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NEW UNIQUENESS CRITERION FOR CAUCHY PROBLEMS OF CAPUTO FRACTIONAL MULTI–TERM DIFFERENTIAL EQUATIONS

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Abstract. The main purpose of this investigation is to revisit solvability process of the Cauchy problems of Caputo fractional two-term initial value problems. To this aim, the Green function technique has chosen to make a bridge between the operator and the fixed point theories. The appeared Green functions in this paper are constructed by the Fox-Wright functions. Our solvability tools include the existence and uniqueness criteria as novel refinements of the Banach contraction principal and Schauder fixed point theorem. This investigation will be finalized by presenting some numerical applications that illustrate proposed solvability criteria.

1. Introduction

In this paper we consider Cauchy problems of the Caputo fractional two-term initial value problems of the form

$$\left({}^{c}\mathscr{D}^{\alpha}_{a^+}y\right)(t) - \lambda\left({}^{c}\mathscr{D}^{\beta}_{a^+}y\right)(t) - \mu y(t) = f(t,y), \quad y(a) = y_0, \tag{1.1}$$

in which, $0 < \alpha \leqslant 1$, $0 < \beta \leqslant \alpha$, $\lambda, \mu \in \mathbb{R}$ and ${}^{c}\mathscr{D}^{\gamma}_{a^+}$ is the Caputo fractional derivative of order $\gamma \in \mathbb{R}^+$ that is defined by

$$\left({}^{c}\mathscr{D}^{\gamma}_{a^+}y\right)(t) := \left(I^{\gamma}_{a^+}y^{(n)}\right)(t), \quad t > a, \ \ n = [\gamma] + 1,$$

where

$$\left(I^{\gamma}_{a^+}z\right)(t) := \int_a^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} z(s)ds, \quad t > a,$$

denotes the fractional integration of order γ. As a general assumption, we consider the function $f : \mathscr{J} \times \mathbb{R} \longrightarrow \mathbb{R}$ to be continuous. We note that $\mathscr{J}$ stands for a closed interval with the starting point a.

Chasing the practical realization of the fractional calculus, it is enough to restrict ourselves into the engineering aspects of the real world phenomena. As popular cases, let us review some few fractional-order engineering problems.

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$(\mathbf{P_1})$ Motion of an immersed plate in a Newtonian fluid. This problem is formulated as:

$$Ay'' + BD_{0^+}^{\frac{3}{2}}y(t) + Cy(t) = f(t), \quad t > 0, \quad A,B,C \in \mathbb{R},$$

where, $f(t)$ is an applied force to the plate.

$(\mathbf{P_2})$ Changing a gas volume in a fluid, that is mathematically expressed as:

$$\frac{d}{dt}(f(t)p(t)) + \lambda D_{0^+}^{\frac{1}{2}}(p(t)-1) = 0, \quad 0 < t < 1, \quad \lambda \in \mathbb{R},$$

such that, $f(t)$ describes changing the volume of gas and $p(t)$ is the gas pressure near the contact surface.

$(\mathbf{P_3})$ Fractional-order controllers. These classes of controllers are commonly known as the $PI^{\lambda}D^{\mu}$-controllers and are efficient controls for fractional-order systems. For instance, the governing equations of such controllers in the time domain for an open-loop system have the following general form:

$$\sum_{k=0}^{n} a_k D^{\beta_k} y(t) = K_P w(t) + K_I D^{-\lambda} w(t) + K_D D^{\mu} w(t), \quad a_k \in \mathbb{R}, \ i = 0,1,\ldots,n,$$

where $K_P, K_I, K_D \in \mathbb{R}$.

$(\mathbf{P_4})$ Fractional-order viscoelastic models. As one of the best engineering fields to study the fractional calculus, one may propose the fractional-order viscoelastic models that describe relationship between the stress and strain in deformable materials with mathematical models such as the multi-parameter generalized Zener model

$$\sigma(t) + a_1 D^{\alpha}\sigma(t) = b_0\varepsilon(t) + b_1 D^{\beta}\varepsilon(t), \quad a_1, b_0, b_1 \in \mathbb{R},$$

in which $\varepsilon(t)$ denotes the strain and $\sigma(t)$ stands for the stress of the considered material.

Detailed estimations for all of the above fractional-order samples are available in [[23], Chaps. 8–10] for the interested followers. A close attention on these problems shows that the governing equations of these engineering models are multi-order differential equations of fractional-order. In this way, one may find so much more mathematical models of the real life phenomena that can be formulated by the multi-order fractional differential equations. This fact implies that prior to the numerical study of such engineering problems we have to organize systematic theoretical investigations on solvability process of this kind of fractional-order differential equations. As sampled collection of research works for the multi-order fractional differential equations we suggest [2], [3], [7], [8], [16], [19], [22], in particular and [1], [4], [5], [9]–[15], [17], [21], [24], [26], in general for more consultation on solvability techniques for differential equations. Because of the necessity described above, we consider the two-term fractional differential equations (1.1) for the basic orders $0 < \alpha, \beta \leqslant 1$.

We devote the remainder space of this section to state the organization of the rest of the paper. Section 2 has an introductory nature including the Fox-Wright functions and some refinements of the Banach contraction principle and the Schauder fixed point theorem that will be considered as the main solvability tools of the main problem (1.1). Section 3 contains some novel statements and related proofs of the existence and uniqueness criteria for the fractional multi-order differential equation (1.1). In this section we introduce corresponding Green function that is essentially constructed by the Fox-Wright functions. Section 4 practically justifies that the obtained theoretical criteria for the existence of at least one solution for (1.1) and its uniqueness. We finalize this investigation with summarize the presented criteria and discussion about the future works.

2. Preliminaries

2.1. The Fox-Wright functions

As previously stated, this section is technical backbone of our investigation and we are going to begin with definition of the Fox-Wright function as follows.

DEFINITION 2.1. [[20], Sec. 1.11, Equ. 1.11.14] Assume $z, a_l, b_j \in \mathbb{C}$ and α_l, $\beta_j \in \mathbb{R}$, $(l = 1,2,\ldots,p,\ j = 1,2,\ldots,q)$. Then, the Fox-Wright function is defined as

$$ {}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{matrix}(a_l,\alpha_l)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{\prod_{l=1}^{p}\Gamma(a_l+\alpha_l)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j)} \frac{z^k}{k!}, \tag{2.1} $$

provided that the infinite series in the right hand side is convergent for any $z \in \mathbb{C}$.

In the following illustration (Figure 1), five plots for various setting of the Fox-Wright function ${}_p\Psi_q(z)$ are given.

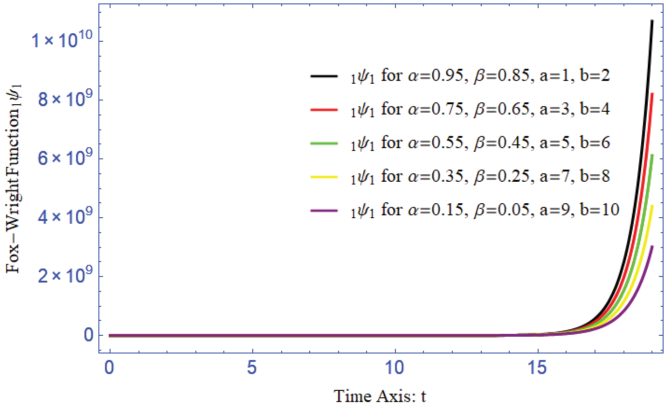

Figure 1: *Numerical simulations of the Fox-Wright function* ${}_1\Psi_1(t)$ *for* $\alpha, \beta \in [0,1]$ *and* $0 \leqslant t \leqslant 19$.

In the light of the following theorem, interpretation of the statement *convergence* in the above definition is presented.

THEOREM 2.2. ([20], Sec. 1.11, Th. 1.5) *Let* $a_l, b_j \in \mathbb{C}$ *and* $\alpha_l, \beta_j \in \mathbb{R}$ $(l = 1,2,\dots,p;\ j = 1,2,\dots,q)$ *and let*

$$\Delta = \sum_{j=1}^{q} \beta_j - \sum_{l=1}^{p} \alpha_l, \tag{2.2}$$

$$\delta = \prod_{l=1}^{p} |\alpha_l|^{-\alpha_l} \prod_{j=1}^{q} |\beta_j|^{\beta_j}, \tag{2.3}$$

and

$$\mu = \sum_{j=1}^{q} b_j - \sum_{l=1}^{p} a_l + \frac{p-q}{2}. \tag{2.4}$$

(a) If $\Delta > -1$*, then the series in* (2.1) *is absolutely convergent for all* $z \in \mathbb{C}$.

(b) If $\Delta = -1$*, then the series in* (2.1) *is absolutely convergent for* $|z| < \delta$ *and for* $|z| = \delta$ *and* $Re(\mu) > \frac{1}{2}$.

2.2. The Green function

When we are dealt with differential equation of linear differential operators, one of the most popular techniques to find global solutions of such differential equations is the Green function technique. This technique makes use of some suitable initial or boundary conditions to transform these kinds of differential problems into their corresponding integral equations with special kernels that are known as the Green functions. For more consultation on the importance of this technique, we suggest the book [6]. Having the convergence conditions of the Fox-Wright functions in hand, it is time to present the general solution of the two-term Caputo fractional Cauchy problem (1.1). To this aim, we have the following.

THEOREM 2.3. ([20], Sec. 5.3, Th. 5.16) *Let* $l-1 < \alpha \leqslant l$, $(l \in \mathbb{N})$, $0 < \beta < \alpha$ *such that* $\alpha - l + 1 \geqslant \beta$. *Let* $\lambda, \mu \in \mathbb{R}$ *and let* $f(t)$ *be a given real function defined on* $\mathbb{R}_+$. *Then, the equation*

$$\left({}^{c}\mathscr{D}^{\alpha}_{a^+} y\right)(t) - \lambda \left({}^{c}\mathscr{D}^{\beta}_{a^+} y\right)(t) - \mu y(t) = f(t) \quad (t > 0,\ \alpha > 0), \tag{2.5}$$

is solvable, and its general solution has the form

$$y(t) := \int_0^t (t-s)^{\alpha-1} G_{\alpha,\beta;\lambda,\mu}(t-s) f(s)\,ds + \sum_{j=1}^{l-1} c_j y_j(x), \tag{2.6}$$

where

$$G_{\alpha,\beta;\lambda,\mu}(t) := \sum_{n=0}^{\infty} \frac{(\mu)^n}{n!} t^{\alpha n}\ {}_1\Psi_1\left[\begin{matrix}(n+1,1)\\(\alpha n+\alpha, \alpha-\beta)\end{matrix}\middle|\lambda t^{\alpha-\beta}\right], \tag{2.7}$$

$$y_j(t) := \sum_{n=0}^{\infty} \frac{(\mu)^n}{n!} t^{\alpha n+j} \, {}_1\Psi_1 \left[\begin{matrix} (n+1,1) \\ (\alpha n + j + 1, \alpha - \beta) \end{matrix} \middle| \lambda t^{\alpha-\beta} \right]$$
$$- \lambda \sum_{n=0}^{\infty} \frac{(\mu)^n}{n!} t^{\alpha n+j+\alpha-\beta} \, {}_1\Psi_1 \left[\begin{matrix} (n+1,1) \\ (\alpha n + j + 1 + \alpha - \beta, \alpha - \beta) \end{matrix} \middle| \lambda t^{\alpha-\beta} \right],$$
$$j = 0, 1, 2, \ldots, m-1, \tag{2.8}$$

$$y_j(t) := \sum_{n=0}^{\infty} \frac{(\mu)^n}{n!} t^{\alpha n+j} \, {}_1\Psi_1 \left[\begin{matrix} (n+1,1) \\ (\alpha n + j + 1, \alpha - \beta) \end{matrix} \middle| \lambda t^{\alpha-\beta} \right], \quad j = m, m+1, \ldots, l-1, \tag{2.9}$$

in which, $m-1 < \beta \leqslant m$, $(m \in \mathbb{N})$, $m \leqslant l$, *and* c_j $(j = 0, 1, 2, \ldots, l-1)$ *are arbitrary real constants.*

Based on the recent theorem, taking $l = 1$, that is $0 < \alpha < \beta \leqslant 1$, we come to the conclusion that the general solution of the two-term fractional Cauchy problem (1.1) takes the following form

$$y(t) := \int_0^t (t-s)^{\alpha-1} G_{\alpha,\beta;\lambda,\mu}(t-s) f(s, y(s)) ds + c_0 y_0(t), \tag{2.10}$$

such that

$$y_0(t) := \sum_{n=0}^{\infty} \frac{(\mu)^n}{n!} t^{\alpha n} \, {}_1\Psi_1 \left[\begin{matrix} (n+1,1) \\ (\alpha n + 1, \alpha - \beta) \end{matrix} \middle| \lambda t^{\alpha-\beta} \right]$$
$$- \lambda \sum_{n=0}^{\infty} \frac{(\mu)^n}{n!} t^{\alpha n+\alpha-\beta} \, {}_1\Psi_1 \left[\begin{matrix} (n+1,1) \\ (\alpha n + 1 + \alpha - \beta, \alpha - \beta) \end{matrix} \middle| \lambda t^{\alpha-\beta} \right], \tag{2.11}$$

As stated, we are interested to find the Green function of the two-term fractional differential equation (1.1). To this aim, we first notify this fact that since fractional derivatives in (1.1) are in the Caputo sense, and since the general solution (2.6)–(2.9) has obtained by the use of the Laplace transform methods, so it is understandable that the coefficients c_j $(j = 0, 1, 2, \ldots, l-1)$ are defined as follows:

$$c_j := y^{(j)}(a), \quad j = 0, 1, 2, \ldots, l-1. \tag{2.12}$$

Thus, considering the general solution (2.10) for the case $l = 1$, it follows that $c_0 = y(a)$. So, the general solution of the main problem (1.1) is of the form

$$y(t) := \int_a^t (t-s)^{\alpha-1} G_{\alpha,\beta;\lambda,\mu}(t-s) f(s, y(s)) ds + y(a) y_0(t), \tag{2.13}$$

where

$$y_0(t) := \sum_{n=0}^{\infty} \frac{(\mu)^n}{n!} (t-a)^{\alpha n} \, {}_1\Psi_1 \left[\begin{matrix} (n+1,1) \\ (\alpha n + 1, \alpha - \beta) \end{matrix} \middle| \lambda (t-a)^{\alpha-\beta} \right]$$
$$- \lambda \sum_{n=0}^{\infty} \frac{(\mu)^n}{n!} (t-a)^{\alpha n+\alpha-\beta} \, {}_1\Psi_1 \left[\begin{matrix} (n+1,1) \\ (\alpha n + 1 + \alpha - \beta, \alpha - \beta) \end{matrix} \middle| \lambda (t-a)^{\alpha-\beta} \right]. \tag{2.14}$$

Concentrating on the initial condition $y(a) = y_0$, we choose the following two important cases that give us the Green function of the two-term fractional Cauchy problem (1.1).

$$y(a) = y_0 \Longrightarrow \begin{cases} \text{case 1.} & y_0 = y(b), \\ \text{case 2.} & y_0 = -y(b), \end{cases} \tag{2.15}$$

in which $b \in \mathbb{R}$, $a < b$ is the endpoint of the interval $\mathscr{J}$, that is $\mathscr{J} := [a,b]$. Having these settings in hand we reach the following conditions:

$$\text{The periodic condition:} \quad y(a) = y(b), \tag{2.16}$$

$$\text{The anti-periodic condition:} \quad y(a) = -y(b). \tag{2.17}$$

Implying the periodic condition (2.16) into the general solution (2.13)–(2.14), gives us

$$y(b) := \frac{\displaystyle\int_a^b (b-s)^{\alpha-1} G_{\alpha,\beta;\lambda,\mu}(b-s) f(s,y(s))ds}{1 - \displaystyle\sum_{n=0}^{\infty} \frac{(\mu)^n}{n!}(b-a)^{\alpha n}\Big(\Theta_1(b) - \lambda(b-a)^{\alpha-\beta}\Theta_2(b)\Big)}. \tag{2.18}$$

where

$$\Theta_1(t) := {}_1\Psi_1\left[\begin{matrix}(n+1,1)\\(\alpha n+1,\alpha-\beta)\end{matrix}\,\middle|\,\lambda(t-a)^{\alpha-\beta}\right], \tag{2.19}$$

$$\Theta_2(t) := {}_1\Psi_1\left[\begin{matrix}(n+1,1)\\(\alpha n+1+\alpha-\beta,\alpha-\beta)\end{matrix}\,\middle|\,\lambda(t-a)^{\alpha-\beta}\right]. \tag{2.20}$$

If we substitute $y(b)$ obtained by (2.18)–(2.20) into the $y(a)$ in the general solution (2.13), then, we arrive at

$$y(t) := \int_a^b G(t,s) f(s,y(s))ds, \tag{2.21}$$

such that the Green function $G(t,s)$ is defined as follows:

$$G(t,s) := \begin{cases} \dfrac{(b-s)^{\alpha-1} G_{\alpha,\beta;\lambda,\mu}(b-s) \displaystyle\sum_{n=0}^{\infty} \frac{(\mu)^n}{n!}(t-a)^{\alpha n}\Big(\Theta_1(t) - \lambda(t-a)^{\alpha-\beta}\Theta_2(t)\Big)}{1 - \displaystyle\sum_{n=0}^{\infty} \frac{(\mu)^n}{n!}(b-a)^{\alpha n}\Big(\Theta_1(b) - \lambda(b-a)^{\alpha-\beta}\Theta_2(b)\Big)} \\ \qquad + (t-s)^{\alpha-1} G_{\alpha,\beta;\lambda,\mu}(t-s), \quad a \leqslant s \leqslant t, \\ \dfrac{(b-s)^{\alpha-1} G_{\alpha,\beta;\lambda,\mu}(b-s) \displaystyle\sum_{n=0}^{\infty} \frac{(\mu)^n}{n!}(t-a)^{\alpha n}\Big(\Theta_1(t) - \lambda(t-a)^{\alpha-\beta}\Theta_2(t)\Big)}{1 - \displaystyle\sum_{n=0}^{\infty} \frac{(\mu)^n}{n!}(b-a)^{\alpha n}\Big(\Theta_1(b) - \lambda(b-a)^{\alpha-\beta}\Theta_2(b)\Big)}, \\ \qquad t \leqslant s \leqslant b. \end{cases} \tag{2.22}$$

REMARK 2.4. Choosing the anti-periodic condition (2.17) instead of the periodic condition (2.16), yields the following Green function

$$G(t,s) := \begin{cases} \dfrac{(b-s)^{\alpha-1}G_{\alpha,\beta;\lambda,\mu}(b-s)\sum_{n=0}^{\infty}\frac{(\mu)^n}{n!}(t-a)^{\alpha n}\Big(\Theta_1(t)-\lambda(t-a)^{\alpha-\beta}\Theta_2(t)\Big)}{1+\sum_{n=0}^{\infty}\frac{(\mu)^n}{n!}(b-a)^{\alpha n}\Big(\Theta_1(b)-\lambda(b-a)^{\alpha-\beta}\Theta_2(b)\Big)} \\ \qquad +(t-s)^{\alpha-1}G_{\alpha,\beta;\lambda,\mu}(t-s), \quad a \leqslant s \leqslant t, \\ \dfrac{(b-s)^{\alpha-1}G_{\alpha,\beta;\lambda,\mu}(b-s)\sum_{n=0}^{\infty}\frac{(\mu)^n}{n!}(t-a)^{\alpha n}\Big(\Theta_1(t)-\lambda(t-a)^{\alpha-\beta}\Theta_2(t)\Big)}{1+\sum_{n=0}^{\infty}\frac{(\mu)^n}{n!}(b-a)^{\alpha n}\Big(\Theta_1(b)-\lambda(b-a)^{\alpha-\beta}\Theta_2(b)\Big)}, \\ \qquad t \leqslant s \leqslant b. \end{cases} \tag{2.23}$$

Now, let us choose the instant setting

$$a := 0, \quad b := 1, \quad \lambda = 0.7, \quad \mu := 0.3, \quad \alpha,\beta \in [0,1], \quad \alpha-\beta := 0.02, \quad s := 0.5.$$

In thiscase, the Green functions $G(t,s)$ given by (2.22) and (2.23) can be illustrated by the figures 2 and 3, respectively.

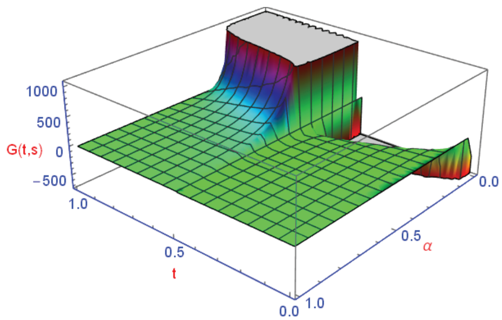

Figure 2: *The periodic Green function $G(t,s)$ given by* (2.22).

By now, we have been able to transform the general solution (2.13) of the main problem (1.1) into the integral operator (2.21) having the Green function $G(t,s)$ given by (2.22) corresponding to the periodic condition (2.16), or the Green function $G(t,s)$ given by (2.23) corresponding to the anti-periodic condition (2.17) as its kernel. In order to present existence and uniqueness tools for the main problem (1.1), we devote the remainder of this section to present some fixed point techniques that include novel refinements of the Banach contraction principal and the Schauder fixed point theorem.

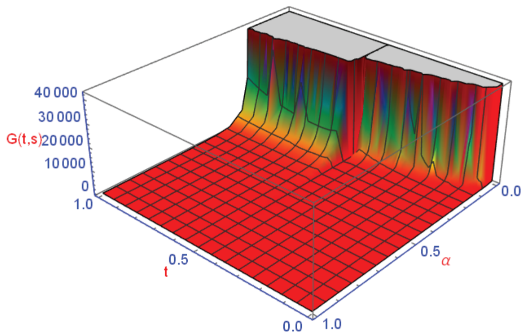

Figure 3: *The anti-periodic Green function* $G(t,s)$ *given by* (2.23).

2.3. The fixed point Theorems

The starting point of this section is to characterize a suitable functional space. To this aim, we choose the Banach space $(C(\mathscr{J},\mathbb{R});\|.\|_\infty)$ in which

$$\|y\|_\infty := \sup_{t\in\mathscr{J}}\{|y(t)|,\ y\in C(\mathscr{J},\mathbb{R})\}.$$

REMARK 2.5. Let us define the mapping $\mathscr{K}:\mathscr{J}\longrightarrow C(\mathscr{J};\mathbb{R})$ as follows:

$$\mathscr{K}(t) := \int_a^b |G(t,s)|ds, \tag{2.24}$$

where, $G(t,s)$ stands for the Green function (2.22) or (2.23). In this case, it is easy to check that $\mathscr{K}$ is a continuous mapping on $\mathscr{J}$ and consequently will be bounded over $\mathscr{J}$. Accordingly, one may conclude the following:

$$\mathscr{K}_{\text{sup}} := \sup\left\{\int_a^b |G(t,s)|ds, \quad t\in\mathscr{J}\right\}. \tag{2.25}$$

DEFINITION 2.6. Suppose (M,d) is a complete metric space and $T: M\longrightarrow M$ is a mapping with the Lipschitz constant $k(T)$, that is

$$d(T(x),T(y)) \leqslant k(T)d(x,y), \qquad x,y\in M,\ \ x\neq y.$$

Then, we define

$$k_\infty(T) := \lim_{n\longrightarrow\infty}\left[k\left(T^n\right)\right]^{\frac{1}{n}}. \tag{2.26}$$

Having the recent data in hand, we are ready to state the following refinement of the Banach contraction principal that will help us to provide a uniqueness criterion for the two-term Cauchy problem (1.1).

THEOREM 2.7. ([18], Th. 3.11) *Let M be a complete metric space and let $T : M \longrightarrow M$ be a continuous mapping for which $k_\infty(T) < 1$. Then, T has a unique fixed point $z \in M$, and for each $x \in M$ the sequence $\{T^n(x)\}$ converges to z.*

Next, the Schauder fixed point theorem is given as follows.

THEOREM 2.8. ([25], Th. 2.A) *Let $\mathbb{K}$ be a nonempty, closed, bounded, convex subset of a Banach space X, and suppose $T : \mathbb{K} \longrightarrow \mathbb{K}$ is a compact operator. Then, T has at least one fixed point.*

3. Main results

We begin the main results with the following existence criterion that constructs an appropriate space for the Schauder fixed point theorem to guarantee the existence of at least one solution for the main problem (1.1). So, we state and prove it as follows.

THEOREM 3.1. *Let the following hypotheses are satisfied:*

(H_1) *There exists an $\theta \in C(J,\mathbb{R}^+)$ and a continuous and nondecreasing function $\rho : \mathbb{R}_+ \longrightarrow \mathbb{R}^+$ for which*

$$|f(t,u)| \leqslant \theta(t)\rho(|u|), \quad t \in \mathscr{J}, \quad u \in \mathbb{R}. \tag{3.1}$$

(H_2) *There exists a positive constant ξ such that*

$$\frac{\xi}{\theta_{\sup}\rho(\xi)\mathscr{K}_{\sup}} > 1, \tag{3.2}$$

in which

$$\theta_{\sup} := \sup\{\theta(t),\, t \in \mathscr{J}\}.$$

Then, the two-term Caputo fractional Cauchy problem (1.1)–(2.16), *(*(1.1)–(2.17)*) has at least one solution.*

Proof. Let us draw the proof map as follows. All we have to do is to show that the assumptions of Theroem 2.8 are satisfied. Thus, as the first step we define the following subset of $C(\mathscr{J};\mathbb{R})$:

$$\mathbb{K} := \{y \in C(\mathscr{J};\mathbb{R}) |\ \|y\|_\infty \leqslant \xi\}. \tag{3.3}$$

The immediate fact is that $\mathbb{K}$ is a nonempty, closed, bounded and convex subset of $C(\mathscr{J};\mathbb{R})$. Here, we define the operator $T : C(\mathscr{J};\mathbb{R}) \longrightarrow C(\mathscr{J};\mathbb{R})$ as follows:

$$(Ty)(t) := \int_a^b G(t,s)f(s,y(s))ds, \tag{3.4}$$

such that $G(t,s)$ is the Green function defined by (2.22), or (2.23). In order to complete the proof we have the following four steps that must be proved.

(S_1) It has to be shown that $T(\mathbb{K}) \subset \mathbb{K}$. To this aim, suppose $y \in \mathbb{K}$. Then, one has

$$\begin{aligned}
|(Ty)(t)| &\leqslant \int_a^b |G(t,s)| f(s,y(s))ds && \text{(according to (3.4))}\\
&\leqslant \theta_{\sup}\rho(\|u\|_\infty)\int_a^b |G(t,s)|ds && \text{(according to } (H_1), (3.1))\\
&\leqslant \theta_{\sup}\rho(\xi)\mathscr{K}_{\sup} && \text{(according to (2.25))}\\
&\leqslant \xi && \text{(according to } (H_2), (3.2)).
\end{aligned}$$

This guarantees that $T\mathbb{K} \subset \mathbb{K}$.

(S_2) It has to be shown that T maps $\mathbb{K}$ into a bounded subset of $C(\mathscr{J};\mathbb{R})$. This is an immediate consequence of the hypothesis (H_1), (2.25) and (3.3), that is for each $y \in \mathbb{K}$ it follows that

$$|(Ty)(t)| \leqslant \theta_{\sup}\rho(\xi)\mathscr{K}_{\sup} \Longrightarrow T\mathbb{K} < \infty.$$

(S_3) It has to be shown that T is continuous. The proof is clear due to the continuity of f and the Lebesgue dominated convergence theorem.

(S_4) It has to be shown that T maps $\mathbb{K}$ into a equicontinuous subset of $C(\mathscr{J};\mathbb{R})$. Thus, suppose $y \in \mathbb{K}$ and $t_1, t_2 \in \mathbb{J}$ with $t_1 < t_2$. Having these assumptions in hand we arrive at

$$\begin{aligned}
|(Ty)(t_2) - (Ty)(t_1)| &= \left|\int_a^b G(t_2,s)f(s,y(s))ds - \int_a^b G(t_1,s)f(s,y(s))ds\right|\\
&\leqslant \int_a^b |G(t_2,s) - G(t_1,s)||f(s,u(s))|ds\\
&\leqslant \theta_{\sup}\rho(\xi)\int_a^b |G(t_2,s) - G(t_1,s)|ds \qquad ((H_1), (3.1)).
\end{aligned}$$

Since the right hand side of the last inequality tends to zero as $t_1 \longrightarrow t_2$, we come to the conclusion that based on the Arzela-Ascoli theorem the operator T is completely continuous.

It is time to complete the proof. First, we have $\mathbb{K}$ that is a nonempty, closed, bounded and convex subset of $C(\mathscr{J};\mathbb{R})$. Secondly, thanks to the four steps $(S_1) - (S_4)$, it has proven that the operator $T : \mathbb{K} \longrightarrow \mathbb{K}$ is compact, that is according to the Schauder fixed point theorem the two-term Caputo fractional Cauchy problem (1.1)–(2.16), ((1.1)–(2.17)) has at least one solution. So, the proof is completed now. □

The next theorem will provide a uniqueness criterion for the main problem (1.1). So, we state and prove it as follows.

THEOREM 3.2. *Suppose there exists a Lipschitz constant $l(f)$ such that*

$$|f(t,y) - f(t,z)| \leqslant l(f)|y - z|, \quad t \in \mathscr{J}, \quad y,z \in C(\mathscr{J};\mathbb{R}). \tag{3.5}$$

Then, the two-term Caputo fractional Cauchy problem (1.1)–(2.16)*, ((*1.1)–(2.17)*) has exactly one solution provided that*

$$l(f)\mathscr{K}_{\sup} < 1, \tag{3.6}$$

in which $\mathscr{K}_{\sup}$ *has given by* (2.25).

Proof. Considering the operator T defined by (3.4), and considering the complete metric space $(C(\mathscr{J};\mathbb{R},d))$ for which

$$d(x,y) = |x-y|, \quad x,y \in C(\mathscr{J};\mathbb{R}),$$

it follows that

$$\begin{aligned}
d(Ty,Tz) &:= |(Ty)(t) - (Tz)(t)| \\
&= \left| \int_a^b G(t,s)f(s,y(s))ds - \int_a^b G(t,s)f(s,z(s))ds \right| \\
&\leqslant \int_a^b G(t,s)d(f(s,y),f(s,z))ds \\
&\leqslant l(f)\mathscr{K}_{\sup} d(y,z) \qquad \text{(according to (2.25) and (3.5))} \\
&:= k(T)d(y,z),
\end{aligned}$$

$$\begin{aligned}
d\left(T^2y,T^2z\right) &:= \left|\left(T^2y\right)(t) - \left(T^2y\right)(t)\right| \\
&= \left| \int_a^b G(t,s)f(s,(Ty)(s))ds - \int_a^b G(t,s)f(s,(Tz)(s))ds \right| \\
&\leqslant \int_a^b G(t,s)d(f(s,Ty),f(s,Tz))ds \\
&\leqslant l^2(f)\mathscr{K}_{\sup}^2 d(y,z) \\
&:= k\left(T^2\right)d(y,z), \\
&\ \ \vdots \\
d\left(T^ny,T^nz\right) &\leqslant l^n(f)\mathscr{K}_{\sup}^n d(y,z) := k\left(T^n\right)d(y,z).
\end{aligned}$$

The recent inequality leads us to the following fact

$$k_\infty(T) := \lim_{n\to\infty}\left[k\left(T^n\right)\right]^{\frac{1}{n}} = \lim_{n\to\infty}\left[l^n(f)\mathscr{K}_{\sup}^n\right]^{\frac{1}{n}} := l(f)\mathscr{K}_{\sup} < 1, \quad ((3.6)). \tag{3.7}$$

Therefore, based on the Theorem 2.7, it has shown that the two-term Caputo fractional Cauchy problem (1.1)–(2.16), ((1.1)–(2.17)) has exactly one solution in the $C(\mathscr{J};\mathbb{R})$. This complete the proof. □

4. Numerical applications

Here we are in such a position that we can examine the practical implementability of the established existence and uniqueness criteria in previous section. To this aim, we first start with a numerical example corresponding to the existence of solutions.

EXAMPLE 4.1. Consider the periodic two-term Caputo fractional Cauchy problem

$$\left({}^{c}\mathscr{D}_{0^+}^{0.92}y\right)(t)-0.7\left({}^{c}\mathscr{D}_{0^+}^{0.9}y\right)(t)-0.3y(t)=\frac{\ln\left(1+t^2\right)}{10^4\left(1+t^2\right)}(1+\exp(2y)),\quad y(0)=y(1). \tag{4.1}$$

In fact, this Cauchy problem is specialized version of the main governing problem (1.1) subject to the following setting:

$$a:=0,\quad b:=1,(\mathscr{J}:=[0,1]),\quad \lambda:=0.7,\quad \mu:=0.3,\quad \alpha:=0.92,\quad \beta:=0.9.$$

Now let us turn back to the Theorem 3.1, where an existence criterion for at least one solution of the Cauchy problem (1.1) has established. So, to reach existence of at least one solution for the periodic two-term fractional Cauchy problem (4.1), we have to prove that all conditions of Theorem 3.1 are satisfied here. Hence, we have the following steps:

Step 1. Consider the nonlinearity

$$f(t,y):=\frac{\ln\left(1+t^2\right)}{10^4\left(1+t^2\right)}(1+\exp(2y)). \tag{4.2}$$

We can illustrate this nonlinearity as follows (Figure 4).

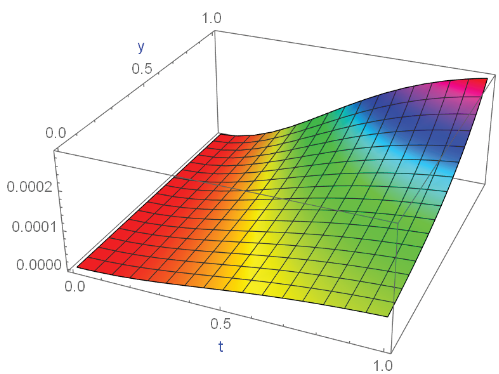

Figure 4: *The nonlinearity $f(t,y)$ given by (4.2).*

A direct consequence of $f(t,y)$ is as follows:

$$|f(t,y)|\leqslant\theta(t)\rho(|y|), \tag{4.3}$$

in which

$$\theta(t) := \frac{\ln\left(1+t^2\right)}{10^4}, \quad \rho(u) := (1+\exp(u))^2.$$

Immediately one may conclude that the condition (H_1) in Theorem 3.1 holds.

Step 2. By the assistance of the numerical simulation corresponding to the periodic Green function $G(t,s)$ defined by (2.22) in Figure 2 (all numerical factors of this illustration are chosen here), it follows that

$$\mathscr{K}_{\sup} := 1000, \quad \text{and} \quad \theta_{\sup} := \ln 2.$$

Thus, choosing $\xi := 1$, we come to the conclusion that

$$\frac{\xi}{\theta_{\sup}\rho(\xi)\mathscr{K}_{\sup}} := \frac{10}{(1+e)^2 \ln 2} \approx \frac{10}{9.58317} > 1,$$

that is the condition (H_2) is also satisfied.

So, Based on Theorem 3.1, the periodic two-term fractional Cauchy problem (4.1) has at least one solution.

EXAMPLE 4.2. Consider the anti-periodic two-term Caputo fractional Cauchy problem

$$\begin{aligned}&\left({}^{c}\mathscr{D}_{0^+}^{0.92}y\right)(t) - 0.7\left({}^{c}\mathscr{D}_{0^+}^{0.9}y\right)(t) - 0.3y(t)\\ &\qquad = \frac{(\exp(0.92t)+\exp(0.9t))}{3\times 10^3\left(1+\exp(0.92t)+\exp(0.9t)\right)^4}\ln(1+|y|), \quad y(0)+y(1)=0.\end{aligned} \tag{4.4}$$

Similar to the example 4.1, the following setting are chosen:

$$a := 0, \quad b := 1, (\mathscr{J} := [0,1]), \quad \lambda := 0.7, \quad \mu := 0.3, \quad \alpha := 0.92, \quad \beta := 0.9.$$

In this position, we have to show that all conditions of Theorem 3.2 are satisfied to reach a unique solution of the anti-periodic two-term Caputo fractional Cauchy problem (4.4). So, we have the following steps:

Step 1. Let us remind once again that

$$f(t,y) := \frac{(\exp(0.92t)+\exp(0.9t))}{3\times 10^3\left(1+\exp(0.92t)+\exp(0.9t)\right)^4}\ln(1+|y|). \tag{4.5}$$

Here is an illustration of the nonlinearity $f(t,y)$ defined by (4.5) (Figure 5).

It is easy to check that

$$|f(t,y)-f(t,z)| \leqslant \frac{1}{3\times 10^3}\frac{6}{81}|y-z| = \frac{2}{81000}|y-z|. \tag{4.6}$$

This shows that the nonlinearity $f(t,y)$ defined in this example is a Lipschitz function with Lipschitz constant $l(f) := 2/81000$. Therefore, the condition (3.5) in Theorem 3.2 holds.

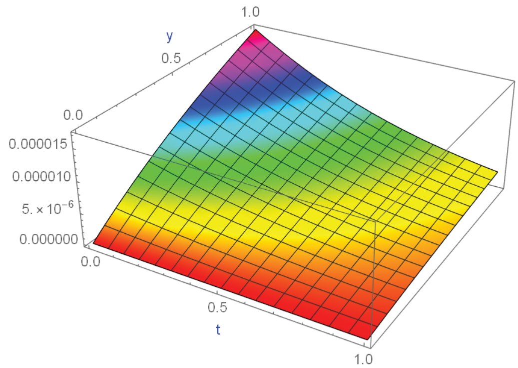

Figure 5: *The nonlinearity $f(t,y)$ given by (4.5).*

Step 2. Considering the numerical illustration corresponding to the anti-periodic Green function $G(t,s)$ defined by (2.23) in Figure 3, it is find out that

$$\mathscr{K}_{\sup} := 4 \times 10^4.$$

In this case, we come to the conclusion that

$$l(f)\mathscr{K}_{\sup} := \frac{2}{81000}.4 \times 10^4 = \frac{80}{81} < 1,$$

which means that the condition (3.6) is also satisfied in Theorem 3.2.

Thus, according to this theorem, the anti-periodic two-term Caputo fractional Cauchy problem (4.4) has exactly one solution.

5. Concluding Remarks

In this investigation as the basic step, general two-term Caputo fractional Cauchy problems have considered to be studied in viewpoint of their solvability. To this aim, we have first considered two particular cases as the periodic and anti-periodic initial conditions and correspondingly, the Green functions of each case have calculated in frame of the Fox-Wright functions as generalizations of the Mittag-Leffler functions. In the next step, some novel refinements of the Banach contraction principal and the Schauder fixed point theorem have chosen to be our solvability tools to establish some existence and uniqueness criteria for the under study two-term fractional Cauchy problems. At the next step, both of the theoretically obtained existence and uniqueness criteria have examined by appropriate numerical applications to guarantee the usefulness of these solvability tools in practice. As continuation, one may suggest to consider the general multi-term fractional Cauchy problems to establish related solvability criteria by the use of the fixed point techniques applied in this article as future research works.

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