if } N^+ = 2s, \\ -|x|^{-N^+ + 2s} & \text{if } N^+ < 2s, \end{cases} \tag{1.19}$$

and

$$\Phi_{N^-}(|x|) = -|x|^{-N^- + 2s}, \tag{1.20}$$

where $N^+ = N^+(\alpha, \Lambda, N)$ and $N^- = N^-(\alpha, \Lambda, N)$ are dimensional constants. Also, the functions given by

$$\Psi_{N^+}(|x|) = -\Phi_{N^+}(|x|) \quad \text{and} \quad \Psi_{N^-}(|x|) = -\Phi_{N^-}(|x|) \tag{1.21}$$

are the fundamental solutions of $\mathscr{M}_{\mathscr{I}}{}^{-}$. We refer to Theorem 1.1 [30] for the details.

Next, consider a *nonlocal analogue of partial trace operators*, $\mathscr{J}_k^{\pm}$ defined as follows:

$$\mathscr{J}_k^{+} := \sup_{V \in \mathscr{W}_k} \mathscr{J}_W u(x), \tag{1.22}$$

and

$$\mathscr{J}_k^{-} := \inf_{V \in \mathscr{W}_k} \mathscr{J}_W u(x), \tag{1.23}$$

where $\mathscr{W}_k$ denotes the family of k-dimensional orthonormal sets in $\mathbb{R}^N$. For any $V \in \mathscr{W}_k$, let $\langle W \rangle$ be denote the k-dimensional subspace generated by W. Then, we define

$$\mathscr{I}_W u(x) := \frac{C(k,s)}{2} \int\limits_{\mathbb{R}^N} \left(u\left(x + \sum_{i=1}^{k} \tau_i \xi_i \right) + u\left(x - \sum_{i=1}^{k} \tau_i \xi_i \right) - 2u(x) \right) \times \left(\sum_{i=1}^{k} \tau_i^2 \right)^{-\frac{(k+2s)}{2}} d\tau_1 \tau_2 \dots \tau_k,$$

where $C(k,s)$ is a positive constant given by (1.15). $\mathscr{I}_W$ is equivalently defined as

$$\mathscr{I}_W u(x) = \frac{C(k,s)}{2} \int_{\langle V \rangle} \Big(u(x+y) + u(x-y) - 2u(x) \Big) |y|^{-(k+2s)} d\mathscr{H}^k(y), \tag{1.24}$$

where $\mathscr{H}^k$ denotes the k-dimensional Hausdorff measure in $\mathbb{R}^N$. It is interesting to note that $\mathscr{I}_k^{\pm} u \longrightarrow \mathscr{P}_k^{\pm}(D^2 u)$ as $s \to 1$. Also, for $k = N$, $J_N^{\pm} u = (-\Delta)^s u$. The fundamental solution of the truncated fractional Laplacian $\mathscr{I}_W$ in $\mathbb{R}^N \setminus \{0\}$ is given by

$$\Phi(x) := |x|^{-(k-2s)}, \tag{1.25}$$

upto a multiplicative constant. For the details, see Corollary 5.2 [12]. The case $k = N$, was studied by Felmer and Quaas [29].

Further, we consider the following *fractional p-Laplace operator*:

$$\mathscr{L} u := P.V. \int_{\mathbb{R}^N} \frac{|u(x,t) - u(y,t)|^{p-2} \big(u(x,t) - u(y,t) \big)}{|x-y|^{N+sp}} dy. \tag{1.26}$$

Vázquez [55] considered the following evolution equation:

$$\begin{cases} \partial_t u + \mathscr{L} u = 0 \text{ in } \mathbb{R}^N, \ N \geqslant 1 \ \text{ and } \ t > 0, \\ \lim\limits_{t \longrightarrow 0} u(x,t) = u_0(x), \end{cases} \tag{1.27}$$

where $u_0 \in L^2(\mathbb{R}^N)$. Author gave the fundamental solution, i.e., a function $u(x,t)$ such that

$$\lim_{t \longrightarrow 0} \int_{\mathbb{R}^N} u(x,t) \phi(x) dx = C\phi(0), \tag{1.28}$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$ and some positive constant C. More precisely, author proved the following result:

THEOREM 1. (see [55, Theorem 1.1]) *For every $C > 0$, there exists a unique self-similar solution, Φ of (1.27) with initial data $C\delta(x)$. Moreover, it is of the form*

$$\Phi(x,t;C) = C^{sp\beta} t^{-\alpha} F(C^{-(p-2)\beta} x t^{-\beta}),$$

where

$$\alpha = \frac{N}{N(p-2)+sp}, \quad \beta = \frac{1}{N(p-2)+sp}$$

for positive, continuous, radially symmetric $(r = |x|t^{-\beta})$ *and decreasing function with the property that*

$$F(r) \approx r^{-(N+sp)} \text{ as } r \longrightarrow \infty.$$

Here, by $f(r) \approx g(r)$, *we mean that* $0 < c_1 \leqslant f(r)g(r) \leqslant c_2$, *for some positive constants* c_1, c_2 *depending on* N, s, p.

Barros-Neto and Gelfand [7] constructed fundamental solutions for the following Tricomi operators:

$$yu_{xx} + u_{yy} \text{ in } \mathbb{R}^2. \tag{1.29}$$

It can be viewed as a prototype of mixed elliptic-hyperbolic operators. Consider the sets D_+ and D_- in $\mathbb{R}^N$ defined as follows:

$$D_\pm = \left\{(x,y) \in \mathbb{R}^2 : \pm(9x^2+4y^3) > 0\right\}.$$

The fundamental solutions $F_\pm$ of the Tricomi operators with pole at a variable point $(a,0)$ on the x-axis are given by

$$F_\pm := C_\pm E_\pm(x,y), \tag{1.30}$$

where

$$E_+(x,y) = \begin{cases} (9x^2+4y^3)^{-\frac{1}{6}} & \text{in } D_+, \\ 0 & \text{otherwise}, \end{cases}$$

$$E_-(x,y) = \begin{cases} |9x^2+4y^3|^{-\frac{1}{6}} & \text{in } D_-, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$C_+ = -\frac{\Gamma(\frac{1}{6})}{2^{\frac{2}{3}} 3\pi^{\frac{1}{2}} \Gamma(\frac{2}{3})},$$
$$C_- = \frac{3\Gamma(\frac{4}{3})}{2^{\frac{2}{3}} \pi^{\frac{1}{2}} \Gamma(\frac{5}{6})}.$$

These solutions satisfy the following equation:

$$yu_{xx} + u_{yy} = \delta(x-a, y), \tag{1.31}$$

where $\delta(x-a,y)$ is the Dirac measure concentrated at $(a,0)$. Further, the same authors [8] calculated fundamental solutions for the Tricomi operator, relative to an arbitrary point, say (a,b) in the plane. Exploiting the invariance of the Tricomi operator under the translations parallel to x-axis, solving (1.31) is equivalent to solving

$$yu_{xx}+u_{yy}=\delta(x,y-b). \tag{1.32}$$

Here, $b\in\mathbb{R}$ is arbitrary and $\delta(x,y-b)$ is the Dirac measure concentrated at $(0,b)$. The authors showed the existence of four fundamental solutions $\{E_i\}_{i=1}^4$, supported in four disjoint regions, say $\{D_i\}_{i=1}^4$. For the details, we refer to Figure 2 [8]. Consider a function

$$E(a,b;a_0,b_0)=(a+a_0)^{-\frac{1}{6}}(a_0-b)^{-\frac{1}{6}}F\left(\frac{1}{6},\frac{1}{6};1;\xi\right),$$

where

$$\xi=\frac{(a-a_0)(b+a_0)}{(a+a_0)(b-a_0)}.$$

The fundamental solutions for the Tricomi operator relative to the point $(0,b)$ are given as follows:

$$E_i(x,y;0,b)=\begin{cases}\frac{1}{2^{\frac{1}{3}}}E(x,y;0,b) & \text{in } D_i,\\ 0 & \text{otherwise,}\end{cases}$$

for $i=1,2$, and

$$E_i(x,y;0,b)=\begin{cases}-\frac{1}{2^{\frac{1}{3}}}E(x,y;0,b) & \text{in } D_i,\\ 0 & \text{otherwise,}\end{cases}$$

for $i=3,4$.

The authors further extended the results of [7, 8] in [9]. They constructed the fundamental solutions for the Tricomi operator relative to any point in the elliptic, parabolic or hyperbolic region of the operator. As mentioned above, the translation invariance of the operator along the x-axis allows us to consider the case when $(x,y)=(0,b)$, where $b\in\mathbb{R}$ is an arbitrary point. Consider the three cases: (i) $b>0$, (ii) $b=0$, and (iii) $b<0$.

Case (i): $b>0$. In this case, the fundamental solution is given by

$$F(x,y;0,b):=\frac{(-v)^{-\frac{1}{6}}}{2^{\frac{1}{3}}}F\left(\frac{1}{6},\frac{1}{6},1;\frac{u}{v}\right),$$

where

$$u(x,y)=\begin{cases}9(x^2+a^2)+4y^3+12ay^{\frac{3}{2}} & \text{for } y\geqslant 0,\\ 9(x^2+a^2)+4y^3-12\imath a(-y)^{\frac{3}{2}} & \text{otherwise,}\end{cases}$$

and

$$\nu(x,y)=\begin{cases}9(x^2+a^2)+4y^3-12ay^{\frac{3}{2}} & \text{for } y\geqslant 0,\\ 9(x^2+a^2)+4y^3+12\imath a(-y)^{\frac{3}{2}} & \text{otherwise,}\end{cases}$$

where $a=\frac{2b^{\frac{3}{2}}}{3}$.

Case (ii): $b=0$. In this case, the fundamental solutions are given by (1.30).

Case (iii): $b<0$. The fundamental solution in this case is given by

$$F(x,y;0,b):=\frac{(-\nu)^{-\frac{1}{6}}}{2^{\frac{1}{3}}}F\left(\frac{1}{6},\frac{1}{6},1;\frac{\mu}{\nu}\right),$$

where

$$\mu(x,y)=\begin{cases}9(x^2-a^2)+4y^3+12\imath ay^{\frac{3}{2}} & \text{for } y\geqslant 0,\\ 9(x^2-a^2)+4y^3+12\imath a(-y)^{\frac{3}{2}} & \text{otherwise,}\end{cases}$$

and

$$\nu(x,y)=\begin{cases}9(x^2-a^2)+4y^3-12\imath ay^{\frac{3}{2}} & \text{for } y\geqslant 0,\\ 9(x^2-a^2)+4y^3-12\imath a(-y)^{\frac{3}{2}} & \text{otherwise,}\end{cases}$$

where $a=\frac{2(-b)^{\frac{3}{2}}}{3}$. Using a method of Delache and Leray [23], Barros-Neto and Cardoso [5] determined the fundamental solutions for a generalized Tricomi operator

$$\mathscr{T}:=y\Delta_x+\frac{\partial^2}{\partial y^2}\ \text{in}\ \mathbb{R}^{N+1}, \tag{1.33}$$

relative to an arbitrary point $(0,b)$, $b<0$, in $\mathbb{R}^{N+1}$. We recall that the case $N=1$ was studied in [7], see (1.30). Consider the case when N is even and look at the region

$$D^N_{b,-}:=\left\{(x,y)\in\mathbb{R}^{N+1}:\ 9(|x|^2-a^2)+12a(-y)^{\frac{3}{2}}+4y^3<0, y<b\right\}.$$

The Fundamental solution of (1.33) with support in the closure of $D^N_{b,-}$ is given by

$$E_-(x,y;0,b)=\begin{cases}\frac{\pi^{\frac{1}{2}-\frac{N}{2}}}{2^{\frac{1}{3}}3^{1-N}\Gamma\left(\frac{3}{2}-\frac{N}{2}\right)}(-u)^{\frac{1}{2}-\frac{N}{2}}(-\nu)^{-\frac{1}{6}}F\left(\frac{2}{3}-\frac{N}{2},\frac{1}{6},\frac{3}{2}-\frac{N}{2};\frac{u}{\nu}\right) & \text{in } D^N_{b,-},\\ 0 & \text{otherwise.}\end{cases} \tag{1.34}$$

Next, consider the case when $N > 1$ is odd. The fundamental solution of (1.33) supported in the closure of $D^N_{b,-}$ is given by

$$E_-(x,y;0,b) = A_m \sum_{j=0}^{m-1} (-1)^j c_j \left(\frac{v-u}{9}\right)^{-j-\frac{1}{6}} \delta^{(m-j-1)}(u(.)) + (-1)^m A_m c_m \left(-\frac{v}{9}\right)^{-m-\frac{1}{6}} F\left(\frac{1}{6}, m+\frac{1}{6}, m+1, \frac{u}{v}\right) \mathbb{1}_{D^N_{b,-}(x,y)}, \tag{1.35}$$

where $\mathbb{1}_{D^N_{b,-}}$ is the indicator function of the set $D^N_{b,-}$. Further, the same authors [6] calculated fundamental solutions for a class of operators given by

$$\frac{1}{2}\Delta_{x,s} + \frac{\alpha}{s}\frac{\partial}{\partial s}, \alpha \in \mathbb{C} \setminus \{0\}, \tag{1.36}$$

where

$$\Delta_{x,s} = \frac{\partial^2}{\partial s^2} - \sum_{i=1}^{N} \frac{\partial^2}{\partial {x_i}^2}.$$

In case when N is even, the fundamental solution of (1.36) with pole at $(0, s_0)$ is given as

$$F_\alpha^{\text{even}}(x,s;0,s_0) = \frac{2r^{1-N}}{(1-N)\omega_{N+1}} \left(\frac{s_0}{s}\right)^\alpha F\left(\alpha, 1-\alpha, \frac{3}{2} - \frac{N}{2}; \frac{-r^2}{4s_0 s}\right), \tag{1.37}$$

where

$$F\left(\alpha, 1-\alpha, \frac{3}{2} - \frac{N}{2}; \frac{-r^2}{4s_0 s}\right) = \sum_{i=0}^{\infty} \frac{(\alpha)_i (1-\alpha)_i}{i! \left(\frac{3}{2} - \frac{N}{2}\right)_i} \left(\frac{-r^2}{4s_0 s}\right)^i$$

for $r = \left(\sum_{i=1}^N x_i^2 + (s-s_0)^2\right)^{\frac{1}{2}}$ and $\omega_{N+1} = \frac{2\pi^{\frac{N+1}{2}}}{\Gamma\left(\frac{N+1}{2}\right)}$. Here, $(a)_i$ denotes the general Pochhammer symbol defined as $(a)_i := \frac{\Gamma(a+i)}{\Gamma(a)}$. For the details, see Theorem 2.1 [6]. Further, when N is odd, say, $N = 2M+1$, for some $M > 0$. The fundamental solution of (1.36) is given as

$$F_\alpha^{\text{odd}}(x,s;0,s_0) = -s_0^{2\alpha} \frac{\Gamma\left(\alpha + \frac{N}{2} - \frac{1}{2}\right)}{\pi^{\frac{N}{2}} \Gamma\left(\alpha + \frac{1}{2}\right)} r^{1-N-2\alpha} F\left(\alpha, \alpha + \frac{N}{2} - \frac{1}{2}, 2\alpha; \frac{-4s_0 s}{r^2}\right), \tag{1.38}$$

for $2\alpha \neq 0, -1, -2, \ldots$. Authors used the Delache and Leray's method [23] to get these explicit representation of fundamental solutions. We mention that in [23], authors gave fundamental solutions to the operators of the form

$$\frac{1}{2}\Box_{x,s} + \frac{\alpha}{s}\frac{\partial}{\partial s}, \quad \alpha \in \mathbb{C} \setminus \{0\}, \tag{1.39}$$

where

$$\Box_{x,s}=\frac{\partial^2}{\partial t^2}-\Delta_{x,s}.$$

Hasanov and Karimov [42] constructed the fundamental solutions in terms of Lauricella's hypergeometric functions of three variables for the following equation:

$$u_{xx}+u_{yy}+u_{zz}+\frac{2\alpha}{x}u_x+\frac{2\beta}{y}u_y+\frac{2\gamma}{z}u_z=0 \tag{1.40}$$

in $\mathbb{R}_3^+ := \{(x,y,z)\colon x>0, y>0, z>0\}$ for constant α,β,γ such that $0<\alpha,\beta,\gamma<\frac{1}{2}$. The fundamental solutions of (1.40) are given by

$$\begin{aligned}
&q_1(x,y,z;x_0,y_0,z_0)\\
&\quad=k_1(r^2)^{-\alpha-\beta-\gamma-\frac{1}{2}}F_A^{(3)}\Big(\alpha+\beta+\gamma+\frac{1}{2};\alpha,\beta,\gamma;2\alpha,2\beta,2\gamma;\xi,\eta,\zeta\Big),\\
&q_2(x,y,z;x_0,y_0,z_0)\\
&\quad=k_2(r^2)^{\alpha-\beta-\gamma-\frac{3}{2}}(xx_0)^{1-2\alpha}\\
&\qquad\times F_A^{(3)}\Big(-\alpha+\beta+\gamma+\frac{3}{2};1-\alpha,\beta,\gamma;2-2\alpha,2\beta,2\gamma;\xi,\eta,\zeta\Big),\\
&q_3(x,y,z;x_0,y_0,z_0)\\
&\quad=k_3(r^2)^{-\alpha+\beta-\gamma-\frac{3}{2}}(yy_0)^{1-2\beta}\\
&\qquad\times F_A^{(3)}\Big(\alpha-\beta+\gamma+\frac{3}{2};\alpha,1-\beta,\gamma;2\alpha,2-2\beta,2\gamma;\xi,\eta,\zeta\Big),\\
&q_4(x,y,z;x_0,y_0,z_0)\\
&\quad=k_4(r^2)^{-\alpha-\beta+\gamma-\frac{3}{2}}(zz_0)^{1-2\gamma}\\
&\qquad\times F_A^{(3)}\Big(\alpha+\beta-\gamma+\frac{3}{2};\alpha,\beta,1-\gamma;2\alpha,2\beta,2-2\gamma;\xi,\eta,\zeta\Big),\\
&q_5(x,y,z;x_0,y_0,z_0)\\
&\quad=k_5(r^2)^{\alpha+\beta-\gamma-\frac{5}{2}}(xx_0)^{1-2\alpha}(yy_0)^{1-2\beta}\\
&\qquad\times F_A^{(3)}\Big(-\alpha-\beta+\gamma+\frac{5}{2};1-\alpha,1-\beta,\gamma;2-2\alpha,2-2\beta,2\gamma;\xi,\eta,\zeta\Big),\\
&q_6(x,y,z;x_0,y_0,z_0)\\
&\quad=k_6(r^2)^{\alpha-\beta+\gamma-\frac{5}{2}}(xx_0)^{1-2\alpha}(zz_0)^{1-2\gamma}\\
&\qquad\times F_A^{(3)}\Big(-\alpha+\beta-\gamma+\frac{5}{2};1-\alpha,\beta,1-\gamma;2-2\alpha,2\beta,2-2\gamma;\xi,\eta,\zeta\Big),\\
&q_7(x,y,z;x_0,y_0,z_0)\\
&\quad=k_7(r^2)^{-\alpha+\beta+\gamma-\frac{5}{2}}(yy_0)^{1-2\beta}(zz_0)^{1-2\gamma}\\
&\qquad\times F_A^{(3)}\Big(\alpha-\beta-\gamma+\frac{5}{2};\alpha,1-\beta,1-\gamma;2\alpha,2-2\beta,2-2\gamma;\xi,\eta,\zeta\Big),
\end{aligned}$$

$$
\begin{aligned}
& q_8(x,y,z;x_0,y_0,z_0) \\
&\quad = k_8(r^2)^{\alpha+\beta+\gamma-\frac{7}{2}}(xx_0)^{1-2\alpha}(yy_0)^{1-2\beta}(zz_0)^{1-2\gamma} \\
&\qquad \times F_A^{(3)}\Big(-\alpha-\beta-\gamma+\frac{7}{2};1-\alpha,1-\beta,1-\gamma;2-2\alpha,2-2\beta,2-2\gamma;\xi,\eta,\zeta\Big),
\end{aligned}
$$

where $k_i,\, i=1,2,\ldots,8$ are constants, $r^2=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2$, and

$$
F_A^{(3)}(a;b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)=\sum_{i,j,k=0}^{\infty}\frac{(a)_{i+j+k}(b_1)_i(b_2)_j(b_3)_k}{(c_1)_i(c_2)_j(c_3)_k i!\,j!\,k!}x^iy^jz^k,
$$
$$
|x|+|y|+|z|<1.
$$

Yagdijan [57] considered a generalized Tricomi operator, also known as Gellerstedt operator,

$$
\mathscr{T}u := u_{tt}-t^m\Delta_x u, \tag{1.41}
$$

for $m\in\mathbb{N},\ x\in\mathbb{R}^N,\ t\in\mathbb{R}$. Let $m=2k$, and $N=1$, the fundamental solution supported in the *forward cone*,

$$
D_1(x_0,t_0):=\left\{(x,t)\in\mathbb{R}^{N+1},|x-x_0|<\frac{1}{k+1}\left(t^{k+1}-t_0^{k+1}\right)\right\},
$$

is given by

$$
E_1(x,t;0,t_0)=\begin{cases} c_kE(x,t;0,t_0) & \text{in } D_1(0,t_0) \\ 0 & \text{otherwise,}\end{cases}
$$

relative to the point $(0,t_0)$, where $c_k=(k+1)^{-\frac{k}{k+1}}2^{-\frac{1}{k+1}}$ and

$$
E(x,t;0,t_0):=(x+\phi(t)+\phi(t_0))^{-\gamma}(-x+\phi(t_0)+\phi(t))^{-\gamma}F(\gamma,\gamma;1;\zeta).
$$

Here $F(\gamma,\gamma;1;\zeta)$ is the hypergeometric function and

$$
\zeta=\frac{(x+\phi(t)-\phi(t_0))(x-\phi(t)+\phi(t_0))}{(x+\phi(t)+\phi(t_0))(x-\phi(t)-\phi(t_0))},\quad \phi(t):=\frac{t^{k+1}}{k+1},\gamma:=\frac{k}{2}\phi(1).
$$

Similarly, when $N>1$ is odd or even, one may see the fundamental solutions to (1.41) in [57].

Garipov and Mavlyaviev [35] gave the fundamental solution for axisymmetric Helmholtz equation:

$$
L_\lambda(u):=\sum_{i=1}^{N-1}\frac{\partial^2u}{\partial x_i^2}+\frac{\partial^2u}{\partial x_N^2}+\frac{k}{x_N}\frac{\partial u}{\partial x_N}+\lambda^2u=0
$$
$$
\text{in } \mathbb{R}_+^N=\Big\{(x_1,x_2\ldots,x_N):\, x_N>0,N>2\Big\},
$$

where $k>0$. The fundamental solutions are given as follows:

$$q_1(M,M_0)=C_1r^{-(k+N-2)}H_3\left(\frac{k+N-2}{2},\frac{k}{2};k;\frac{-4x_Nx_N^{(0)}}{r^2},\frac{\lambda^2r^2}{4}\right),$$
$$q_2(M,M_0)=C_2r^{-(k+N-2)}\left(\frac{-4x_Nx_N^{(0)}}{r^2}\right)^{1-k}H_3\left(\frac{N}{2}-\frac{k}{2},1-\frac{k}{2};2-k;\frac{-4x_Nx_N^{(0)}}{r^2},\frac{\lambda^2r^2}{4}\right),$$

where $r^2=\sum_{i=1}^{N}\left(x_i-x_i^{(0)}\right)^2$ and H_3 is the confluent Horn-Kummer function given by

$$H_3(a,b;c;\alpha,\beta)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a)_{i-j}(b)_i}{(c)_i}\frac{\alpha^i}{i!}\frac{\beta^j}{j!}. \tag{1.42}$$

The solutions have power-singularity r^{2-N} as $r\longrightarrow 0$.

Further, the same authors obtained the fundamental solutions for two multidimensional elliptic equations in [36]. In particular, they considered the following equations:

$$L_\lambda u:=x_N^m\sum_{i=1}^{N}\left(\frac{\partial^2u}{\partial x_i^2}+\lambda^2u\right)+\frac{\partial^2u}{\partial x_N^2}=0 \text{ for } m>0,\ N>2, \tag{1.43}$$

and

$$T_\lambda u:=e^{x_N}\sum_{i=1}^{N}\left(\frac{\partial^2u}{\partial x_i^2}+\lambda^2u\right)+\frac{\partial^2u}{\partial x_N^2}=0 \text{ for } N>2. \tag{1.44}$$

Fundamental solutions for (1.43) are given as

$$q_{\lambda_1}(M,M_0)=C_3r^{-(\mu+N-2)}H_3\left(\frac{\mu+N-2}{2},\frac{\mu}{2};\mu;\frac{r^2-r_1^2}{\rho^2},\frac{\lambda^2r^2}{4}\right),$$

and

$$q_{\lambda_2}(M,M_0)=C_4r^{-(\mu+N-2)}\left(\frac{r^2-r_1^2}{r^2}\right)^{1-\mu}H_3\left(\frac{N}{2}-\frac{\mu}{2},1-\frac{\mu}{2};2-\mu;\frac{r^2-r_1^2}{r^2},\frac{\lambda^2r^2}{4}\right), \tag{1.45}$$

for some positive constants C_3 and C_4 and $\mu=\frac{m}{m+2}$. Here, r and r_1 are as follows:

$$r^2=\sum_{i=1}^{N-1}\left(x_i-x_i^{(0)}\right)^2+\frac{4}{(m+2)^2}\left(x_N^{\frac{m+2}{2}}-\left(x_N^{(0)}\right)^{\frac{m+2}{2}}\right)^2,$$
$$r_1^2=\sum_{i=1}^{N-1}\left(x_i-x_i^{(0)}\right)^2+\frac{4}{(m+2)^2}\left(x_N^{\frac{m+2}{2}}+\left(x_N^{(0)}\right)^{\frac{m+2}{2}}\right)^2.$$

Also, these solutions have power singularity r^{2-N} as $r \longrightarrow 0$. Next, the fundamental solutions of (1.44) are provided by the functions

$$q_{\lambda_1}(M,M_0) = C_5 r_1^{-(\mu-1)} H_3\left(\frac{\mu-1}{2}, \frac{1}{2}; \mu; \frac{r_1^2-r^2}{r_1^2}, \frac{\lambda^2 r_1^2}{4}\right),$$

and

$$\begin{aligned} q_{\lambda_2}(M,M_0) \\ = C_6 r_1^{-(\mu-1)} \left(\frac{r^2-r_1^2}{r^2}\right)^{1-\mu} \Bigg(& H_3\left(\frac{N}{2}-\frac{\mu}{2}, \frac{1}{2}; 1; \frac{r_1^2-r^2}{r_1^2}, \frac{\lambda^2 r_1^2}{4}\right) \log\left(\frac{r_1^2-r^2}{r_1^2}\right) \\ + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\left(\frac{N-1}{2}\right)_{i-j}\left(\frac{1}{2}\right)_i}{(1)_i} \left(\psi\left(\frac{n-1}{2}+i-j\right)+\psi\left(\frac{1}{2}+i\right)-2\psi(1+i)\right) \\ \times \frac{\left(\frac{r_1^2-r^2}{r_1^2}\right)}{i!} \frac{\left(\frac{\lambda^2 r_1^2}{4}\right)}{j!}\Bigg), \end{aligned} \tag{1.46}$$

where H_3 is given by (1.42) and $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$.

Next, consider the following equation studied by Ergashev [26]:

$$\begin{aligned} &\mathscr{L}_{(a)}^N * := \sum_{i=1}^{N} u_{x_i x_i} + \sum_{j=1}^{m} \frac{2a_j}{x_j} u_{x_j} = 0 \\ &\text{in } \mathbb{R}_m^N := \{(x_1,x_2,\ldots,x_N) \colon x_1>0, x_2>0,\ldots,x_m>0\}, \end{aligned} \tag{1.47}$$

where $N \geqslant 2$ is the Euclidean dimension and m is the number of singular coefficients of (1.47) such that $0<m\leqslant N$ and $a_j \in \mathbb{R}$ with $0<2a_j<1$, $j=1,\ldots,m$, $(a)=(a_1,a_2,\ldots,a_m)$. Let $x:=(x_1,x_2,\ldots,x_N)$ be any point and $\xi:=(\xi_1,\xi_2,\ldots,\xi_N)$ be a fixed point in $\mathbb{R}_m^N$. The 2^m fundamental solutions of (1.47) are given by:

(i)

$$F_A^{(m)}\left[\begin{matrix} a, b_1, \ldots, b_m; \\ c_1, \ldots c_m; \end{matrix}\ \sigma\right],$$

(ii)

$$\begin{cases} (x_1\xi_1)^{1-c_1} F_A^{(m)}\left[\begin{matrix} a+1-c_1, b_1+1-c_1, b_2, \ldots, b_m; \\ 2-c_1, c_2, \ldots c_m; \end{matrix}\ \sigma\right], \\ \vdots \\ (x_m\xi_m)^{1-c_m} F_A^{(m)}\left[\begin{matrix} a+1-c_m, b_1, b_2, \ldots, b_{m-1}, b_m+1-c_m; \\ c_1, c_2, \ldots, c_{m-1}, 2-c_m; \end{matrix}\ \sigma\right], \end{cases}$$

(iii)

$$\begin{cases} (x_1\xi_1)^{1-c_1}(x_2\xi_2)^{1-c_2}F_A^{(m)}\left[\begin{matrix} a+2-c_1-c_2,b_1+1-c_1,b_2+1-c_2,b_3,\dots,b_m; \\ 2-c_1,2-c_2,c_3,\dots c_m; \end{matrix} \sigma\right], \\ \vdots \\ (x_1\xi_1)^{1-c_1}(x_m\xi_m)^{1-c_m}F_A^{(m)}\left[\begin{matrix} a+2-c_1-c_m,b_1+1-c_1,b_2,\dots,b_m+1-c_m; \\ 2-c_1,2-c_2,c_3,\dots,c_m; \end{matrix} \sigma\right], \\ (x_2\xi_2)^{1-c_2}(x_3\xi_3)^{1-c_3}F_A^{(m)}\left[\begin{matrix} a+2-c_2-c_3,b_1,b_2+1-c_2,b_3+1-c_3,b_4\dots,b_m; \\ c_1,2-c_2,2-c_3,\dots,c_m; \end{matrix} \sigma\right], \\ \vdots \\ (x_{m-1}\xi_{m-1})^{1-c_{m-1}}(x_m\xi_m)^{1-c_m}F_A^{(m)}\left[\begin{matrix} a+2-c_{m-1}-c_m,b_1,b_2,\dots,b_{m-2},b_{m-1}+1-c_{m-1},b_m+1-c_m; \\ c_1,c_2,\dots,c_{m-2},2-c_{m-1},2-c_m; \end{matrix} \sigma\right] \end{cases}$$

⋮

(n)

$$(x_1\xi_1)^{1-c_1}\dots(x_m\xi_m)^{1-c_m}F_A^{(m)}\left[\begin{matrix} a+m-c_1-c_2,\dots-c_m,b_1+1-c_1,\dots,b_m+1-c_m; \\ 2-c_1,2-c_2,\dots 2-c_m; \end{matrix} \sigma\right],$$

where $a = a_1+\dots a_n - 1 + \frac{N}{2}$, $b_i = a_i$, $c_i = 2a_i$, $1 \leqslant i \leqslant m$. Also, the fundamental solutions have singularity at $r = 0$. For the details, we refer to [26]. Fundamental solutions for the generalized Euler-Poisson-Darboux equation were given by Hasanov et al. [43]:

$$u_{tt} + \frac{2\gamma}{t}u_t = u_{xx} + u_{yy} + \frac{2a}{x}u_x + \frac{2\beta}{y}u_y, \quad x > 0,\ y > 0,\ t > 0, \tag{1.48}$$

are the following:

$$\begin{aligned} &q_1(x,y,t;x_0,y_0,z_0) \\ &\quad = k_1(r^2)^{-\alpha-\beta-\gamma-\frac{1}{2}}F_A\Big(\alpha+\beta+\gamma+\frac{1}{2};\alpha,\beta,\gamma,2\alpha,2\beta,2\gamma;\xi,\eta,\zeta\Big), \\ &q_2(x,y,t;x_0,y_0,z_0) \\ &\quad = k_2(r^2)^{\alpha-\beta-\gamma-\frac{3}{2}}(xx_0)^{1-2\alpha} \\ &\qquad \times F_A^{(3)}\Big(-\alpha+\beta+\gamma+\frac{3}{2};1-\alpha,\beta,\gamma,2-2\alpha,2\beta,2\gamma;\xi,\eta,\zeta\Big), \\ &q_3(x,y,t;x_0,y_0,z_0) \\ &\quad = k_3(r^2)^{-\alpha+\beta-\gamma-\frac{3}{2}}(yy_0)^{1-2\beta} \\ &\qquad \times F_A^{(3)}\Big(\alpha-\beta+\gamma+\frac{3}{2};\alpha,1-\beta,\gamma,2\alpha,2-2\beta,2\gamma;\xi,\eta,\zeta\Big), \\ &q_4(x,y,t;x_0,y_0,z_0) \\ &\quad = k_4(r^2)^{-\alpha-\beta+\gamma-\frac{3}{2}}(tt_0)^{1-2\gamma} \\ &\qquad \times F_A^{(3)}\Big(\alpha+\beta-\gamma+\frac{3}{2};\alpha,\beta,1-\gamma,2\alpha,2\beta,2-2\gamma;\xi,\eta,\zeta\Big), \end{aligned}$$

$$
\begin{aligned}
&q_5(x,y,t;x_0,y_0,z_0)\\
&\quad = k_5(r^2)^{\alpha+\beta-\gamma-\frac{5}{2}}(xx_0)^{1-2\alpha}(yy_0)^{1-2\beta}\\
&\qquad \times F_A^{(3)}\Big(-\alpha-\beta+\gamma+\frac{5}{2};1-\alpha,1-\beta,\gamma,2-2\alpha,2-2\beta,2\gamma;\xi,\eta,\zeta\Big),\\
&q_6(x,y,t;x_0,y_0,z_0)\\
&\quad = k_6(r^2)^{\alpha-\beta+\gamma-\frac{5}{2}}(xx_0)^{1-2\alpha}(tt_0)^{1-2\gamma}\\
&\qquad \times F_A^{(3)}\Big(-\alpha+\beta-\gamma+\frac{5}{2};1-\alpha,\beta,1-\gamma,2-2\alpha,2\beta,2-2\gamma;\xi,\eta,\zeta\Big),\\
&q_7(x,y,t;x_0,y_0,z_0)\\
&\quad = k_7(r^2)^{-\alpha+\beta+\gamma-\frac{5}{2}}(yy_0)^{1-2\beta}(tt_0)^{1-2\gamma}\\
&\qquad \times F_A^{(3)}\Big(\alpha-\beta-\gamma+\frac{5}{2};\alpha,1-\beta,1-\gamma,2\alpha,2-2\beta,2-2\gamma;\xi,\eta,\zeta\Big),\\
&q_8(x,y,t;x_0,y_0,z_0)\\
&\quad = k_8(r^2)^{\alpha+\beta+\gamma-\frac{7}{2}}(xx_0)^{1-2\alpha}(yy_0)^{1-2\beta}(tt_0)^{1-2\gamma}\\
&\qquad \times F_A^{(3)}\Big(-\alpha-\beta-\gamma+\frac{7}{2};1-\alpha,1-\beta,1-\gamma;2-2\alpha,2-2\beta,2-2\gamma;\xi,\eta,\zeta\Big),
\end{aligned}
$$

where $k_1,k_2,\dots k_8$ are constants and $r^2=(x-x_0)^2+(y-y_0)^2-(t-t_0)^2$. For the details, we refer to Section 3 [43].

Ergashev [27] determined the fundamental solutions for the equation

$$
\sum_{i=1}^{N} u_{x_i x_i} + \frac{2\alpha_1}{x_1}u_{x_1} + \frac{2\alpha_2}{x_2}u_{x_2} + \frac{2\alpha_3}{x_3}u_{x_3} - \lambda^2 u = 0 \tag{1.49}
$$

in $\mathbb{R}_N^{3+} := \big\{(x_1,x_2,\dots,x_N)\colon x_1>0, x_2>0, x_3>0\big\}$, where N is the dimension of the Euclidean space, $N \geqslant 3$, $\alpha_i \in \mathbb{R}$ are constants such that $0<\alpha_i<\frac{1}{2}$, for $i=1,2,3$. $\alpha = \alpha_1+\alpha_1+\alpha_3-1$. Also, $\sigma_i = \frac{4x_k x_{0k}}{r^2}$, $k=1,2,3$, $\sigma_4 = -\frac{1}{4}\lambda^2 r^2$ and $\sigma=(\sigma_1,\sigma_2,\sigma_3,\sigma_4)$ for some real or pure imaginary constant λ. Let $r^2=\sum_{i=1}^{N}(x_i-x_{01})^2$. The fundamental solutions of (1.49) are given as follows:

$$
\begin{aligned}
q_1(x,x_0) &= k_1(r^2)^{-\alpha}H_{4,3}^0(\alpha,\alpha_1,\alpha_2,\alpha_3;2\alpha_1,2\alpha_2,2\alpha_3;\sigma),\\
q_2(x,x_0) &= k_2(r^2)^{2\alpha_1-\alpha-1}(x_1x_{01})^{1-2\alpha_1}\\
&\qquad \times H_{4,3}^0(1+\alpha-2\alpha_1,1-\alpha_1,\alpha_2,\alpha_3;2-2\alpha_1,2\alpha_2,2\alpha_3;\sigma),\\
q_3(x,x_0) &= k_3(r^2)^{2\alpha_2-\alpha-1}(x_2x_{02})^{1-2\alpha_2}\\
&\qquad \times H_{4,3}^0(1+\alpha-2\alpha_2,\alpha_1,1-\alpha_2,\alpha_3;2\alpha_1,2-2\alpha_2,2\alpha_3;\sigma),\\
q_4(x,x_0) &= k_4(r^2)^{2\alpha_3-\alpha-1}(x_3x_{03})^{1-2\alpha_3}\\
&\qquad \times H_{4,3}^0(1+\alpha-2\alpha_3,\alpha_1,\alpha_2,1-\alpha_3;2\alpha_1,2\alpha_2,2-2\alpha_3;\sigma),\\
q_5(x,x_0) &= k_5(r^2)^{2\alpha_1+2\alpha_2-\alpha-2}(x_1x_{01})^{1-2\alpha_1}(x_2x_{02})^{1-2\alpha_2}\\
&\qquad \times H_{4,3}^0(2+\alpha-2\alpha_1-2\alpha_2,1-\alpha_1,1-\alpha_2,\alpha_3;2-2\alpha_1,2-2\alpha_2,2\alpha_3;\sigma),
\end{aligned}
$$

$$q_6(x,x_0) = k_6(r^2)^{2\alpha_1+2\alpha_3-\alpha-2}(x_1x_{01})^{1-2\alpha_1}(x_3x_{03})^{1-2\alpha_3}$$
$$\times H^0_{4,3}(2+\alpha-2\alpha_1-2\alpha_3, 1-\alpha_1, \alpha_2, 1-\alpha_3; 2-2\alpha_1, 2\alpha_2, 2-2\alpha_3; \sigma),$$
$$q_7(x,x_0) = k_7(r^2)^{2\alpha_2+2\alpha_3-\alpha-2}(x_2x_{02})^{1-2\alpha_2}(x_3x_{03})^{1-2\alpha_3}$$
$$\times H^0_{4,3}(2+\alpha-2\alpha_2-2\alpha_3, \alpha_1, 1-\alpha_2, 1-\alpha_3; 2\alpha_1, 2-2\alpha_2, 2-2\alpha_3; \sigma),$$
$$q_8(x,x_0) = k_8(r^2)^{2\alpha_1+2\alpha_2+2\alpha_3-\alpha-2}(x_1x_{01})^{1-2\alpha_1}(x_2x_{02})^{1-2\alpha_2}(x_3x_{03})^{1-2\alpha_3}$$
$$\times H^0_{4,3}(3+\alpha-2\alpha_1-2\alpha_2-2\alpha_3, 1-\alpha_1, 1-\alpha_2, 1-\alpha_3;$$
$$2-2\alpha_1, 2-2\alpha_2, 2-2\alpha_3; \sigma),$$

where $k_i = 1,2,\ldots,8$ are constants and $H^0_{4,3}$ is a confluent hypergeometric function of four variables given by

$$H^0_{4,3}(a,b_1,b_2,b_3;d_1,d_2,d_3;x,y,z,t) = \sum_{m,n,k,l=0}^{\infty} (a)_{m+n+k-l} \frac{(b_1)_m(b_2)_n(b_3)_k}{(d_1)_m(d_2)_n(d_3)_k} \frac{x^m}{m!} \frac{y^m}{n!} \frac{z^k}{k!} \frac{t^l}{l!},$$

for $|x|+|y|+|z|<1$.

Recently, Hasanov et al. [44] studied the following generalized Gellerstedt equation:

$$y^m z^k t^l u_{xx} + x^n z^k t^l u_{yy} + x^n y^m t^l u_{zz} + x^n y^m z^k u_{tt} = 0 \tag{1.50}$$

in $\mathbb{R}^4_+ = \{(x,y,z,t) \colon x>0, y>0, z>0, t>0\}$ for positive constants k,l,m,n. Consider

$$P(r) = r^{-\alpha-\beta-\gamma-\delta-1},$$

where $\alpha = \frac{n}{2(n+2)}$, $\beta = \frac{m}{2(m+2)}$, $\gamma = \frac{k}{2(k+2)}$, $\delta = \frac{l}{2(l+2)}$, and

$$r^2 = \left(\frac{2}{n+2}x^{\frac{n+2}{2}} - \frac{2}{n+2}x_0^{\frac{n+2}{2}}\right)^2 + \left(\frac{2}{m+2}y^{\frac{m+2}{2}} - \frac{2}{m+2}y_0^{\frac{m+2}{2}}\right)^2$$
$$+ \left(\frac{2}{k+2}z^{\frac{k+2}{2}} - \frac{2}{k+2}z_0^{\frac{k+2}{2}}\right)^2 + \left(\frac{2}{l+2}t^{\frac{l+2}{2}} - \frac{2}{l+2}t_0^{\frac{l+2}{2}}\right)^2.$$

Let

$$F^{(4)}_A(a;b_1,b_2,b_3,b_4;c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$$
$$= \sum_{m,n,p,q}^{\infty} \frac{(a)_{m+n+p+q}(b_1)_m(b_2)_n(b_3)_p(b_4)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q m!n!p!q!} x^m y^n z^p t^q$$

for $|x|+|y|+|z|+|t|<1$. The fundamental solutions of (1.50) are the following:

$$g_1(x,y,z,t;x_0,y_0,z_0,t_0)$$
$$= k_1 P \times F^{(4)}_A(\alpha+\beta+\gamma+\delta+1; \alpha,\beta,\gamma,\delta; 2\alpha,2\beta,2\gamma,2\delta; \xi,\eta,\tau,\zeta),$$
$$g_2(x,y,z,t;x_0,y_0,z_0,t_0)$$
$$= k_2 P \xi^{1-2\alpha} \times F^{(4)}_A(2-\alpha+\beta+\gamma+\delta; 1-\alpha,\beta,\gamma,\delta; 2-2\alpha,2\beta,2\gamma,2\delta; \xi,\eta,\tau,\zeta),$$

$$
\begin{aligned}
&g_3(x,y,z,t;x_0,y_0,z_0,t_0)\\
&\ = k_3P\eta^{1-2\beta}F_A^{(4)}(2+\alpha-\beta+\gamma+\delta;\alpha,1-\beta,\gamma,\delta;2\alpha,2-2\beta,2\gamma,2\delta;\xi,\eta,\tau,\zeta),\\
&g_4(x,y,z,t;x_0,y_0,z_0,t_0)\\
&\ = k_4P\eta^{1-2\gamma}F_A^{(4)}(2+\alpha+\beta-\gamma+\delta;\alpha,\beta,1-\gamma,\delta;2\alpha,2\beta,2-2\gamma,2\delta;\xi,\eta,\tau,\zeta)\\
&g_5(x,y,z,t;x_0,y_0,z_0,t_0)\\
&\ = k_5P\tau^{1-2\delta}\\
&\quad\times F_A^{(4)}(2+\alpha+\beta+\gamma-\delta;\alpha,\beta,\gamma,1-\delta;2\alpha,2\beta,2\gamma,2-2\delta;\xi^m,\eta^n,\tau^p,\zeta^q),\\
&g_6(x,y,z,t;x_0,y_0,z_0,t_0)\\
&\ = k_6P\xi^{1-2\alpha}\eta^{1-2\beta}\\
&\quad\times F_A^{(4)}(3-\alpha-\beta+\gamma+\delta;1-\alpha,1-\beta,\gamma,\delta;2-2\alpha,2-2\beta,2\gamma,2\delta;\xi,\eta,\tau,\zeta),\\
&g_7(x,y,z,t;x_0,y_0,z_0,t_0)\\
&\ = k_7P\xi^{1-2\alpha}\zeta^{1-2\gamma}\\
&\quad\times F_A^{(4)}(3-\alpha+\beta-\gamma+\delta;1-\alpha,\beta,1-\gamma,\delta;2-2\alpha,2\beta,2-2\gamma,2\delta;\xi,\eta,\tau,\zeta),\\
&g_8(x,y,z,t;x_0,y_0,z_0,t_0)\\
&\ = k_8P\xi^{1-2\alpha}\tau^{1-2\delta}\\
&\quad\times F_A^{(4)}(3-\alpha+\beta+\gamma-\delta;1-\alpha,\beta,\gamma,1-\delta;2-2\alpha,2\beta,2\gamma,2-2\delta;\xi,\eta,\tau,\zeta),\\
&g_9(x,y,z,t;x_0,y_0,z_0,t_0)\\
&\ = k_9P\eta^{1-2\beta}\zeta^{1-2\gamma}\\
&\quad\times F_A^{(4)}(3+\alpha-\beta-\gamma+\delta;\alpha,1-\beta,1-\gamma,\delta;2\alpha,2-2\beta,2-2\gamma,2\delta;\xi,\eta,\tau,\zeta),\\
&g_{10}(x,y,z,t;x_0,y_0,z_0,t_0)\\
&\ = k_{10}P\eta^{1-2\beta}\zeta^{1-2\delta}\\
&\quad\times F_A^{(4)}(3+\alpha-\beta+\gamma-\delta;\alpha,1-\beta,\gamma,1-\delta;2\alpha,2-2\beta,2\gamma,2-2\delta;\xi,\eta,\tau,\zeta),\\
&g_{11}(x,y,z,t;x_0,y_0,z_0,t_0)\\
&\ = k_{11}P\tau^{1-2\gamma}\zeta^{1-2\delta}\\
&\quad\times F_A^{(4)}(3+\alpha+\beta-\gamma-\delta;\alpha,\beta,1-\gamma,1-\delta;2\alpha,2\beta,2-2\gamma,2-2\delta;\xi,\eta,\tau,\zeta),\\
&g_{12}(x,y,z,t;x_0,y_0,z_0,t_0)\\
&\ = k_{12}P\xi^{1-2\alpha}\eta^{1-2\beta}\tau^{1-2\gamma}\\
&\quad\times F_A^{(4)}(4-\alpha-\beta-\gamma+\delta;1-\alpha,1-\beta,1-\gamma,\delta;2-2\alpha,2-2\beta,2-2\gamma,2\delta;\xi,\eta,\tau,\zeta),\\
&g_{13}(x,y,z,t;x_0,y_0,z_0,t_0)\\
&\ = k_{13}P\xi^{1-2\alpha}\eta^{1-2\beta}\zeta^{1-2\delta}\\
&\quad\times F_A^{(4)}(4-\alpha-\beta+\gamma-\delta;1-\alpha,1-\beta,\gamma,1-\delta;2-2\alpha,2-2\beta,2\gamma,2-2\delta;\xi,\eta,\tau,\zeta),
\end{aligned}
$$

$$g_{14}(x,y,z,t;x_0,y_0,z_0,t_0)$$
$$= k_{14}P\xi^{1-2\alpha}\eta^{1-2\beta}\tau^{1-2\gamma}$$
$$\times F_A^{(4)}(4-\alpha+\beta-\gamma-\delta;1-\alpha,1-\beta,1-\gamma,\delta;2-2\alpha,2-2\beta,2-2\gamma,2\delta;\xi,\eta,\tau,\zeta),$$
$$g_{15}(x,y,z,t;x_0,y_0,z_0,t_0)$$
$$= k_{15}P\eta^{1-2\beta}\tau^{1-2\gamma}\zeta^{1-2\delta}$$
$$\times F_A^{(4)}(4+\alpha-\beta-\gamma-\delta;\alpha,1-\beta,1-\gamma,1-\delta;2\alpha,2-2\beta,2-2\gamma,2-2\delta;\xi,\eta,\tau,\zeta),$$
$$g_{16}(x,y,z,t;x_0,y_0,z_0,t_0)$$
$$= k_{16}P\xi^{1-2\alpha}\eta^{1-2\beta}\tau^{1-2\gamma}\zeta^{1-2\delta}$$
$$\times F_A^{(4)}(5-\alpha-\beta-\gamma-\delta;1-\alpha,1-\beta,1-\gamma,1-\delta;2-2\alpha,2-2\beta,2-2\gamma,2-2\delta;\xi,\eta,\tau,\zeta),$$

where $k_i, i = 1,2,\ldots,16$ are constants and ξ,η,τ,ζ defined using r,l,m,n,k. For the details, we refer to Theorem 4.1 [44].

2. Fundamental solution to fully nonlinear parabolic PDE

In this section we will state a result which helps us in computing the Pucci's extremal operator in the case of radial solutions. For the details, see [21, 31]. Since this is short and interesting, so we are presenting it here for the sake of completeness.

LEMMA 1. *Let $\psi\colon (0,\infty) \longrightarrow \mathbb{R}$ ba a C^2 function. For $x \in \mathbb{R}^N \setminus \{0\}$ define $u(x) = \psi(|x|)$, then the eigenvalues of D^2u (the Hessian of u), are $\psi''(|x|)$, which is simple, and $\frac{\psi'(|x|)}{|x|}$ which has multiplicity $N-1$.*

Proof. By an easy computation, we get

$$D^2u(x) = \frac{\psi'(|x|)}{|x|}I + \left(\frac{\psi''(|x|)}{|x|^2} - \frac{\psi'(|x|)}{|x|^3}\right)x\otimes x,$$

where I is an $N\times N$ identity matrix and $x\otimes x$ is an $N\times N$ matrix whose (i,j)-th entries are x_ix_j. Hence we have

$$D^2u(x)\frac{x}{|x|} = \psi''(|x|)\frac{x}{|x|} \quad \text{and} \quad D^2u(x)\xi = \frac{\psi'(|x|)}{|x|}\xi,$$

for every vector $\xi \in \mathbb{R}^N$ such that $\xi.x = 0$ and thus the lemma follows. □

We prove the existence of fundamental solution Φ_h of the fully nonlinear parabolic partial differential equation

$$u_t - F(D^2u) = 0 \text{ in } \mathbb{R}^N \times \mathbb{R} \tag{2.1}$$

for any positively homogeneous uniformly elliptic operator F, where $F\colon S_N \longrightarrow \mathbb{R}$ is a function satisfying the following condition:

(H1) F is convex and positively homogeneous of degree 1, i.e.,

$$\forall M \in S_N \text{ and } t \geqslant 0,\ F(tM) = tF(M).$$

(H2) There exist numbers $0 < \lambda \leqslant \Lambda$ such that

$$\mathcal{M}^-_{\lambda,\Lambda}(M) \leqslant F(M) \leqslant \mathcal{M}^+_{\lambda,\Lambda}(M),\ \forall\, M \in S_N,$$

where $\mathcal{M}^-_{\lambda,\Lambda}$ and $\mathcal{M}^+_{\lambda,\Lambda}$ are Pucci's extremal operators defined as above.

(H3) F is invariant with respect to orthogonal changes of coordinates, i.e,

$$F(Q^t MQ) = F(M), \text{ for every real orthogonal matrix } Q \text{ and } M \in S_N.$$

Felmer and Quaas [29] proved that if F is a function satisfying (H1)–(H3), then

$$F(M) = \mathcal{M}^+_{\lambda,\Lambda}(M),\ \forall\, M \in S_N.$$

We consider the viscosity solution u of (2.1). Since by (H1), $F(M)$ is a convex function in M and by the celebrated work of Krylov [45, 46], $u \in C^{2,\alpha}(\Omega)$, $0 < \alpha < 1$.

PROPOSITION 1. *Let F satisfy* (H1)–(H3). Then the fundamental solution Φ_h of (2.1) is given by

$$\Phi_h(x,t) = \begin{cases} \frac{c_2}{t^{\frac{1}{2}(\gamma+1)}} e^{-\frac{|x|^2}{4\Lambda t}}, & \text{for } x \in \mathbb{R}^N,\ t > 0, \\ 0, \ \text{for } x \in \mathbb{R}^N,\ t < 0, \end{cases} \tag{2.2}$$

where $\gamma = \frac{(N-1)\lambda}{\Lambda}$.

Proof. By (H1)–(H3), from Lemma 2.2 [29], (2.1) converts into

$$u_t - \mathcal{M}^+_{\lambda,\Lambda}(D^2u) = 0 \text{ in } \mathbb{R}^N \times \mathbb{R}, \tag{2.3}$$

where $0 < \lambda \leqslant \Lambda < \infty$. Let us look at the radial solution of (2.3) for $t > 0$, i.e., $u(x,t) = u(|x|,t) = u(r,t)$ for $t > 0$. Now in the light of Lemma 1, (2.3) converts into

$$u_t - \left(\Lambda u_{rr} + \lambda \frac{(N-1)}{r} u_r \right) = 0 \text{ in } (0,\infty) \times (0,\infty). \tag{2.4}$$

Let us seek a solution $u(r,t)$ of (2.4) having the form

$$u(r,t) = \frac{1}{t^\alpha} v\left(\frac{r}{t^\beta}\right),\ r > 0, t > 0,$$

where the constants α, β and the function $v\colon \mathbb{R} \longrightarrow \mathbb{R}$ have to be determined and u, $u_r \longrightarrow 0$ as $r \longrightarrow \infty$ for each fixed $t > 0$. An easy computation yields that

$$\beta v'(s) s t^{-\alpha-1} + \alpha v(s) t^{-\alpha-1} + \Lambda t^{-\alpha-2\beta} v''(s) + \frac{(N-1)\lambda v'(s) t^{-\alpha-\beta}}{r} = 0,$$

where $s = \frac{r}{t^\beta}$. Let us take $\beta = \frac{1}{2}$ in the above equation and an easy simplification gives that

$$v''(s) + v'(s)\left[\frac{(N-1)\lambda}{\Lambda s} + \frac{s}{2\Lambda}\right] + \frac{\alpha}{\Lambda}v(s) = 0. \tag{2.5}$$

Further, (2.5) can be rewritten as

$$v''(s) + v'(s)\left[\frac{\gamma}{s} + \frac{s}{2\Lambda}\right] + \frac{\alpha}{\Lambda}v(s) = 0, \tag{2.6}$$

where $\gamma = \frac{(N-1)\lambda}{\Lambda}$. On multiplying (2.6) by s^γ and choosing $\alpha = \frac{\gamma+1}{2}$ yields that

$$(s^\gamma v'(s))' + \frac{1}{2\Lambda}(s^{\gamma+1}v(s))' = 0.$$

An integration yields that

$$s^\gamma v'(s) + \frac{1}{2\Lambda}s^{\gamma+1}v(s) = c_1,$$

for some constant c_1. Since $v(s), v'(s) \longrightarrow 0$ as $s \longrightarrow \infty$ so this implies that $c_1 = 0$ and again by a simple integration, we have

$$v(s) = c_2 e^{-\frac{s^2}{4\Lambda}},$$

where c_2 is some constant and therefore

$$u(x,t) = u(|x|,t) = \frac{c_2}{t^{\frac{1}{2}(\gamma+1)}} e^{-\frac{|x|^2}{4\Lambda t}}$$

and we call it a fundamental solution of (2.1) and denote it by

$$\Phi_h(x,t) = \begin{cases} \frac{c_2}{t^{\frac{1}{2}(\gamma+1)}} e^{-\frac{|x|^2}{4\Lambda t}}, & \text{for } x \in \mathbb{R}^N,\ t > 0, \\ 0, \quad \text{for } x \in \mathbb{R}^N,\ t < 0. \end{cases}$$

It is easy to observe that Φ_h is singular at the point $(0,0)$. □

REMARK 1. It is an easy observation that the fundamental solution (2.2) of (2.1) is monotone nonincreasing in r for each fixed $t > 0$.

REMARK 2. Let $\lambda = 1 = \Lambda$ in (2.3), then $\mathscr{M}^+_{\lambda,\Lambda}(D^2u) = \Delta u$ and the function given by (2.2) is the fundamental solution of

$$u_t - \Delta u = 0 \text{ in } \mathbb{R}^N \times \mathbb{R}.$$

For the details, see [28].

REMARK 3. It is remarkable that (2.1) has special solutions of the form

$$u(x,t) = u(r,t) = t^{-\alpha} v(t^{-\beta} r),\ r > 0,\ t > 0$$

with the exponent $\beta = \frac{1}{2},\ \alpha = \frac{(N-1)\lambda + \Lambda}{2\Lambda}$.

We mention here that using the arguments similar to 2.3.1 [28], solution to the following initial-value problem:

$$\begin{cases} u_t - \mathscr{M}^+_{\lambda,\Lambda}(D^2 u) = 0 & \text{in } \mathbb{R}^N \times \{t > 0\}, \\ u = g & \text{on } \mathbb{R}^N \times \{t = 0\} \end{cases} \tag{2.7}$$

is given by

$$u(x,t) = \frac{c_2}{t^{\frac{1}{2}(\gamma+1)}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4\Lambda t}} g(y)dy, \ \text{for } x \in \mathbb{R}^N \text{ and } t > 0. \tag{2.8}$$

Further, we study the rate of the error in terms of the representation formula. We prove the following results in the spirit of J. L. Vázquez, see Theorem 4.1 [54].

THEOREM 2. *Let $g \in L^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} g(y)dy = M$ be its mass. Also, assume that*

$$\mathscr{N}(g) := \int_{\mathbb{R}^N} |g(y)y|dy < \infty. \tag{2.9}$$

Then for

$$P(x,t) := \frac{c_2}{t^{\frac{1}{2}(\gamma+1)}} e^{-\frac{|x|^2}{4\Lambda t}} \ \text{and} \ \gamma = \frac{(N-1)\lambda}{\Lambda}, \tag{2.10}$$

we get

$$t^{\frac{1}{2}(\gamma+1)} |u(x,t) - MP(x,t)| \leqslant C\mathscr{N}(g) t^{-\frac{1}{2}} \tag{2.11}$$

and

$$\|u(x,t) - MP(x,t)\|_{L^1(\mathbb{R}^N)} \leqslant C_1 \mathscr{N}(g) t^{-\frac{1}{2}(\gamma+2-N)}, \tag{2.12}$$

where C and C_1 are positive constants.

Proof. Using (2.8), we have that

$$\begin{aligned} u(x,t) - MP(x,t) &= \int_{\mathbb{R}^N} P(x-y,t) g(y)dy - P(x,t) \int_{\mathbb{R}^N} g(y)dy \\ &= \int_{\mathbb{R}^N} \big(P(x-y,t) - P(x,t)\big) g(y)dy \\ &= \int_{\mathbb{R}^N} \left(\int_0^1 P(x-sy,t)ds \right) g(y)dy \end{aligned}$$

$$= \int_{\mathbb{R}^N} \left(\int_0^1 P(x - sy, t) ds \right) g(y) dy$$
$$= \frac{c_2}{t^{\frac{1}{2}(\gamma+1)}} \int_{\mathbb{R}^N} dy \int_0^1 g(y) \left\langle y, \frac{x - sy}{2t} \right\rangle e^{-\frac{|x-sy|^2}{4\Lambda t}} ds$$
$$= \frac{c_2}{t^{\frac{1}{2}(\gamma+2)}} \int_{\mathbb{R}^N} dy \int_0^1 g(y) \left\langle y, \frac{x - sy}{2t^{\frac{1}{2}}} \right\rangle e^{-\frac{|x-sy|^2}{4\Lambda t}} ds. \tag{2.13}$$

Further, using the bound on function

$$f\left(\frac{x - sy}{t^{\frac{1}{2}}}\right) := \frac{x - sy}{t^{\frac{1}{2}}} e^{-\frac{|x-sy|^2}{4\Lambda t}}$$

yields

$$\left|u(x,t) - MP(x,t)\right| \leqslant C \frac{1}{t^{\frac{1}{2}(\gamma+2)}} \int_{\mathbb{R}^N} |g(y)y| dy$$
$$= C \frac{1}{t^{\frac{1}{2}(\gamma+2)}} \mathcal{N}(g),$$

where $C > 0$ is a constant. Now, since $P(x,t)$ is of the order $\frac{1}{t^{\frac{1}{2}(\gamma+1)}}$ in sup norm, thus we have

$$t^{\frac{1}{2}(\gamma+1)} \left|u(x,t) - MP(x,t)\right| \leqslant C t^{-\frac{1}{2}} \mathcal{N}(g),$$

for some constant $C = C(N)$. This yields (2.11). Next, in order to get (2.12), we consider (2.13)

$$u(x,t) - MP(x,t) = \frac{c_2}{t^{\frac{1}{2}(\gamma+1)}} \int_{\mathbb{R}^N} dy \int_0^1 g(y) \left\langle y, \frac{x - sy}{2t} \right\rangle e^{-\frac{|x-sy|^2}{4\Lambda t}} ds.$$

For fixed $t > 0$, integrating with respect to variable x, we get

$$\|u(x,t) - MP(x,t)\|_{L^1(\mathbb{R}^N)} \leqslant \frac{c_2}{t^{\frac{1}{2}(\gamma+1)}} \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} dy \int_0^1 |g(y)y| \frac{|x - sy|}{2t} e^{-\frac{|x-sy|^2}{4\Lambda t}} ds. \tag{2.14}$$

Further, using the change of variable

$$\xi = \frac{x - sy}{t^{\frac{1}{2}}},$$

yields

$$\|u(x,t) - MP(x,t)\|_{L^1(\mathbb{R}^N)} \leqslant c_2 t^{-\frac{1}{2}(\gamma+2-N)} \int_{\mathbb{R}^N} dy \int_0^1 |g(y)y| \left(\int_{\mathbb{R}^N} t^{-\frac{N}{2}} \frac{|\xi|}{2t^{\frac{1}{2}}} e^{-\frac{|\xi|^2}{4\Lambda}} d\xi \right) ds.$$

The last integral is a constant independent of u, we get

$$\|u(x,t) - MP(x,t)\|_{L^1(\mathbb{R}^N)} \leqslant C_1 \mathcal{N}(g) t^{-\frac{1}{2}(\gamma+2-N)},$$

for some positive constant C_1. □

REMARK 4. For the long time asymptotics for (2.7), we refer to Theorem 1.2 [3]. We mention that in particular, in our case $\Phi(x,t)$ appearing in that result is given by $P(x,t)$, see (2.10).

3. Fundamental solutions in Heisenberg group

We review the fundamental solution to some important operators in Heisenberg group $\mathbb{H}^N$. We first recall the briefs about the Heisenberg group $\mathbb{H}^N$. The points in $\mathbb{H}^N$ are denoted by

$$\xi := (z,t) = (x_1,\dots,x_N,y_1,\dots,y_N,t)$$

and the group $\mathbb{H}^N$ is defined as the triplet $\left(\mathbb{R}^{N+1}, o, \{\Phi_\lambda\}\right)$, where the group law o is defined as follows:

$$\begin{aligned}\xi\, o\, \xi' &= \left(x+x', y+y', t+t'+2\langle y,x'\rangle - 2\langle x,y'\rangle\right)\\ &= \left(x_1+x_1',\dots,x_N+x_N', y_1+y_1',\dots,y_N+y_N', t+t'+2\sum_{i=1}^{N}\left(y_i x_i' - x_i y_i'\right)\right).\end{aligned}$$

Here, $\langle .,.\rangle$ denotes the standard inner product in $\mathbb{R}^N$. $(\mathbb{R}^{2N+1}, o)$ is a Lie group with identity element the origin $\mathbf{0}$ and inverse $\xi^{-1} = -\xi$. The dilation group $\{\Phi_\lambda\}_{\lambda>0}$ is given by

$$\Phi(\lambda)\colon \mathbb{R}^{2N+1} \longrightarrow \mathbb{R}^{2N+1}$$

such that

$$\xi \mapsto \Phi_\lambda(\xi) := \left(\lambda x, \lambda y, \lambda^2 t\right).$$

$\mathbb{H}^N$ is also known as Heisenberg-Weyl group in $\mathbb{R}^{2N+1}$. The Jacobian basis of the Heisenberg Lie algebra of $\mathbb{H}^N$ is given by

$$X_i = \partial_{x_i} + 2y_i\partial_t, X_{i+N} = \partial_{y_i} - 2x_i\partial_t,\ 1 \leqslant i \leqslant N,\ T = \partial_t.$$

Given a domain $\Omega \subset \mathbb{H}^N$, for $u \in C^1(\Omega,\mathbb{R})$, the subgradient or the Heisenberg gradient $\nabla_{\mathbb{H}^N} u$ is defined as follows:

$$\nabla_{\mathbb{H}^N} u(\xi) := \left(X_1 u(\xi),\dots,X_N u(\xi), X_{N+1}u(\xi),\dots,X_{2N}u(\xi)\right).$$

Also,

$$D^2_{\mathbb{H}^N,S}u := \begin{bmatrix} X_1X_1u & \cdots & X_NX_1u & X_{N+1}X_1u & \cdots & X_{2N}X_1u\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ X_1X_Nu & \dots & X_NX_Nu & X_{N+1}X_Nu & \dots & X_{2N}X_Nu\\ X_1X_{N+1}u & \dots & X_NX_{N+1}u & X_{N+1}X_{N+1}u & \dots & X_{2N}X_{N+1}u\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ X_1X_{2N}u & \dots & X_NX_{2N}u & X_{N+1}X_{2N}u & \dots & X_{2N}X_{2N}u \end{bmatrix}_{Sym},$$

where

$$A_{\mathrm{Sym}} = \frac{1}{2}\left[A + A^T\right], \text{ for any matrix } A,$$

i.e., symmetric part of the matrix A. Now, since

$$\begin{aligned}[X_i, X_{i+N}] &= X_i X_{i+N} - X_{i+N} X_i \\ &= (\partial_{x_i} + 2y_i\partial_t)(\partial_{y_i} - 2x_i\partial_t) - (\partial_{y_i} - 2x_i\partial_t)(\partial_{x_i} + 2y_i\partial_t) \\ &= -4\partial_t,\end{aligned}$$

so it follows that

$$\mathrm{rank}\big(\mathrm{Lie}\{X_1, X_2, \ldots, X_N, X_{N+1}, X_{N+2}, \ldots, X_{2N}, T\}(0,0)\big) = 2N+1,$$

which is the Euclidean dimension of $\mathbb{H}^N$. We denote by Q, the *homogeneous dimension* of $\mathbb{H}^N$, which is $Q = 2N+2$. The norm on $\mathbb{H}^N$ is defined by

$$|\xi|_{\mathbb{H}^N} := \left[\left(\sum_{i=1}^{N}\left(x_i^2 + y_i^2\right)^2\right) + t^2\right]^{\frac{1}{4}}.$$

The corresponding distance on $\mathbb{H}^N$ is defined as follows:

$$d_{\mathbb{H}^N}(\xi, \hat{\xi}) := |\hat{\xi}^{-1} o\, \xi|_{\mathbb{H}^N},$$

where $\hat{\xi}^{-1}$ is the inverse of $\hat{\xi}$ w.r. to o, i.e., $\hat{\xi}^{-1} = -\hat{\xi}$. The sub-Laplacian or the Heisenberg Laplacian (also known as Laplacian-Kohn operator), $\Delta_{\mathbb{H}^N}$ is the self-adjoint operator defined as

$$\begin{aligned}\Delta_{\mathbb{H}^N} &:= \sum_{i=1}^{2N} X_i^2 \\ &= \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial t} - 4x_i \frac{\partial^2}{\partial y_i \partial t} + 4\left(x_i^2 + y_i^2\right)\frac{\partial^2}{\partial t^2}.\end{aligned}$$

It is useful to observe that

$$\Delta_{\mathbb{H}^N} = \mathrm{div}\left(\sigma^T \sigma \nabla u\right),$$

where

$$\sigma = \begin{bmatrix} I_N & 0 & 2y \\ 0 & I_N & -2x \end{bmatrix}$$

and σ^T is its transpose. Note that

$$A = \sigma^T \sigma = \begin{bmatrix} I_N & 0 & 2y \\ 0 & I_N & -2x \\ 2y & -2x & 4\left(|x|^2 + |y|^2\right) \end{bmatrix}$$

is a positive semi-definite matrix with $\det(A) = 0$, $\forall \xi \in \mathbb{H}^N$.

Let $1 < p < \infty$. The *sub-elliptic analogue of p-Laplacian* (1.3) is given by

$$-\Delta_{p,\mathbb{H}^N} = \sum_{i=1}^{2N} X_j^* \left(|Xu|^{p-2} X_j u \right), \tag{3.1}$$

where X_j^* is the adjoint of X_j. The fundamental solution of (3.1) is given as follows:

$$\Phi_{p,\mathbb{H}^N} := \begin{cases} C_p \left(|.|_{\mathbb{H}^N} \right)^{\frac{(p-Q)}{p-1}} & \text{if } p \neq Q \\ C_p \log(|.|_{\mathbb{H}^N}) & \text{if } p = Q, \end{cases} \tag{3.2}$$

with singularity at the identity element $\mathbf{0} \in \mathbb{H}^N$. Here, C_p denotes a constant given by

$$C_p := \begin{cases} \frac{p-1}{p-Q} (Q\omega_p)^{-\frac{1}{p-1}} & \text{if } p \neq Q \\ (Q\omega_p)^{-\frac{1}{p-1}} & \text{if } p = Q. \end{cases}$$

For the details, we refer to Theorem 2.1 [16].

Next, we consider the *Pucci-Heisenberg operators*, $\mathscr{M}^{\pm}_{\lambda,\Lambda}\left(D^2_{\mathbb{H}^N} u\right)$ defined as

$$\begin{aligned} \mathscr{M}^{-}_{\lambda,\Lambda}\left(D^2_{\mathbb{H}^N} u\right) &= -\Lambda \sum_{e_i > 0} e_i - \lambda \sum_{e_i < 0} e_i, \\ \mathscr{M}^{+}_{\lambda,\Lambda}\left(D^2_{\mathbb{H}^N} u\right) &= -\Lambda \sum_{e_i < 0} e_i - \lambda \sum_{e_i > 0} e_i, \end{aligned} \tag{3.3}$$

where $\{e_i\}_{i=1}^{2N}$ are the eigenvalues of $D^2_{\mathbb{H}^N} u$. It is immediate to see that for $\lambda = \Lambda = 1$, above operators reduce to the Heisenberg Laplacian. The fundamental solutions of

$$\mathscr{M}^{+}_{\lambda,\Lambda}\left(D^2_{\mathbb{H}^N} u\right) \text{ in } \mathbb{R}^{2N+1} \setminus \{0\} \tag{3.4}$$

are given by

$$\Phi_1(\xi) := \begin{cases} C_1 |\xi|^{2-\alpha}_{\mathbb{H}^N} + C_2 & \text{if } \alpha < 2, \\ C_1 \log|\xi|_{\mathbb{H}^N} + C_2 & \text{if } \alpha = 2, \\ -C_1 |\xi|^{2-\alpha}_{\mathbb{H}^N} + C_2 & \text{if } \alpha < 2, \end{cases}$$

and

$$\Phi_2(\xi) := C_1 |\xi|^{2-\beta}_{\mathbb{H}^N} + C_2, \tag{3.5}$$

with constants $C_1 \geqslant 0$, $C_2 \in \mathbb{R}$, where

$$\alpha = \frac{\lambda}{\Lambda}(Q-1) + 1,$$

$$\beta = \frac{\Lambda}{\lambda}(Q-1)+1.$$

Also, the fundamental solutions of

$$\mathscr{M}^{-}_{\lambda,\Lambda}\left(D^2_{\mathbb{H}^N}u\right) \text{ in } \mathbb{R}^{2N+1}\setminus\{0\} \tag{3.6}$$

are given by

$$\begin{aligned}\Psi_1(\xi) &= -\Phi_2(\xi)\\ \Psi_2(\xi) &= -\Phi_1(\xi).\end{aligned} \tag{3.7}$$

For the details, we refer to Theorem 3.3 [22]. Moreover, $\lambda = \Lambda$ yields $\alpha = \beta = Q$ and $\Phi_1 \equiv \Phi_2$ give the fundamental solution for the Heisenberg Laplacian. One may see the book [33] for the details.

4. Fundamental solutions in Grushin-type spaces

In this section, we give the fundamental solution to the p-Laplace operator in a class of Grushin-type spaces for $1 < p < \infty$. These are sub-Riemannian spaces without an algebraic group law. First, we recall the construction of such spaces. Consider the Euclidean space $\mathbb{R}^N$. Let $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$ and X_i be the vector fields given by

$$X_i := P_i(x_1, x_2, \ldots, x_{i-1})\frac{\partial}{\partial x_i} \tag{4.1}$$

for $i = 1, 2, \ldots, N$. Here, $P_i(x_1, x_2, \ldots, x_{i-1})$ is a polynomial. We enforce $P_1 = 1$, which immediately gives

$$X_1 = \frac{\partial}{\partial x_1}.$$

It can be seen that for $i < j$, P_j are differentiable and the Lie bracket is given by

$$X_{ij} := [X_i, X_j] = P_i(x_1, x_2, \ldots, x_{i-1})\frac{\partial P_j(x_1, x_2, \ldots, x_{i-1})}{\partial x_i}\frac{\partial}{\partial_{x_j}}.$$

We endow $\mathbb{R}^N$ with an inner product with singularity at the vanishing points of the polynomial so that $\{X_i\}_{i=1}^N$ forms an orthonormal basis. This produces a sub-Riemannian manifold g_N, which is the tangent space to a generalized Grushin-type space G_N. Points in G_N are denoted by $p = (x_1, x_2, \ldots, x_N)$. G_N is a metric space with its metric given by the Carnot-Carathéodory distance, which is defined as follows:

$$d_C(p,q) := \inf_{\Gamma}\int_0^1 \|\gamma'(t)\|\,dt, \tag{4.2}$$

where Γ denotes the set of all curves γ such that $\gamma(0) = p$, $\gamma(1) = q$ and

$$\gamma'(t) \in \operatorname{span}\{\{X_i(\gamma(t))\}_{i=1}^N\}.$$

For fixed $M \in \mathbb{N}$, $a \in \mathbb{R}$ and $0 \neq c \in \mathbb{R}$, consider the following vector fields:

$$\begin{cases} X_1 &= \frac{\partial}{\partial x_1}, \\ X_i &= c(x_1 - a)^M \frac{\partial}{\partial x_1} \ \text{ for } \ 2 \leqslant i \leqslant N. \end{cases} \tag{4.3}$$

Given a smooth function f on G_N, the horizontal gradient of f is given as follows:

$$\nabla_G :== (X_1 f(p), X_2 f(p), \ldots, X_N f(p)).$$

Now, we state the result concerning fundamental solution of the p-Laplacian for the vector fields defined by (4.3) and $1 < p < \infty$. The result is as follows:

THEOREM 3. (see [11, Theorem 3.1]) *Fix some point $p_0 = (a_1, b_2, b_3, \ldots, b_N) \in G_N$. Let $1 < p < \infty$. Consider the following terms:*

$$\begin{aligned} Q &= (M+1)(N-1)+1, \\ \omega &= \frac{Q-p}{(2M+2)(1-p)}, \\ \alpha &= \frac{Q-p}{1-p}, \\ h(x_1, x_2, \ldots, x_N) &= c^2(x_1 - a)^{2M+2} + (M+1)^2 \sum_{i=2}^{N} (x_i - b_i)^2, \\ f(x_1, x_2, \ldots, x_N) &= [h(x_1, x_2, \ldots, x_N)]^{\omega}, \\ \psi(x_1, x_2, \ldots, x_N) &= [h(x_1, x_2, \ldots, x_N)]^{\frac{1}{2M+2}}, \\ \sigma_p &= \int_{B_1} \|\nabla_G \psi\|^p d\mathscr{L}_N, \\ C_1 &= \alpha^{-1}(Q\sigma_p)^{\frac{1}{1-p}}, \\ C_2 &= (Q\sigma_p)^{\frac{1}{1-p}}. \end{aligned}$$

Then

$$\Phi(x_1, x_2, \ldots, x_N) = \begin{cases} C_1 f(x_1, x_2, \ldots, x_N) & \text{if } p \neq Q, \\ C_2 \log \psi(x_1, x_2, \ldots, x_N) & \text{if } p = Q, \end{cases} \tag{4.4}$$

is the fundamental solution of the p-Laplacian for the vector fields given by (4.3)*. In particular, we have*

$$\Delta_p \Phi(x_1, x_2, \ldots, x_N) = \delta_{p_0}$$

in the sense of distributions.

5. Summary

We finally summarize that in this paper, we have made an attempt to present fundamental solutions to some well-known differential operators. One may also see the citations of the works referred to in this article for long-time asymptotic behaviour of solutions to heat equations including the operators stated in sections 1 and 2. We emphasize that these operators have a wide range of theoretical and practical applications. We only scratch the surface of the topic here.

We start with the Laplace operator, which occurs in a wide range of physical contexts, for instance, Fick's law of diffusion, Fourier's law of heat conduction and Ohm's law of electrical conduction, see Chapter 12 [32] for the details. We also refer to a recent note by Dipierro and Valdinoci [24] for an exhaustive list of scenarios, where elliptic equations, in particular, Laplace equation occur naturally. It is well known that p-Laplace equation is the model equation for nonlinear potential theory. One of the well exposed fields, where the p-Laplace operator occurs is image enhancement. The variational approach to image restoration problem consists of minimizing the energy functional associated with p-Laplacian, see [17] for the details. Infinity Laplacian has important applications in image processing, shape metamorphism and differential games. Pucci's extremal operators occur in the study of stochastic control problems, where the diffusion coefficient is a control variable. We refer to the book [10] for the details. Partial trace operators, $\mathscr{P}_k^{\pm}$ appear in the context of mean curvature flow analysis in arbitrary codimensions through a level set approach by Ambrosio and Soner [1]. Pseudo-differential operators such as fractional Laplacian occur naturally in the study of fluid dynamics, quantum mechanics, population dynamics quasi-geostrophic equation and many more. One may see [14, 19, 37] for these aspects. In particular, [37] briefly lists the areas where these operators appear. Next, we mention that Frankl [34] explored the Tricomi problem concerning (1.29) initiated by [53] is related to the study of gas flows at nearly sonic speeds. More precisely, Tricomi equation characterizes the transition from subsonic flow (elliptic region) to supersonic flow (hyperbolic region). Grushin operator [40] can be naturally thought of as the Tricomi operator for transonic flow subjected to subsonic regions. Also, PDEs in non-Euclidean contexts, such as the Heisenberg group, naturally emerge in quantum mechanics [56], non-Markovian coupling of Brownian motions [51] and many others.

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