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SOLVABILITY FOR A COUPLED SYSTEM OF 4–SEQUENTIAL FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The present work deals with a coupled system of fractional differential equations involving four sequential Caputo derivatives in each of its components. The fractional differential system gives rise to a standard coupled system of two ordinary differential equations of order four, which has practical applications in some real-world phenomena such as robotics, aerospace, and electrical engineering. The existence of a unique vector solution for our sequential system is studied. The existence of at least one vector solution for the considered system is also investigated. Some illustrative examples are discussed in detail to show the main results' applicability. The stabilities in the sense of Ulam Hyers for the system is discussed. A conclusion follows at the end.

1. Introduction

Over the last three decades, fractional differential equations have been attractive to many researchers in the past decades due to the non-localization properties of the fractional derivatives contrary to the integer-order derivatives [12, 16, 18]. It has been discovered that this subject has applications in a wide range of technical and physical sciences, including complex media electrodynamics, control theory ecology, viscoelasticity, biomathematics, and electrical circuits, we refer the reader to [2, 3, 5, 6, 7, 10, 15, 17, 20, 21, 22, 23] for some important applications. Other important results can be found in the following references:

We begin by citing the paper [11] where the authors investigated the existence of unique maximal and minimal solutions for the following coupled differential system in terms of the generalized fractional derivative

$$\begin{cases} D_{a_+}^{\alpha,\Im}u(t)+F_1(t,v(t))=0, \;\; t\in[a,b], \\ D_{a_+}^{\beta,\Im}v(t)+F_2(t,u(t))=0, \;\; t\in[a,b], \\ u(b)+\lambda_\alpha u(a)=I_{a^+}^{\alpha,\Im}F_3(b,v(b)), \;\; u'^{n-1}(a)=0, \\ v(b)+\lambda_\beta v(a)=I_{a^+}^{\beta,\Im}F_4(b,u(b)), \;\; v'^{n-1}(a)=0, \end{cases}$$

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where $D_{a+}^{\alpha,\Im}u$ and $D_{a+}^{\beta,\Im}$, $n-1<\alpha,\beta<n$, $n\geqslant 2$ are the fractional derivatives of a "function u concerning another function": $\Im$ and $-1<\lambda_\alpha,\lambda_\beta\leqslant 0$.

We cite also the work in [24] where it can be found that the authors discussed the existence, uniqueness, and some Ulam stability results for the following fractional coupled system

$$\begin{cases} D^{\alpha_1}x_1(t)=f_1(t,x_1(t)),\\ D^{\alpha_2}x_2(t)=f_2(t,x_1(t),x_2(t)),\\ \qquad\qquad\vdots\\ D^{\alpha_n}x_n(t)=f_n(t,x_1(t),x_2(t),\dots,x_n(t)),\\ 0<t\leqslant 1,\;\; k-1<\alpha_k<k,\;\; k=1,2,\dots,n,\\ x_1(0)=a_0^1,\;\; k=1,\;\; x_k^{(j)}(0)=a_j^k,\;\; j=0,1,\dots,k-2,\;\; k=2,3,\dots,n,\\ D^{\delta_{k-1}}x_k(1)=0,\;\; k-1<\delta_{k-1}<k,\;\; k=2,3,\dots,n, \end{cases}$$

where, $n\in\mathbb{N}-\{0,1\}$. For all $k=1,2,\dots,n$, the functions $f_k:(0,1]\times\mathbb{R}^k\to\mathbb{R}$ are continuous, singular at $t=0$, $\lim_{t\to 0^+}f_k(t)=\infty$ and there exist $\beta_k\in(0,1)$, $k=1,2,\dots,n$, such that $t^{\beta_k}f_k$, $k=1,2,\dots,n$ are continuous on $[0,1]$.

In [8], the two-Beddani studied the existence and uniqueness of solutions for the coupled system of Caputo fractional differential equations

$$\begin{cases} D^{\beta_1}(D^{\alpha_1}+g_1(t))u(t)+f_1(t,u(t),v(t),D^{\delta_1}u(t),D^{\delta_2}v(t))=h_1(t,u(t),v(t)),\\ D^{\beta_2}(D^{\alpha_2}+g_2(t))u(t)+f_2(t,u(t),v(t),D^{\delta_1}u(t),D^{\delta_2}v(t))=h_2(t,u(t),v(t)),\\ \qquad u(0)=a_1,\;\; v(0)=a_2,\;\; u(1)=b_1,\;\; v(1)=b_2,\;\; t\in J \end{cases}$$

where, $J=[0,1]$, $0<\alpha_k,\beta_k<1$, $0<\delta_k<\alpha_k<1$, $k=1$; the functions $f_k:[0,1]\times\mathbb{R}^4\to\mathbb{R}$, $k=1,2$ are continuous $g_k:(0,1]\to[0,+\infty)$ are continuous functions singular at $t=0$, and $\lim_{t\to 0^+}g_k(t)=\infty$ and the operators D^{β_k},D^{α_k} and D^{δ_k}, $k=1,2$ are the derivatives in the sense of Caputo and the constants a_k, b_k are reals.

In [14], the authors used fixed point theorems to investigate the following coupled system

$$\begin{cases} D^{\alpha}u(t)=f(t,u(t),v(t)),\;\; t\in[0,T],\\ D^{\beta}u(t)=f(t,u(t),v(t)),\;\; t\in[0,T],\\ (u+v)(0)=-(u+v)(T),\;\; \int_\zeta^\eta(u-v)(s)ds=A \end{cases}$$

where D^χ is the Caputo fractional derivatives of order $\chi\in\{\alpha,\beta\}$, here α and β are the orders of these fractional derivatives (see [15]); $f,g:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions, and A is a non-negative constant.

In [1], by applying both Schaefer and Krasnoselskii fixed point theorems, the authors proved the existence and uniqueness of solutions for the problem

$$\begin{cases} D^{\alpha}u(t) = f_1(t,v(t),D^{\alpha-1}v(t),D^{\alpha-2}v(t),\dots,D^{\alpha-(n-1)}v(t)), \ \ t\in[0,1], \\ D^{\beta}v(t) = f_2(t,u(t),D^{\beta-1}u(t),D^{\beta-2}u(t),\dots,D^{\beta-(n-1)}u(t)), \ \ t\in[0,1], \\ \qquad u(0)=u_0^*, \ \ u'(0)=u''^{n-2}(0)=0 \\ \qquad u^{n-1}(0)=\gamma I^p u(\eta), \ \ \eta\in\,]0,1[, \\ \qquad v(0)=v_0^*, \ \ v'(0)=v''^{n-2}(0)=0, \\ \qquad v^{n-1}(0)=\delta I^p u(0), \ \ \zeta\in\,]0,1[, \end{cases}$$

where D^{α} and D^{β} denote the Caputo fractional derivatives, p and q are non negative reals numbers, $n-1<\alpha<n$, $n-1<\beta<n$, with $n\in\mathbb{N}^*$, $n\neq 1$, $u_0^*,v_0^*\in\mathbb{R}$, f_1 and f_2 are two functions.

In [9], the authors studied the existence of some unique solutions for a new problem of fractional differential equations involving Caputo derivatives. Also, using the Adam-Bashforth method, some numerical simulations for the proposed illustrative examples have been presented. The new problem of [9] is the following:

$$\begin{cases} {}^cD^{\alpha_1}\,{}^cD^{\alpha_2}\,{}^cD^{\alpha_3}\left[{}^cD^{\alpha_4}u(t)-\lambda f(t)u(t)\right] \\ \qquad = g(t,u(t),{}^cD^{\alpha_2}u(t),{}^cD^{\alpha_3}u(t),{}^cD^{\alpha_4}u(t)), \ \ t\in J=[0,1], \\ u(0))=0, \\ u(1)=a_1, \\ {}^cD^{\alpha_4}u(0)=a_2, \\ {}^cD^{\alpha_4}u(1)=0 \end{cases}$$

where, ${}^cD^{\alpha_i}$, also the three others the derivatives are Caputo fractional derivatives, $0<\alpha_i\leqslant 1$, $i=1,\dots,4$, $\alpha_2<\alpha_4$, $\alpha_3<\alpha_4$, $\lambda>0$, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ and $g:[0,1]\times\mathbb{R}^4\to\mathbb{R}$ are continuous.

In this article, we are concerned with the existence and uniqueness of solutions for the following coupled system of nonlinear fractional differential equations with four sequential Caputo derivatives:

$$\begin{cases} D^{\alpha_1}D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}x(t) = H_1(t,x(t),y(t))+a_1f_1(x(t))+b_1g_1(D^{\alpha_1}D^{\alpha_2}x(t)), \\ \qquad t\in J=[0,1] \\ D^{\beta_1}D^{\beta_2}D^{\beta_3}D^{\beta_4}y(t) = H_2(t,x(t),y(t))+a_2f_2(y(t))+b_2g_2(D^{\beta_1}D^{\beta_2}y(t)), \\ \qquad t\in J=[0,1] \\ x(0)=x(1)=D^{\alpha_1}D^{\alpha_2}x(1)=D^{\alpha_4}x(0)=0, \\ y(0)=y(1)=D^{\beta_1}D^{\beta_2}y(1)=D^{\beta_4}y(0)=0, \end{cases} \tag{1}$$

where, $D^{\alpha_1}, D^{\alpha_2}, D^{\alpha_3}, D^{\alpha_4}, D^{\beta_1}, D^{\beta_2}, D^{\beta_3}, D^{\beta_4}$ are Caputo fractional derivatives, $0 < \alpha_i \leqslant 1$, $0 < \beta_i \leqslant 1$, $i = 1, \ldots, 4$, $\alpha_2 + \alpha_1 < \alpha_4$, $\beta_1 + \beta_2 < \beta_4$, $f_j : \mathbb{R} \to \mathbb{R}$, $g_j : \mathbb{R} \to \mathbb{R}$ and $H_j : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$, $j = 1, 2$ are continuous functions, and $H_i(t,0,0) \neq 0$, $f_i(0) \neq 0$, $g_i(0) \neq 0$, $i = 1, 2$. We note in passing that the classical case of the above-considered sequential system gives rise to a coupled system of two ordinary differential equations of order four, which has important practical applications in elastic beams [4, 13, 19, 25].

We think that the present research paper on this topic has the potential to contribute to the development of more accurate and efficient modeling techniques, as well as to the design of new control strategies for complex systems since the standard coupled system of order four can be seen as a limiting-case for the above fractional sequential system.

2. Preliminaries

We recall some important notions for studying the above-coupled sequential fractional system.

DEFINITION 1. The Riemann-Liouville fractional integral operator of order $\varepsilon > 0$, for a continuous function l defined over $[a,b]$ is defined as:

$$J^{\varepsilon} l(t) = \frac{1}{\Gamma(\varepsilon)} \int_a^t (t-\rho)^{\varepsilon-1} l(\rho) d\rho, \quad \varepsilon > 0, \quad a \leqslant t \leqslant b.$$

DEFINITION 2. The fractional derivative of $l \in C^n([a,b]$ in the sense of Caputo is defined as:

$$D^{\varepsilon} l(t) = \frac{1}{\Gamma(n-\varepsilon)} \int_a^t (t-\rho)^{n-\varepsilon-1} l^{(n)}(\rho) d\rho, \quad n-1 < \varepsilon < n, \quad n \in \mathbb{N}^*, \quad t \in [a,b].$$

LEMMA 1. *Let $p, q > 0$, $h \in C([a,b])$. Then*

$$I^p I^q h(t) = I^{p+q} h(t), \quad D^p I^p h(t) = h(t), \quad t \in [a,b]. \tag{2}$$

LEMMA 2. *Let $q > p > 0$ and $h \in C([a,b])$. Then, $D^p I^q h = I^{q-p} h$.*

LEMMA 3. *Let $\varepsilon > 0$. The set of solutions of the equation $D^{\varepsilon} u(t) = 0$ is given by*

$$u(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}, \quad t \in [0,1] \tag{3}$$

where, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\varepsilon] + 1$.

LEMMA 4. *For any $\varepsilon > 0$, we have*

$$I^{\varepsilon} D^{\varepsilon} u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}, \quad t \in [0,1]$$

with, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\varepsilon] + 1$.

Let us now consider the notations:

$$k_1(t) = H_1(t,x(t),y(t)) + a_1 f_1(x(t)) + b_1 g_1(D^{\alpha_1}D^{\alpha_2}x(t)),$$
$$k_2(t) = H_2(t,x(t),y(t)) + a_2 f_2(y(t)) + b_2 g_2(D^{\alpha_1}D^{\alpha_2}y(t)),$$

$$\lambda_1 = J^{\alpha_4+\alpha_3+\alpha_2+\alpha_1}k_1(1), \quad \lambda_2 = \frac{1}{\Gamma(\alpha_4+\alpha_3+\alpha_2+1)}, \quad \lambda_3 = \frac{1}{\Gamma(\alpha_4+\alpha_3+1)},$$
$$\lambda_4 = J^{\alpha_4+\alpha_3}k_1(1), \quad \lambda_5 = \frac{1}{\Gamma(\alpha_4+\alpha_3-\alpha_1+1)},$$
$$\lambda_6 = \frac{1}{\Gamma(\alpha_4+\alpha_3-\alpha_2-\alpha_1+1)}, \quad \lambda_7 = \frac{1}{\Gamma(\alpha_4+\alpha_3+\alpha_2+\alpha_1+1)},$$
$$\delta_1 = J^{\beta_4+\beta_3+\beta_2+\beta_1}k_2(1), \quad \delta_2 = \frac{1}{\Gamma(\beta_4+\beta_3+\beta_2+1)}, \quad \delta_3 = \frac{1}{\Gamma(\beta_4+\beta_3+1)},$$
$$\delta_4 = J^{\alpha_4+\alpha_3}k_2(1), \quad \delta_5 = \frac{1}{\Gamma(\beta_4+\beta_3-\beta_1+1)},$$
$$\delta_6 = \frac{1}{\Gamma(\beta_4+\beta_3-\beta_2-\beta_1+1)}, \quad \delta_7 = \frac{1}{\Gamma(\beta_4+\beta_3+\beta_2+\beta_1+1)},$$
$$\eta_1 = \frac{-\lambda_1}{\lambda_3}+\frac{\lambda_4}{\lambda_6}, \quad \eta_2 = \frac{\lambda_2}{\lambda_3}-\frac{\lambda_5}{\lambda_6}, \quad \eta_3 = \frac{-\lambda_1}{\lambda_2}+\frac{\lambda_4}{\lambda_5}, \quad \eta_4 = \frac{\lambda_3}{\lambda_2}-\frac{\lambda_6}{\lambda_5},$$
$$\mu_1 = \frac{-\delta_1}{\delta_3}+\frac{\delta_4}{\delta_6}, \quad \mu_2 = \frac{\delta_2}{\delta_3}-\frac{\delta_5}{\delta_6}, \quad \mu_3 = \frac{-\delta_1}{\delta_2}+\frac{\delta_4}{\delta_5}, \quad \mu_4 = \frac{\delta_3}{\delta_2}-\frac{\delta_6}{\lambda_5}.$$

LEMMA 5. *The equation*

$$\begin{cases} D^{\alpha_1}D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}x(t) = k_1(t), \;\; t \in J, \\ D^{\beta_1}D^{\beta_2}D^{\beta_3}D^{\beta_4}y(t) = k_2(t), \end{cases} \tag{4}$$

with the conditions:

$$\begin{cases} x(0) = x(1) = D^{\alpha_1}D^{\alpha_2}x(1) = D^{\alpha_4}x(0) = 0, \\ y(0) = y(1) = D^{\beta_1}D^{\beta_2}y(1) = D^{\beta_4}y(0) = 0, \end{cases} \tag{5}$$

admits as a solution the expression given by:

$$\begin{cases} x(t) = J^{\alpha_4+\alpha_3+\alpha_2+\alpha_1}k_1(t) + \lambda_2\frac{\eta_1}{\eta_2}t^{\alpha_4+\alpha_3+\alpha_2} + \lambda_3\frac{\eta_3}{\eta_4}t^{\alpha_4+\alpha_3}, \\ y(t) = J^{\beta_4+\beta_3+\beta_2+\beta_1}k_2(t) + \delta_2\frac{\mu_1}{\mu_2}t^{\beta_4+\beta_3+\beta_2} + \delta_3\frac{\mu_3}{\mu_4}t^{\beta_4+\beta_3}, \end{cases} \tag{6}$$

such that, $\mu_4 \neq 0,\ \mu_2 \neq 0,\ \eta_4 \neq 0,\ \eta_2 \neq 0.$

Proof. First of all, we have

$$\begin{cases} x(t) = J^{\alpha_4+\alpha_3+\alpha_2+\alpha_1}k_1(t) + J^{\alpha_4+\alpha_3+\alpha_2}c_1 + J^{\alpha_4+\alpha_3}c_2 + J^{\alpha_4}c_3 + c_4, \\ y(t) = J^{\beta_4+\beta_3+\beta_2+\beta_1}k_2(t) + J^{\beta_4+\beta_3+\beta_2}d_1 + J^{\beta_4+\beta_3}d_2 + J^{\beta_4}d_3 + d_4. \end{cases} \tag{7}$$

Thanks to (7), we can write

$$c_4 = c_3 = d_4 = d_4 = 0,$$

$$c_1 = \frac{\eta_1}{\eta_2}, \quad c_2 = \frac{\eta_3}{\eta_4},$$

$$d_1 = \frac{\mu_1}{\mu_2}, \quad d_2 = \frac{\mu_3}{\mu_4}.$$

The solution of (6) is expressed as follows:

$$\begin{cases} x(t) = J^{\alpha_4+\alpha_3+\alpha_2+\alpha_1} k_1(t) + \lambda_2 \frac{\eta_1}{\eta_2} t^{\alpha_4+\alpha_3+\alpha_2} + \lambda_3 \frac{\eta_3}{\eta_4} t^{\alpha_4+\alpha_3}, \\ y(t) = J^{\beta_4+\beta_3+\beta_2+\beta_1} k_2(t) + \delta_2 \frac{\mu_1}{\mu_2} t^{\beta_4+\beta_3+\beta_2} + \delta_3 \frac{\mu_3}{\mu_4} t^{\beta_4+\beta_3}. \end{cases}$$

This ends the proof. $\square$

Let us now be placed in the fixed point theory by considering, first, the following space:

$$X := \{x \in C([0,1],\mathbb{R}),\ D^{\alpha_1} D^{\alpha_2} x \in C([0,1],\mathbb{R})\}.$$

$$Y := \left\{y \in C([0,1],\mathbb{R}),\ D^{\beta_1} D^{\beta_2} y \in C([0,1],\mathbb{R})\right\}.$$

and the norm, for each $0 < \alpha_i \leqslant 1,\ 0 < \beta_i \leqslant 1,\ i = 1,2:$

$$\| x \|_X = \| x \|_\infty + \| D^{\alpha_1} D^{\alpha_2} x \|_\infty, \tag{8}$$

$$\| x \|_\infty = \sup_{t\in[0,1]} | x(t) |, \quad \| D^{\alpha_1} D^{\alpha_2} x \|_\infty = \sup_{t\in[0,1]} | D^{\alpha_1} D^{\alpha_2} x(t) |. \tag{9}$$

Then, we define on Y the norm

$$\| y \|_Y = \| y \|_\infty + \| D^{\beta_1} D^{\beta_2} y \|_\infty, \tag{10}$$

$$\| y \|_\infty = \sup_{t\in[0,1]} | y(t) |, \quad \| D^{\beta_1} D^{\beta_2} y \|_\infty = \sup_{t\in[0,1]} | D^{\beta_1} D^{\beta_2} y(t) |. \tag{11}$$

Finally, we define:

$$\| (x,y) \|_{X\times Y} = \| x \|_X + \| y \|_Y. \tag{12}$$

3. Main results

We consider the following sufficient conditions:

$(H1)$: Suppose the existence of non negative reals numbers R_i, $i = 1,2,3,4$, such that for all $t \in J$ and $(u,v),(w,z) \in \mathbb{R}^2$, we have,

$$| H_1(t,u,v) - H_1(t,w,z) | \leqslant R_1 | u - w | + R_2 | v - z |,$$
$$| H_2(t,u,v) - H_2(t,w,z) | \leqslant R_3 | u - w | + R_4 | v - z |.$$

$(H2)$: There are constants m_i, $i=1,2$ such that for all $t \in J$ and $(u,v) \in \mathbb{R}^2$, we have:

$$\begin{aligned} &\mid f_1(u) - f_1(v) \mid \leqslant m_1 \mid u - v \mid, \\ &\mid f_2(u) - f_2(v) \mid \leqslant m_2 \mid u - v \mid. \end{aligned}$$

$(H3)$: There are constants n_i, $i=1,2$, such that for all $t \in J$ and $(u,v) \in \mathbb{R}^2$, we have:

$$\begin{aligned} &\mid g_1(u) - g_1(v) \mid \leqslant n_1 \mid u - v \mid, \\ &\mid g_2(u) - g_2(v) \mid \leqslant n_2 \mid u - v \mid. \end{aligned}$$

$(H4)$: There exist positive constants Λ_i, $i=1,2,\ldots,6$, such that for all $t \in J$ and $(u,v) \in \mathbb{R}^2$, we have

$$\begin{aligned} \mid H_1(t,u,v) \mid \leqslant \Lambda_1, &\quad \mid H_2(t,u,v) \mid \leqslant \Lambda_2, \\ \mid f_1(u) \mid \leqslant \Lambda_3, &\quad \mid f_2(u) \mid \leqslant \Lambda_4, \\ \mid g_1(u) \mid \leqslant \Lambda_5, &\quad \mid g_2(u) \mid \leqslant \Lambda_6. \end{aligned}$$

Let us finally put the notations:

$$T_1 := \max\left\{(R_1 + a_1 m_1 + b_1 n_1)(\lambda_3 + \lambda_7)\ ;\ R_2(\lambda_3 + \lambda_7)\right\}, \tag{13}$$

$$T_2 := \max\left\{(R_3 + a_2 m_2 + b_2 n_2)(\delta_3 + \delta_7)\ ;\ R_4(\delta_3 + \delta_7)\right\}, \tag{14}$$

$$T := T_1 + T_2. \tag{15}$$

THEOREM 1. *Assume that $(H1)$, $(H2)$ and $(H3)$ are satisfied. If $T < 1$, then,* (1) *admits a unique solution.*

Proof. Consider the operator $F : X \times Y \to X \times Y$ defined by

$$F(x(t),y(t)) = (F_1(x(t),y(t)), F_2(x(t),y(t)))$$

where,

$$\begin{aligned} &F_1(x(t),y(t)) \\ &= J^{\alpha_4+\alpha_3+\alpha_2+\alpha_1}(H_1(t,x(t),y(t)) + a_1 f_1(x(t)) + b_1 g_1(D^{\alpha_1} D^{\alpha_2} x(t))) \\ &\quad + \lambda_2 \frac{\eta_1}{\eta_2} t^{\alpha_4+\alpha_3+\alpha_2} + \lambda_3 \frac{\eta_3}{\eta_4} t^{\alpha_4+\alpha_3}, \end{aligned} \tag{16}$$

$$\begin{aligned} &F_2(x(t),y(t)) \\ &= J^{\beta_4+\beta_3+\beta_2+\beta_1}(H_2(t,x(t),y(t)) + a_2 f_2(y(t)) + b_2 g_2(D^{\beta_1} D^{\beta_2} y(t))) \\ &\quad + \delta_2 \frac{\mu_1}{\mu_2} t^{\beta_4+\beta_3+\beta_2} + \delta_3 \frac{\mu_3}{\mu_4} t^{\beta_4+\beta_3}. \end{aligned} \tag{17}$$

We prove that F is an application that satisfies the Banach contraction principle. We take two arbitrary elements $(x_1,y_1),(x_2,y_2)\in X\times Y$. So, we have

$$\begin{aligned}
&\mid F_1(x_2,y_2)(t)-F_1(x_1,y_1)(t)\mid\\
\leqslant\ & J^{\alpha_4+\alpha_3+\alpha_2+\alpha_1}\mid H_1(t,x_2(t),y_2(t))-H_1(t,x_1(t),y_1(t))\mid\\
&+J^{\alpha_4+\alpha_3+\alpha_2+\alpha_1}a_1\mid f_1(x_2(t))-f_1(x_1(t))\mid\\
&+J^{\alpha_4+\alpha_3+\alpha_2+\alpha_1}b_1\mid g_1(D^{\alpha_1}D^{\alpha_2}x_2(t))-g_1(D^{\alpha_1}D^{\alpha_2}x_1(t))\\
\leqslant\ & \frac{R_1}{\Gamma(\alpha_4+\alpha_3+\alpha_2+\alpha_1+1)}\parallel x_2-x_1\parallel_\infty\\
&+\frac{R_2}{\Gamma(\alpha_4+\alpha_3+\alpha_2+\alpha_1+1)}\parallel y_2-y_1\parallel_\infty\\
&+\frac{a_1m_1}{\Gamma(\alpha_4+\alpha_3+\alpha_2+\alpha_1+1)}\parallel x_2-x_1\parallel_\infty\\
&+\frac{b_1n_1}{\Gamma(\alpha_4+\alpha_3+\alpha_2+\alpha_1+1)}\parallel D^{\alpha_1}D^{\alpha_2}x_2-D^{\alpha_1}D^{\alpha_2}x_1\parallel_\infty.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\parallel (F_1(x_2,y_2)(t)-F_1(x_1,y_1))\parallel_\infty\\
\leqslant\ & \lambda_7(R_1+a_1m_1)\parallel x_2-x_1\parallel_\infty+\lambda_7b_1n_1\parallel D^{\alpha_1}D^{\alpha_2}x_2-D^{\alpha_1}D^{\alpha_2}x_1\parallel_\infty\\
&+\lambda_7R_2\parallel y_2-y_1\parallel_\infty.
\end{aligned}\tag{18}$$

On the other hand, one can state that

$$\begin{aligned}
&\mid D^{\alpha_1}D^{\alpha_2}F_1(x_2,y_2)(t)-D^{\alpha_1}D^{\alpha_2}F_1(x_1,y_1)(t)\mid\\
\leqslant\ & J^{\alpha_4+\alpha_3}\mid H_1(t,x_2(t),y_2(t))-H_1(t,x_1(t),y_1(t))\mid\\
&+J^{\alpha_4+\alpha_3}a_1\mid f_1(x_2(t))-f_1(x_1(t))\mid\\
&+J^{\alpha_4+\alpha_3}b_1\mid g_1(D^{\alpha_1}D^{\alpha_2}x_2(t)))-g_1(D^{\alpha_1}D^{\alpha_2}x_1(t)))\\
\leqslant\ & \frac{R_1}{\Gamma(\alpha_4+\alpha_3+1)}\parallel x_2-x_1\parallel_\infty\\
&+\frac{R_2}{\Gamma(\alpha_4+\alpha_3+1)}\parallel y_2-y_1\parallel_\infty\\
&+\frac{a_1m_1}{\Gamma(\alpha_4+\alpha_3+1)}\parallel x_2-x_1\parallel_\infty\\
&+\frac{b_1n_1}{\Gamma(\alpha_4+\alpha_3+1)}\parallel D^{\alpha_1}D^{\alpha_2}x_2-D^{\alpha_1}D^{\alpha_2}x_1\parallel_\infty,
\end{aligned}$$

$$\begin{aligned}
&\parallel D^{\alpha_1}D^{\alpha_2}F_1(x_2,y_2)-D^{\alpha_1}D^{\alpha_2}F_1(x_1,y_1)\parallel_\infty\\
\leqslant\ & \lambda_3(R_1+a_1m_1)\parallel x_2-x_1\parallel_\infty+\lambda_3b_1n_1\parallel D^{\alpha_1}D^{\alpha_2}x_2-D^{\alpha_1}D^{\alpha_2}x_1\parallel_\infty\\
&+\lambda_3R_2\parallel y_2-y_1\parallel_\infty.
\end{aligned}\tag{19}$$

Thanks to (13), (18), (19) and (11), we find

$$\begin{aligned}
&\| F_1(x_2,y_2)(t)-F_1(x_1,y_1) \| \\
=\ & \| F_1(x_2,y_2)-F_1(x_1,y_1) \|_\infty + \| D^{\alpha_1}D^{\alpha_2}F_1(x_2)-D^{\alpha_1}D^{\alpha_2}F_1(x_1) \|_\infty \\
\leqslant\ & (R_1+a_1m_1)(\lambda_3+\lambda_7) \| x_2-x_1 \|_\infty \\
& +(b_1n_1)(\lambda_3+\lambda_7) \| x_2-x_1 \|_\infty + R_2(\lambda_3+\lambda_7) \| y_2-y_1 \|_\infty \\
\leqslant\ & (R_1+a_1m_1+b_1n_1)(\lambda_3+\lambda_7) \| x_2-x_1 \|_X \\
& +R_2(\lambda_3+\lambda_7) \| y_2-y_1 \|_Y .
\end{aligned} \tag{20}$$

Hence, we obtain

$$\| F_1(x_2,y_2)(t)-F_1(x_1,y_1) \|_X \leqslant T_1 \left[\| x_2-x_1 \|_X + \| y_2-y_1 \|_Y\right]. \tag{21}$$

In the same way, we have the following two inequalities

$$\begin{aligned}
&\| F_2(x_2,y_2)(t)-F_2(x_1,y_1) \|_\infty \\
\leqslant\ & \delta_7(R_3+a_2m_2) \| y_2-y_1 \|_\infty + \delta_7 b_2n_2 \| D^{\beta_1}D^{\beta_2}y_2-D^{\beta_1}D^{\beta_2}y_1 \|_\infty \\
& +\delta_7R_4 \| x_2-x_1 \|_\infty
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
&\| D^{\beta_1}D^{\beta_2}F_2(x_2,y_2)-D^{\beta_1}D^{\beta_2}F_2(x_1,y_1) \|_\infty \\
\leqslant\ & \delta_3(R_3+a_2m_2) \| y_2-y_1 \|_\infty + \delta_3 b_2n_2 \| D^{\beta_1}D^{\beta_2}y_2-D^{\beta_1}D^{\beta_2}y_1 \|_\infty \\
& +\delta_3R_4 \| x_2-x_1 \|_\infty .
\end{aligned} \tag{23}$$

Thanks to (14), (22), (23) and (12), we get

$$\| F_2(x_2,y_2)(t)-F_2(x_1,y_1) \|_Y \leqslant T_2 \left[\| (y_2-y_1) \|_Y + \| (x_2-x_1) \|_X\right]. \tag{24}$$

Therefore,

$$\| F(x_2,y_2)-F(x_1,y_1) \|_{X\times Y} \leqslant T \| (x_2,y_2)-(x_1,y_1) \|_{X\times Y} . \tag{25}$$

We have then proved that F is contractive which achieves the proof. We present to the reader the following theorem that concerns the existence of at least a solution. Before doing that we need the notations:

$$\Theta_1 := \lambda_7(\Lambda_1+a_1\Lambda_3+b_1\Lambda_5)+\lambda_2\frac{\eta_1}{\eta_2}+\lambda_3\frac{\eta_3}{\eta_4},$$

$$\Theta_2 := \lambda_3(\Lambda_1+a_1\Lambda_3+b_1\Lambda_5)+\lambda_5\frac{\eta_1}{\eta_2}+\lambda_6\frac{\eta_3}{\eta_4},$$

$$\Theta_3 := \delta_7(\Lambda_2+a_2\Lambda_4+b_2\Lambda_6)+\delta_2\frac{\mu_1}{\mu_2}+\delta_3\frac{\mu_3}{\mu_4},$$

$$\Theta_4 := \delta_3(\Lambda_2+a_2\Lambda_4+b_2\Lambda_6)+\delta_5\frac{\mu_1}{\mu_2}+\delta_6\frac{\mu_3}{\mu_4}. \quad \square$$

THEOREM 2. *Suppose that* $(H1),(H2)$ *and* $(H4)$ *are satisfied, and* $T<1$. *If there exists* $\rho>0$ *such that*

$$\Theta_1+\Theta_2+\Theta_3+\Theta_4 \leqslant \rho, \tag{26}$$

then problem (1) *has at least one solution.*

Proof. We define the following operators:

$$F_1 := P_1+Q_1, F_2 := P_2+Q_2,$$

$$F := (F_1,F_2) = P+Q, P := (P_1,P_2), Q := (Q_1,Q_2).$$

For $(x_1,x_2)\in X\times Y$ and $t\in J$,

$$P_1(x,y)(t) = J^{\alpha_4+\alpha_3+\alpha_2+\alpha_1}(H_1(t,x(t),y(t))+a_1f_1(x(t)))+\lambda_2\frac{\eta_1}{\eta_2}t^{\alpha_4+\alpha_3+\alpha_2}, \tag{27}$$

$$Q_1(x,y)(t) = J^{\alpha_4+\alpha_3+\alpha_2+\alpha_1}b_1g_1(D^{\alpha_1}D^{\alpha_2}x(t))+\lambda_3\frac{\eta_3}{\eta_4}t^{\alpha_4+\alpha_3}, \tag{28}$$

$$P_2(x,y)(t) = J^{\beta_4+\beta_3+\beta_2+\beta_1}(H_2(t,x(t),y(t))+a_2f_2(y(t)))+\delta_2\frac{\mu_1}{\mu_2}t^{\beta_4+\beta_3+\beta_2}, \tag{29}$$

and

$$Q_2(x,y)(t) = J^{\beta_4+\beta_3+\beta_2+\beta_1}b_2g_2(D^{\beta_1}D^{\beta_2}y(t))+\delta_3\frac{\mu_3}{\mu_4}t^{\beta_4+\beta_3}. \tag{30}$$

• Let $B=\{(x,y)\in X\times Y : \| (x,y) \|_{X\times Y}\leqslant \rho\}$. We will prove that $P(x_1,y_1)+Q(x_2,y_2)\in B$, for any $(x_1,y_1),(x_2,y_2)\in B$. Let $(x_1,y_1),(x_2,y_2)\in B$ and $t\in J$. We have

$$\begin{aligned}
&\mid P_1(x_1,y_1)+Q_1(x_2,y_2)\mid \\
&\leqslant J^{\alpha_4+\alpha_3+\alpha_2+\alpha_1}(\mid H_1(t,x_1(t),y_1(t))\mid + \mid a_1f_1(x_1(t))\mid)+\lambda_2\frac{\eta_1}{\eta_2} \\
&\quad + J^{\alpha_4+\alpha_3+\alpha_2+\alpha_1}\mid b_1g_1(D^{\alpha_1}D^{\alpha_2}x_2(t))\mid +\lambda_3\frac{\eta_3}{\eta_4} \\
&\leqslant \lambda_7(\Lambda_1+a_1\Lambda_3+b_1\Lambda_5)+\lambda_2\frac{\eta_1}{\eta_2}+\lambda_3\frac{\eta_3}{\eta_4}.
\end{aligned}$$

So,

$$\| P_1(x_1,y_1)+Q_1(x_2,y_2) \|_\infty \leqslant \lambda_7(\Lambda_1+a_1\Lambda_3+b_1\Lambda_5)+\lambda_2\frac{\eta_1}{\eta_2}+\lambda_3\frac{\eta_3}{\eta_4}$$

and

$$\begin{aligned}
&\mid D^{\alpha_1}D^{\alpha_2}P_1(x_1,y_1)(t)+D^{\alpha_1}D^{\alpha_2}Q_1(x_2,y_2)(t)\mid \\
&\leqslant J^{\alpha_4+\alpha_3}(\mid H_1(t,x_1(t),y_1(t))\mid + \mid a_1f_1(x_1(t))\mid)+\lambda_5\frac{\eta_1}{\eta_2} \\
&\quad + J^{\alpha_4+\alpha_3}\mid b_1g_1(D^{\alpha_1}D^{\alpha_2}x_2(t))\mid +\lambda_6\frac{\eta_3}{\eta_4} \\
&\leqslant \lambda_3(\Lambda_1+a_1\Lambda_3+b_1\Lambda_5)+\lambda_5\frac{\eta_1}{\eta_2}+\lambda_6\frac{\eta_3}{\eta_4}.
\end{aligned}$$

Hence,

$$\begin{aligned}&\| D^{\alpha_1} D^{\alpha_2} P_1(x_1,y_1)(t) + D^{\alpha_1} D^{\alpha_2} Q_1(x_2,y_2)(t) \|_\infty \\ \leqslant\ & \lambda_3(\Lambda_1 + a_1\Lambda_3 + b_1\Lambda_5) + \lambda_5 \frac{\eta_1}{\eta_2} + \lambda_6 \frac{\eta_3}{\eta_4}.\end{aligned}$$

Then, it yields that

$$\| P_1(x_1,y_1) + Q_1(x_2,y_2) \|_X \leqslant \Theta_1 + \Theta_2.$$

In the same way, we find both

$$\| P_2(x_1,y_1) + Q_2(x_2,y_2) \|_\infty \leqslant \delta_7(\Lambda_2 + a_2\Lambda_4 + b_2\Lambda_6) + \delta_2 \frac{\mu_1}{\mu_2} + \delta_3 \frac{\mu_3}{\mu_4}$$

and

$$\begin{aligned}&\| D^{\beta_1} D^{\beta_2} P_2(x_1,y_1)(t) + D^{\alpha_1} D^{\alpha_2} Q_2(x_2,y_2)(t) \|_\infty \\ \leqslant\ & \delta_3(\Lambda_2 + a_2\Lambda_4 + b_2\Lambda_6) + \delta_5 \frac{\mu_1}{\mu_2} + \delta_6 \frac{\mu_3}{\mu_4}.\end{aligned}$$

Consequently,

$$\| P_2(x_1,y_1) + Q_2(x_2,y_2) \|_Y \leqslant \Theta_3 + \Theta_4.$$

This implies that the following inequality is valid:

$$\| P(x_1,y_1) + Q(x_2,y_2) \|_{X\times Y} \leqslant \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 \leqslant \rho,$$

which ends the proof of the fact that $P(x_1,y_1) + Q(x_2,y_2) \in B$ for any $(x_1,y_1),(x_2,y_2) \in B$.

- Now, we will prove that P is a contraction mapping on $X \times Y$.

Let $(x_1,y_1),(x_2,y_2) \in X \times Y$ and $t \in J$. We have

$$\begin{aligned}&| P_1(x_2,y_2) - P_1(x_1,y_1) | \\ \leqslant\ & J^{\alpha_4+\alpha_3+\alpha_2+\alpha_1} | (H_1(t,x_2(t),y_2(t)) - H_1(t,x_1(t),y_1(t)) | \\ &+ J^{\alpha_4+\alpha_3+\alpha_2+\alpha_1} | a_1 f_1(x_2(t)) - a_1 f_1(x_1(t)) | \\ \leqslant\ & \lambda_7(R_1 | x_2 - x_1 | + R_2 | y_2 - y_1 | + \lambda_7 m_1 a_1 | x_2 - x_1 | \\ \leqslant\ & \lambda_7(R_1 + a_1 m_1) | x_2 - x_1 | + \lambda_7 R_2 | y_2 - y_1 |,\end{aligned}$$

$$\begin{aligned}&\| P_1(x_2,y_2) - P_1(x_1,y_1) \|_\infty \\ \leqslant\ & \lambda_7(R_1 + a_1 m_1) \| x_2 - x_1 \|_\infty + \lambda_7 R_2 \| y_2 - y_1 \|_\infty\end{aligned}$$

and

$$\begin{aligned}&| D^{\alpha_1} D^{\alpha_2} P_1(x_2,y_2) - D^{\alpha_1} D^{\alpha_2} P_1(x_1,y_1) | \\ \leqslant\ & J^{\alpha_4+\alpha_3} | (H_1(t,x_2(t),y_2(t)) - H_1(t,x_1(t),y_1(t)) | \\ &+ J^{\alpha_4+\alpha_3} | a_1 f_1(x_2(t)) - a_1 f_1(x_1(t)) | \\ \leqslant\ & \lambda_3 R_1 | x_2 - x_1 | + \lambda_3 R_2 | y_2 - y_1 | + \lambda_3 m_1 a_1 | x_2 - x_1 | \\ \leqslant\ & \lambda_3(R_1 + a_1 m_1) | x_2 - x_1 | + \lambda_3 R_2 | y_2 - y_1 |,\end{aligned}$$

$$\begin{aligned}&\| D^{\alpha_1}D^{\alpha_2}P_1(x_2,y_2)-D^{\alpha_1}D^{\alpha_2}P_1(x_1,y_1)\|_\infty\\ &\leqslant \lambda_3(R_1+a_1m_1)\| x_2-x_1\|_\infty+\lambda_3R_2\| y_2-y_1\|_\infty .\end{aligned}$$

Therefore,

$$\begin{aligned}&\| P_1(x_2,y_2)-P_1(x_1,y_1)\|_X\\ &\leqslant (\lambda_7+\lambda_3)(R_1+a_1m_1)\| x_2-x_1\|_X+R_2(\lambda_7+\lambda_3)\| y_2-y_1\|_Y .\end{aligned}$$

Then,

$$\| P_1(x_2,y_2)-P_1(x_1,y_1)\|_X\leqslant l_1(\| x_2-x_1\|_X+\| y_2-y_1\|_Y),$$

where, $l_1=\max\{(\lambda_7+\lambda_3)(R_1+a_1m_1);R_2(\lambda_7+\lambda_3)\}$. With the same arguments as before, we have

$$\| P_2(x_2,y_2)-P_2(x_1,y_1)\|_Y\leqslant l_2(\| x_2-x_1\|_X+\| y_2-y_1\|_Y),$$

where, $l_2=\max\{(\delta_7+\delta_3)(R_3+a_2m_2);R_4(\delta_7+\delta_3)\}$. Therefore,

$$\| P(x_2,y_2)-P(x_1,y_1)\|_{X\times Y}\leqslant (l_1+l_2)(\| (x_2,y_2)-(x_1,y_1)\|_{X\times Y} .$$

Using Theorem 1 and remarking that $l_1+l_2<T$, we conclude that P is contractive.

We will prove that Q is continuous. Let $(x_n,y_n)_n$ be a sequence, such that $(x_n,y_n)\to(x,y)$ in $X\times Y$. For $t\in J$, we have

$$\begin{aligned}&| Q_1(x_n,y_n)-Q_1(x,y)|\\ &\leqslant J^{\alpha_4+\alpha_3+\alpha_2+\alpha_1}b_1g_1\,| D^{\alpha_1}D^{\alpha_2}(x_n(t))-x(t))|\\ &\leqslant b_1\lambda_7\| D^{\alpha_1}D^{\alpha_2}(x_n(t))-x(t))\|_\infty\end{aligned}$$

and

$$\begin{aligned}&| D^{\alpha_1}D^{\alpha_2}Q_1(x_n,y_n)-D^{\alpha_1}D^{\alpha_2}Q_1(x,y)|\\ &\leqslant J^{\alpha_4+\alpha_3}b_1g_1\,| D^{\alpha_1}D^{\alpha_2}(x_n(t))-x(t))|\\ &\leqslant b_1\lambda_3\| D^{\alpha_1}D^{\alpha_2}(x_n(t))-x(t))\|_\infty .\end{aligned}$$

Hence, we obtain

$$\| Q_1(x_n,y_n)-Q_1(x,y)\|_X\leqslant b_1(\lambda_7+\lambda_3)\| D^{\alpha_1}D^{\alpha_2}(x_n(t))-x(t))\|_X . \tag{31}$$

Also, we have

$$\| Q_2(x_n,y_n)-Q_2(x,y)\|_Y\leqslant b_2(\delta_7+\delta_3)\| D^{\alpha_1}D^{\alpha_2}(y_n(t))-y(t))\|_Y . \tag{32}$$

Thanks to (31) and (32), we can write

$$\| Q(x_n,y_n)-Q(x,y)\|_{X\times Y}\leqslant (b_1(\lambda_7+\lambda_3)+b_2(\delta_7+\delta_3))\| D^{\alpha_1}D^{\alpha_2}((x_n,y_n)-(x,y))\|_{X\times Y} .$$

Therefore, $\| Q(x_n,y_n)-Q(x,y) \|_{X\times Y} \to 0$ as $\| (x_n,y_n)-(x,y) \|_{X\times Y} \to 0$. This means that Q is continuous. We prove that QB is a bounded subset of $X\times Y$. Let $(x,y)\in B$ and $t\in J$. We have

$$\begin{aligned} | Q_1(x(t),y(t)) | &\leqslant J^{\alpha_4+\alpha_3+\alpha_2+\alpha_1} b_1 g_1 \mid D^{\alpha_1} D^{\alpha_2}(x(t)) \mid +\lambda_3 \frac{\eta_3}{\eta_4} t^{\alpha_4+\alpha_3} \\ &\leqslant b_1\lambda_7\Lambda_5+\lambda_3\frac{\eta_3}{\eta_4} \end{aligned}$$

and

$$\begin{aligned} | D^{\alpha_1}D^{\alpha_2}Q_1(x(t),y(t)) | &\leqslant J^{\alpha_4+\alpha_3} b_1 g_1 \mid D^{\alpha_1} D^{\alpha_2}(x(t)) \mid +\lambda_6 \frac{\eta_3}{\eta_4} t^{\alpha_4+\alpha_3-\alpha_2-\alpha_1} \\ &\leqslant b_1\lambda_3\Lambda_5+\lambda_6\frac{\eta_3}{\eta_4}. \end{aligned}$$

Therefore,

$$\| Q_1(x,y) \|_X \leqslant b_1\Lambda_5(\lambda_7+\lambda_3)+(\lambda_3+\lambda_6)\frac{\eta_3}{\eta_4}. \tag{33}$$

In the same way, we obtain

$$\| Q_2(x(t),y(t)) \|_Y \leqslant b_2\Lambda_6(\delta_7+\delta_3)+(\delta_3+\delta_6)\frac{\mu_3}{\mu_4} \tag{34}$$

and using (33) and (34), we find

$$\| Q(x,y) \|_{X\times Y} \leqslant b_1\Lambda_5(\lambda_7+\lambda_3)+(\lambda_3+\lambda_6)\frac{\eta_3}{\eta_4}+b_2\Lambda_6(\delta_7+\delta_3)+(\delta_3+\delta_6)\frac{\mu_3}{\mu_4}\leqslant\rho.$$

Thus QB is a bounded subset of $X\times Y$. Now, we prove that Q is equicontinuous. Let $(x,y)\in X\times Y$ and $t_1,t_2\in J$, with $t_1<t_2$. We have

$$\begin{aligned} &| Q_1(x(t_2),y(t_2))-Q_1(x(t_1),y(t_1)) | \\ &\leqslant \frac{b_1}{\Gamma(\alpha_4+\alpha_3+\alpha_2+\alpha_1)}\int_0^{t_2}(t_2-s)^{\alpha_4+\alpha_3+\alpha_2+\alpha_1-1} \mid g_1(D^{\alpha_1}D^{\alpha_2}x(t_2)) \mid ds \\ &\quad -\frac{b_1}{\Gamma(\alpha_4+\alpha_3+\alpha_2+\alpha_1)}\int_0^{t_1}(t_1-s)^{\alpha_4+\alpha_3+\alpha_2+\alpha_1-1} \mid g_1(D^{\alpha_1}D^{\alpha_2}x(t_2)) \mid ds \\ &\quad +\lambda_3\frac{\eta_3}{\eta_4} \mid t_2^{\alpha_4+\alpha_3}-t_1^{\alpha_4+\alpha_3} \mid \\ &\leqslant \lambda_7 b_1\Lambda_5(t_2^{\alpha_4+\alpha_3+\alpha_2+\alpha_1}-t_1^{\alpha_4+\alpha_3+\alpha_2+\alpha_1})+\lambda_3\frac{\eta_3}{\eta_4}(t_2^{\alpha_4+\alpha_3}-t_1^{\alpha_4+\alpha_3}) \end{aligned} \tag{35}$$

and

$$
\begin{aligned}
&| D^{\alpha_1} D^{\alpha_2} Q_1(x(t_2), y(t_2)) - D^{\alpha_1} D^{\alpha_2} Q_1(x(t_1), y(t_1)) | \\
\leqslant\ & \frac{b_1}{\Gamma(\alpha_4+\alpha_3)} \int_0^{t_2} (t_2 - s)^{\alpha_4+\alpha_3-1} | g_1(D^{\alpha_1} D^{\alpha_2} x(t_2)) | ds \\
&- \frac{b_1}{\Gamma(\alpha_4+\alpha_3)} \int_0^{t_1} (t_1 - s)^{\alpha_4+\alpha_3-1} | g_1(D^{\alpha_1} D^{\alpha_2} x(t_2)) | ds \\
&+ \lambda_6 \frac{\eta_3}{\eta_4} | t_2^{\alpha_4+\alpha_3-\alpha_2-\alpha_1} - t_1^{\alpha_4+\alpha_3-\alpha_2-\alpha_1} | \\
\leqslant\ & \lambda_3 b_1 \Lambda_5 (t_2^{\alpha_4+\alpha_3} - t_1^{\alpha_4+\alpha_3}) \\
&+ \lambda_6 \frac{\eta_3}{\eta_4} (t_2^{\alpha_4+\alpha_3-\alpha_2-\alpha_1} - t_1^{\alpha_4+\alpha_3-\alpha_2-\alpha_1}).
\end{aligned} \tag{36}
$$

With the same arguments as before, we observe that the following inequalities

$$
\begin{aligned}
&| Q_2(x(t_2), y(t_2)) - Q_2(x(t_1), y(t_1)) | \\
\leqslant\ & \frac{b_2}{\Gamma(\beta_4+\beta_3+\beta_2+\beta_1)} \int_0^{t_2} (t_2 - s)^{\beta_4+\beta_3+\beta_2+\beta_1-1} | g_2(D^{\beta_1} D^{\beta_2} y(t_2)) | ds \\
&- \frac{b_2}{\Gamma(\beta_4+\beta_3+\beta_2+\beta_1)} \int_0^{t_1} (t_1 - s)^{\beta_4+\beta_3+\beta_2+\beta_1-1} | g_2(D^{\beta_1} D^{\beta_2} x(t_2)) | ds \\
&+ \delta_3 \frac{\mu_3}{\mu_4} | t_2^{\beta_4+\beta_3} - t_1^{\beta_4+\beta_3} | \\
\leqslant\ & \delta_7 b_2 \Lambda_6 (t_2^{\beta_4+\beta_3+\beta_2+\beta_1} - t_1^{\beta_4+\beta_3+\beta_2+\beta_1}) + \delta_3 \frac{\mu_3}{\mu_4} (t_2^{\beta_4+\beta_3} - t_1^{\beta_4+\beta_3})
\end{aligned} \tag{37}
$$

and

$$
\begin{aligned}
&| D^{\beta_1} D^{\beta_2} Q_2(x(t_2), y(t_2)) - D^{\beta_1} D^{\beta_2} Q_2(x(t_1), y(t_1)) | \\
\leqslant\ & \frac{b_2}{\Gamma(\beta_4+\beta_3)} \int_0^{t_2} (t_2 - s)^{\beta_4+\beta_3-1} | g_2(D^{\beta_1} D^{\beta_2} x(t_2)) | ds \\
&- \frac{b_2}{\Gamma(\beta_4+\beta_3)} \int_0^{t_1} (t_1 - s)^{\beta_4+\beta_3-1} | g_2(D^{\beta_1} D^{\beta_2} x(t_2)) | ds \\
&+ \delta_6 \frac{\mu_3}{\mu_4} | t_2^{\beta_4+\beta_3-\beta_2-\beta_1} - t_1^{\beta_4+\beta_3-\beta_2-\beta_1} | \\
\leqslant\ & \delta_3 b_1 \Lambda_6 (t_2^{\beta_4+\beta_3} - t_1^{\beta_4+\beta_3}) + \delta_6 \frac{\mu_3}{\mu_4} (t_2^{\beta_4+\beta_3-\beta_2-\beta_1} - t_1^{\beta_4+\beta_3-\beta_2-\beta_1})
\end{aligned} \tag{38}
$$

are satisfied.

Under the conditions $t_1 \to t_2$, one can observe that (35)–(36)–(37)–(38) tend to 0. Then Q is equicontinuous. Thanks to the fixed-point theorem of Krasnoselskii, we state that problem (1) has a solution. □

EXAMPLE 1. We consider (1) under the following particular cases:

$$
\begin{aligned}
& f_1(x)=\frac{1}{x+3}, \quad g_1(x)=\frac{1}{x^2+3}, \quad H_1(t,x,y)=\frac{e^{-2t}}{t+10}x+\frac{\cos t}{12e^t}y, \\
& \quad a_1=0.1, \quad b_1=0.2, \\
& f_2(y)=\frac{1}{y+5}, \quad g_2(y)=\frac{1}{y^2+9}, \quad H_2(t,x,y)=\frac{e^{-t}}{t^2+15}x+\frac{e^{-3t}\sin t}{10}y, \\
& \quad a_2=\frac{1}{2}, \quad b_2=\frac{1}{3}, \\
& \quad \alpha_1=0.5, \quad \alpha_2=0.55, \quad \alpha_3=0.52, \quad \alpha_4=0.58, \\
& \quad \beta_1=0.37, \quad \beta_2=0.72, \quad \beta_3=0.31, \quad \beta_4=0.8,
\end{aligned}
$$

we find

$$
\begin{aligned}
& R_1=\frac{1}{10}, \quad R_2=\frac{1}{12}, \quad R_3=\frac{1}{3}, \quad R_4=\frac{1}{10}, \\
& m_1=\frac{1}{9}, \quad m_2=\frac{1}{4}, \quad n_1=\frac{1}{4}, \quad n_2=\frac{1}{9}, \\
& T_1=\max\{0.2237;0.1157\}=0.2237, \\
& T_2=\max\{0.1363;0.1685\}=0.1685, \\
& T=T_1+T_2=0.3922<1.
\end{aligned}
$$

Therefore, by Theorem 1, we state that the above example has a unique solution.

EXAMPLE 2. We consider (1), with the conditions:

$$
\begin{aligned}
& f_1(x)=\frac{e^{-3t}}{x^2+8}, \quad g_1(x)=\frac{e^{-t^2}}{e^t+16}x, \quad H_1(t,x,y)=\frac{\sin t+\cos t}{t+20}x+\frac{\sin t}{t^2+10}y, \\
& \quad a_1=0.1, \quad b_1=0.3, \\
& f_2(y)=\frac{1}{7(e^t+1)}y, \quad g_2(y)=\frac{e^{-t^2}}{t^2+2}, \quad H_2(t,x,y)=\frac{3\sin t\cos t}{e^{-t}+15}x+\frac{\cos t}{e^{t^2}+10}y, \\
& \quad a_2=\frac{1}{5}, \quad b_2=\frac{1}{7}, \\
& \quad \alpha_1=0.51, \quad \alpha_2=0.57, \quad \alpha_3=0.53, \quad \alpha_4=0.58, \\
& \quad \beta_1=0.36, \quad \beta_2=0.7, \quad \beta_3=0.38, \quad \beta_4=0.71.
\end{aligned}
$$

We have

$$
\begin{aligned}
& R_1=\frac{1}{10}, \quad R_2=0.1, \quad R_3=0.2, \quad R_4=\frac{1}{10}, \\
& m_1=\frac{1}{8}, \quad m_2=\frac{1}{7}, \quad n_1=\frac{1}{16}, \quad n_2=\frac{1}{7}, \\
& T_1=\max\{0.1794;0.1367\}=0.1794, \\
& T_2=\max\{0.4180;0.1393\}=0.4180, \\
& T=T_1+T_2=0.5974<1.
\end{aligned}
$$

Thanks to Theorem 1, we state that the above example has a unique solution.

4. Ulam stability results

We start this section by presenting the Ulam-Hyers stability definitions.Then, we prove some results regarding the introduced concepts.

DEFINITION 3. The System (1) has the Ulam Hyers stability if there exists a real number $\Theta_{H,f,g} > 0$, such that for all: $\varepsilon_1, \varepsilon_2 > 0$, $:t \in J$ and for each $(x;y) \in X \times Y$ solution of the inequality

$$\begin{cases} D^{\alpha_1}D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}x(t) - H_1(t,x(t),y(t)) - a_1 f_1(x(t)) - b_1 g_1(D^{\alpha_1}D^{\alpha_2}x(t)) \leqslant \varepsilon_1, \\ \\ D^{\beta_1}D^{\beta_2}D^{\beta_3}D^{\beta_4}y(t) - H_2(t,x(t),y(t)) - a_2 f_2(y(t)) - b_2 g_2(D^{\beta_1}D^{\beta_2}y(t)) \leqslant \varepsilon_2, \end{cases} \tag{39}$$

under the conditions:

$$\begin{cases} x(0) = x(1) = D^{\alpha_1}D^{\alpha_2}x(1) = D^{\alpha_4}x(0) = 0, \\ \\ y(0) = y(1) = D^{\beta_1}D^{\beta_2}y(1) = D^{\beta_4}y(0) = 0, \end{cases} \tag{40}$$

there exists $(x^*;y^*) \in X \times Y$ a solution of system (1) such that

$$\| (x - x^*, y - y^*) \|_{X\times Y} \leqslant \varepsilon\Theta_{H,f,g},\ \varepsilon > 0.$$

DEFINITION 4. The System (1) has the Ulam Hyers stability in the generalized sense if there is $\nabla_{H,f,g} \in C(\mathbb{R}^+,\mathbb{R}^+)$; $\nabla_{H,f,g}(0) = 0$ such that for all: $\varepsilon > 0$, : and for each $(x;y) \in X \times Y$ solution of (39)–(40), there exists $(x^*;y^*) \in X \times Y$ a solution of system (1) such that

$$\| (x - x^*, y - y^*) \|_{X\times Y} \leqslant \nabla_{H,f,g}(\varepsilon).$$

THEOREM 3. *If the conditions of Theorem 1 are satisfied, then problem (1) is Ulam Hyers stable.*

Proof. Let $(x;y) \in X \times Y$ be a solution of (39)–(40), and let, by Theorem 1 $(x^*;y^*) \in X \times Y$ be the unique solution of (1). We integrate (39), we can write

$$\begin{aligned} &| x(t) - \lambda_7 \int_0^t (1-\nu)^{\alpha_4+\alpha_3+\alpha_2+\alpha_1-1} (H_1(t,x(t),y(t)) \\ &+ a_1 f_1(x(t)) + b_1 g_1(D^{\alpha_1}D^{\alpha_2}x(t)))d\nu \\ &- \lambda_2 \frac{\eta_1}{\eta_2} t^{\alpha_4+\alpha_3+\alpha_2} - \lambda_3 \frac{\eta_3}{\eta_4} t^{\alpha_4+\alpha_3} \ |: \leqslant \lambda_7 \varepsilon_1, \end{aligned} \tag{41}$$

and

$$\begin{aligned} &| y(t) - \delta_7 \int_0^t (1-\nu)^{\beta_4+\beta_3+\beta_2+\beta_1-1} (H_2(t,x(t),y(t)) + a_2 f_2(x(t)) \\ &+ b_2 g_2(D^{\beta_1}D^{\beta_2}x(t)))d\nu - \lambda_2 \frac{\mu_1}{\mu_2} t^{\beta_4+\beta_3+\beta_2} - \delta_3 \frac{\mu_3}{\mu_4} t^{\beta_4+\beta_3} \ |: \leqslant \delta_7 \varepsilon_2, \end{aligned} \tag{42}$$

Using (39), (41) and (42), we have

$$\begin{aligned}\| x-x^* \|_\infty \leqslant{}& \varepsilon_1\lambda_7+\lambda_7(R_1+a_1m_1) \| x-x^* \|_\infty \\ &+\lambda_7R_2 \| y-y^* \|_\infty +\lambda_7 b_1n_1 \| D^{\alpha_1}D^{\alpha_2}x-D^{\alpha_1}D^{\alpha_2}x^* \|_\infty,\end{aligned}$$

and

$$\begin{aligned}\| y-y^* \|_\infty \leqslant{}& \varepsilon_2\delta_7+\delta_7(R_4+a_2m_2) \| y-y^* \|_\infty \\ &+\delta_7R_3 \| x-x^* \|_\infty +\delta_7 b_2n_2 \| D^{\beta_1}D^{\beta_2}y-D^{\beta_1}D^{\beta_2}y^* \|_\infty .\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\| D^{\alpha_1}D^{\alpha_2}x-D^{\alpha_1}D^{\alpha_2}x^* \|_\infty \\ &\leqslant \varepsilon_1\lambda_3+\lambda_3(R_1+a_1m_1) \| x-x^* \|_\infty +\lambda_3R_2 \| y-y^* \|_\infty \\ &\quad +\lambda_3 b_1n_1 \| D^{\alpha_1}D^{\alpha_2}x-D^{\alpha_1}D^{\alpha_2}x^* \|_\infty,\end{aligned}$$

and

$$\begin{aligned}&\| D^{\beta_1}D^{\beta_2}y-D^{\beta_1}D^{\beta_2}y^* \|_\infty \\ &\leqslant \varepsilon_2\delta_3+\delta_3(R_4+a_2m_2) \| y-y^* \|_\infty +\delta_3R_3 \| x-x^* \|_\infty \\ &\quad +\delta_3 b_2n_2 \| D^{\beta_1}D^{\beta_2}y-D^{\beta_1}D^{\beta_2}y^* \|_\infty .\end{aligned}$$

So, it yields that

$$\begin{aligned}\| x-x^* \|_X \leqslant{}& \varepsilon_1(\lambda_7+\lambda_3)+(\lambda_7+\lambda_3)(R_1+a_1m_1+b_1n_1) \| x-x^* \|_X \\ &+(\lambda_7+\lambda_3)R_2 \| y-y^* \|_Y\end{aligned}$$

and

$$\begin{aligned}\| y-y^* \|_Y \leqslant{}& \varepsilon_2(\delta_7+\delta_3)+(\delta_7+\delta_3)(R_4+a_2m_2+b_2n_2) \| y-y^* \|_Y \\ &+(\delta_7+\delta_3)R_3 \| x-x^* \|_X,\end{aligned}$$

$$\| (x-x^*,y-y^*) \|_{X\times Y}\leqslant \varepsilon\Xi+T \| (x-x^*,y-y^*) \|_{X\times Y},$$

where

$$\varepsilon=\max\{\varepsilon_1,\varepsilon_2\}$$

and

$$\Xi=\max\{(\lambda_7+\lambda_3) :,: (\delta_7+\delta_3)\},$$

Hence,

$$\| (x-x^*,y-y^*) \|_{X\times Y}\leqslant \frac{\varepsilon\Xi}{1-T} := \varepsilon\Theta_{H,f,g}, \quad \Theta_{H,f,g}=\frac{\Xi}{1-T}$$

Thus, the solution of (1) is Ulam Hyers stable. □

REMARK 1. If we consider the case $\nabla_{H,f,g}(\varepsilon)=\frac{\varepsilon\Xi}{1-T}$, then, we obtain the generalised Ulam Hyers stability for (1).

5. Conclusion

We have analyzed a coupled system of sequential differential equations in the sense of Caputo. We first established the existence of a unique solution for the sequential differential system. Subsequently, we extended our investigation to explore the existence of at least one solution for the same system. Our analysis and examples presented in this paper support the existence of solutions to various hypotheses that have been imposed in the paper. The obtained results have implications for applications in diverse fields, engineering, and mathematical modeling, where sequential systems play a crucial role.

Further research can build upon this work by considering additional properties, stability analysis, or exploring specific applications in real-world problems.

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