Differential
Equations
& Applications
Volume 16, Number 3 (2024), 227–249
doi:10.7153/dea-2024-16-14

ON AN INVERSE BOUNDARY–VALUE PROBLEM FOR THE EQUATION OF MOTION OF A HOMOGENEOUS ELASTIC BEAM WITH PINNED ENDS

ELVIN I. AZIZBAYOV*, AFAG I. HASANOVA AND YASHAR T. MEHRALIYEV

(*Communicated by I. Velčić*)

Abstract. This paper is devoted to the study of the inverse boundary-value problem for the linearized equation of motion of a homogeneous beam with pinned ends. The primary goal of the work is to study the existence and uniqueness of the classical solution of the considered inverse boundary-value problem. To investigate the solvability of the considered problem, we carried out a transformation from the original problem to some auxiliary equivalent problem with trivial boundary conditions. Furthermore, we prove the existence and uniqueness theorem for the auxiliary problem by the contraction mappings principle. Based on the equivalency of these problems is shown the existence and uniqueness of the classical solution of the original problem.

1. Introduction

It is known that mathematical modeling of many real processes occurring during experiments in the field of some natural sciences leads to the study of inverse problems for partial differential equations. In the theory of equations of mathematical physics, inverse problems are understood as problems of simultaneous determination of unknown coefficients and right-hand side of partial differential equations from some additional measurements.

Inverse problems arise in various fields of human activity, such as seismology, mineral exploration, biology, medical visualization, computed tomography, Earth remote sensing, spectral analysis, nondestructive control, etc. From historical background can be seen, that the theoretical foundations of the study of inverse problems were established and developed in the works by Tikhonov [29], Lavrentiev [15], Ivanov [8], and their followers (see for example, [7, 14, 16, 23, 24, 25, 26], and the references therein).

Actually, inverse boundary-value problems for second-order partial differential equations extensively studied with different methods and different boundary conditions, notably in [5, 10, 13, 20, 21], etc. But it should be noted that inverse problems for pseudohyperbolic equations, namely for the equation of motion of a homogeneous beam,

Mathematics subject classification (2020): Primary 35R30, 35L25, 35Q74; Secondary 35A01, 35A02, 35A09.

Keywords and phrases: Inverse problem, two dimensional pseudohyperbolic equation, homogenous elastic beam, integral overdetermination condition, classical solution, existence, uniqueness.

* Corresponding author.

Paper DEA-16-14

are developed much less than for second-order equations. This can be explained by the fact that the corresponding direct problems are less studied. Let us now browse the content of some works devoted to direct and inverse coefficient problems for the equation of motion of a homogeneous beam: In the article published by Sabitov [27] investigated the Cauchy problem for the equation of the beam's motion with clamped ends. The authors Goy, Negrych and Savka [6] established conditions for the solvability of the boundary-value problem for the equation of motion of a homogeneous elastic beam with nonlocal two-point conditions and local boundary conditions. The works [2, 3] are devoted to the investigation of direct boundary-value problems for the one-dimensional equation of motion of a homogeneous elastic beam. But in the papers [1, 18, 19], one-dimensional nonlocal inverse boundary-value problems are studied for the equation of motion of a homogeneous elastic beam with different boundary and different overdetermination conditions. The authors proved the theorems for existence and uniqueness of classical solution of the one-dimensional inverse coefficient problem. A distinctive feature of this article is the consideration of a two-dimensional inverse boundary-value problem for a linearized equation of motion of a homogeneous beam with pinned ends.

Moreover, the vibrations and wave movements of an elastic beam on an elastic base investigated by Mitropolsky and Moseenkov [22], Thompson [28], Bardin and Furta [4], Vlasov and Leont'ev [30], Kostin [12] et al. The simplest nonlinear model of the motion of a homogeneous beam described by the equation

$$\frac{\partial^2 w}{\partial t^2}+\frac{\partial^4 w}{\partial x^4}+k\frac{\partial^2 w}{\partial x^2}+\alpha w+w^3=0,$$

where w is beam deflection. Note that a similar equation also arises in the theory of crystals, in which w is parameter of order [9].

2. Mathematical formulation of the problem

Let $T>0$ be a fixed time moment and let $D_T=Q_{xy}\times\{0\leqslant t\leqslant T\}$ denotes a closed bounded region in space, where $Q_{xy}=\{(x,y):\ 0<x<1,\ 0<y<1\}$. We further suppose that $f(x,y,t),\ g(x,y,t),\ \varphi(x,y),\ \psi(x,y),\ p_i(t)$, and $h_i(t)\ (i=1,2)$ are given functions of $x,y\in[0,1]$ and $t\in[0,T]$. Consider the two-dimensional inverse boundary-value problem of identifying an unknown triple of functions $u(x,y,t), a(t)$, and $b(t)$ for the equation

$$\begin{aligned} &u_{tt}(x,y,t)+\Delta^2 u(x,y,t)+\beta\Delta u(x,y,t)\\ &=a(t)u(x,y,t)+b(t)g(x,y,t)+f(x,y,t)\ (x,y,t)\in D_T, \end{aligned} \tag{1}$$

with the nonlocal initial conditions

$$\begin{aligned} u(x,y,0)&=\int_0^T p_1(t)u(x,y,t)dt+\varphi(x,y),\\ u_t(x,y,0)&=\int_0^T p_2(t)u(x,y,t)dt+\psi(x,y),\ 0\leqslant x,y\leqslant 1, \end{aligned} \tag{2}$$

the boundary conditions

$$u_x(0,y,t) = u(1,y,t) = u_{xxx}(0,y,t) = u_{xx}(1,y,t) = 0,\ \ 0 \leqslant y \leqslant 1,\ 0 \leqslant t \leqslant T, \tag{3}$$

$$u(x,0,t) = u_y(x,1,t) = u_{yy}(x,0,t) = u_{yyy}(x,1,t) = 0,\ \ 0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant T, \tag{4}$$

and the overdetermination conditions

$$u(0,1,t) = h_1(t),\ 0 \leqslant t \leqslant T, \tag{5}$$

and

$$\int_0^1 \int_0^1 u(x,y,t)dxdy = h_2(t),\ 0 \leqslant t \leqslant T, \tag{6}$$

where $\beta > 0$ is known fixed number and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

DEFINITION 1. The triple $\{u(x,y,t), a(t), b(t)\}$ defined in D_T is said to be a classical solution of the problem (1)–(6), if the functions $u(x,y,t) \in C^{2,2,4}(\bar{D}_T)$ and $a(t), b(t) \in C[0,T]$ satisfies the relations (1)–(6) in classical (usual) sense, where

$$\tilde{C}^2(\bar{D}_T) = \{u(x,y,t) : u(x,y,t) \in C^2(\bar{D}_T), u_{xxxx}(x,y,t), u_{yyyy}(x,y,t) \in C(\bar{D}_T)\}.$$

In order to investigate the problem (1)–(6), first we consider the following auxiliary problem

$$y''(t) = a(t)y(t),\ 0 \leqslant t \leqslant T, \tag{7}$$

$$y(0) = \int_0^T p_1(t)y(t)dt,\ \ y'(0) = \int_0^T p_2(t)y(t)dt, \tag{8}$$

where $p_1(t), p_2(t), a(t) \in C[0,T]$ are given functions, and $y = y(t)$ is desired function. Moreover, by the solution of the problem (7), (8), we mean a function $y(t)$ belonging to $C^2[0,T]$ and satisfying the conditions (7), (8) in the usual sense.

The following lemma is valid. But we omit the proof of the following lemma to avoid a lengthy digression (see Lemma 1 in [17]).

LEMMA 1. *Assume that $a(t), p_1(t), p_2(t) \in C[0,T]$, $\|a(t)\|_{C[0,T]} = const$, and the condition*

$$\left(T\,\|P_2(t)\|_{C[0,T]} + \|P_1(t)\|_{C[0,T]} + \frac{T}{2}R\right)T < 1$$

hold. Then the problem (7), (8) *has a unique trivial solution.*

Now along with the inverse boundary-value problem (1)–(6), we consider the following auxiliary inverse boundary-value problem: It is required to determine a triple $\{u(x,y,t), a(t), b(t)\}$ of functions $u(x,y,t) \in \tilde{C}^2(\bar{D}_T)$, and $a(t), b(t) \in C[0,T]$ from relations (1)–(3), and

$$\begin{aligned} h_1''(t) + u_{xxxx}(0,1,t) + 2u_{xxyy}(0,1,t) + u_{yyyy}(0,1,t) + \beta(u_{xx}(0,1,t) + u_{yy}(0,1,t)) \\ = a(t)h_1(t) + b(t)g(0,1,t) + f(0,1,t),\ 0 \leqslant t \leqslant T, \end{aligned} \tag{9}$$

$$h_2''(t)+\int_0^1 u_{xxx}(1,y,t)dy-\int_0^1 u_{yyy}(x,0,t)dx$$

$$-2u_{xy}(1,0,t)+\beta\left(\int_0^1 u_x(1,y,t)dy-\int_0^1 u_y(x,0,t)dx\right)$$

$$=a(t)h_2(t)+b(t)\int_0^1\int_0^1 g(x,y,t)dxdy+\int_0^1\int_0^1 f(x,y,t)dxdy,\ 0\leqslant t\leqslant T. \tag{10}$$

THEOREM 1. *Suppose that* $\varphi(x,y),\ \psi(x,y)\in C(\bar{Q}_{xy}),\ f(x,y,t),\ g(x,y,t)\in C(\bar{D}_T)$, $h_1(t),h_2(t)\in C^2[0,T],\ h(t)\equiv h_1(t)\int_0^1\int_0^1 g(x,y,t)dxdy-h_2(t)g(0,1,t)\neq 0,\ 0\leqslant t\leqslant T$, *and the compatibility conditions*

$$\varphi(0,1)=h_1(0)-\int_0^T p_1(t)h_1(t)dt,\psi(0,1)=h_1'(0)-\int_0^T p_2(t)h_1(t)dt, \tag{11}$$

$$\begin{gathered}\int_0^1\int_0^1 \varphi(x,y)dxdy=h_2(0)-\int_0^T p_1(t)h_2(t)dt,\\ \int_0^1\int_0^1 \psi(x,y)dxdy=h_2'(0)-\int_0^T p_2(t)h_2(t)dt,\end{gathered} \tag{12}$$

holds. Then the following assertions are valid:

i) *each classical solution* $\{u(x,y,t),a(t),b(t)\}$ *of the problem* (1)–(6) *is a solution of problem* (1)–(4), (9), (10), *as well;*

ii) *each solution* $\{u(x,y,t),a(t),b(t)\}$ *of the problem* (1)–(4), (9), (10) *is a classical solution of problem* (1)–(6), *if*

$$\left(T\,\|p_2(t)\|_{C[0,T]}+\|p_1(t)\|_{C[0,T]}+\frac{T}{2}\,\|a(t)\|_{C[0,T]}\right)T<1. \tag{13}$$

Proof. Let $\{u(x,y,t),a(t),b(t)\}$ be any classical solution to problem (1)–(6). Taking into account the condition $h_i(t)\in C^2[0,T]\ (i=1,2)$, and differentiating twice both sides of (5) and (6) with respect to t gives

$$u_t(0,1,t)=h_1'(t),\ \ u_{tt}(0,1,t)=h_1''(t),\ 0\leqslant t\leqslant T, \tag{14}$$

$$\int_0^1\int_0^1 u_t(x,y,t)dxdy=h_2'(t),\ \ \int_0^1\int_0^1 u_{tt}(x,y,t)dxdy=h_2''(t),\ 0\leqslant t\leqslant T. \tag{15}$$

Now, from equation (1), we find:

$$\begin{aligned} &\frac{d^2}{dt^2}u(0,1,t)+u_{xxxx}(0,1,t)+2u_{xxyy}(0,1,t)\\ &+u_{yyyy}(0,1,t)+\beta(u_{xx}(0,1,t)+u_{yy}(0,1,t))\\ =&\,a(t)u(0,1,t)+b(t)g(0,1,t)+f(0,1,t),\ 0\leqslant t\leqslant T. \end{aligned} \tag{16}$$

From (16), taking into account (5) and (14), we conclude that the relation (9) is fulfilled.

Further, integrating Eq. (1) with respect to x and y over the interval $[0,1]$ gives

$$\begin{aligned} &\frac{d^2}{dt^2}\int_0^1\int_0^1 u(x,y,t)dxdy+\int_0^1(u_{xxx}(1,y,t)-u_{xxx}(0,y,t))dy\\ &+2(u_{xy}(1,1,t)-u_{xy}(1,0,t)-u_{xy}(0,1,t)+u_{xy}(0,0,t))\\ &-\int_0^1(u_{yyy}(x,1,t)-u_{yyy}(x,0,t))dx\\ &+\beta\int_0^1(u_x(1,y,t)-u_x(0,y,t))dy+\beta\int_0^1(u_y(x,1,t)-u_y(x,0,t))dx\\ =&\,a(t)\int_0^1\int_0^1 u(x,y,t)dxdy+b(t)\int_0^1\int_0^1 g(x,y,t)dxdy\\ &+\int_0^1\int_0^1 f(x,y,t)dxdy,\ 0\leqslant t\leqslant T. \end{aligned}$$

By allowing the last relation and taking into account (3), (4), we obtain:

$$\begin{aligned} &\frac{d^2}{dt^2}\int_0^1\int_0^1 u(x,y,t)dxdy+\int_0^1 u_{xxx}(1,y,t)dy-\int_0^1 u_{yyy}(x,0,t))dx\\ &-2u_{xy}(1,0,t)+\beta\int_0^1 u_x(1,y,t)dy-\beta\int_0^1 u_y(x,0,t)dx\\ =&\,a(t)\int_0^1\int_0^1 u(x,y,t)dxdy+b(t)\int_0^1\int_0^1 g(x,y,t)dxdy\\ &+\int_0^1\int_0^1 f(x,y,t)dxdy,\ 0\leqslant t\leqslant T. \end{aligned} \tag{17}$$

Hence, from (17), taking into account (6) and (15), we arrive at (10).

Now suppose that the triple $\{u(x,y,t),a(t),b(t)\}$ is a solution to the problem (1)–(4), (9), (10). Then from (9) and (16), we get

$$\frac{d^2}{dt^2}(u(0,1,t)-h_1(t))=a(t)(u(0,1,t)-h_1(t)),\ 0\leqslant t\leqslant T. \tag{18}$$

Using (2) and the compatibility condition (11), we have

$$\begin{aligned}
&u(0,1,0)-h_1(0)-\int_0^T p_1(t)(u(0,1,t)-h_1(t))dt\\
&=u(0,1,0)-\int_0^T p_1(t)u(0,1,t)dt-\left(h_1(0)-\int_0^T p_1(t)h_1(t)dt\right)\\
&=\varphi(0,1)-\left(h_1(0)-\int_0^T p_1(t)h_1(t)dt\right)=0,\\
&u_t(0,1,0)-h_1'(0)-\int_0^T p_2(t)(u(0,1,t)-h_1(t))dt\\
&=u_t(0,1,0)-\int_0^T p_2(t)u(0,1,t)dt-\left(h'{}_1(0)-\int_0^T p_2(t)h_1(t)dt\right)\\
&=\psi(0,1)-\left(h_1(0)-\int_0^T p_2(t)h_1(t)dt\right)=0.
\end{aligned} \tag{19}$$

Since, by Lemma 1, problem (18), (19) has only a trivial solution, so from $u(0,1,t)-h_1(t)=0\ \ (0\leqslant t\leqslant T)$, we obtain that the condition (2.5) is satisfied.

Now, from (10) and (17) we find:

$$\begin{aligned}
&\frac{d^2}{dt^2}\left(\int_0^1\int_0^1 u(x,y,t)dxdy-h_2(t)\right)\\
&=a(t)\left(\int_0^1\int_0^1 u(x,y,t)dxdy-h_2(t)\right),\ 0\leqslant t\leqslant T.
\end{aligned} \tag{20}$$

By using the initial conditions (2) and the compatibility conditions (12), we may write

$$\begin{aligned}
&\int_0^1\int_0^1 u(x,y,0)dxdy-h_2(0)-\int_0^T p_1(t)\left(\int_0^1\int_0^1 u(x,y,t)dxdy-h_2(t)\right)dt\\
&=\int_0^1\int_0^1\left(u(x,y,0)-\int_0^T p_1(t)u(x,y,t)dt\right)dxdy-\left(h_2(0)-\int_0^T p_1(t)h_2(t)dt\right)\\
&=\int_0^1\int_0^1 \varphi(x,y)dxdy-\left(h_2(0)-\int_0^T p_1(t)h_2(t)dt\right)=0,
\end{aligned}$$

$$\int_0^1\int_0^1 u_t(x,y,0)dxdy - h_2'(0) - \int_0^T p_2(t)\left(\int_0^1\int_0^1 u(x,y,t)dxdy - h_2(t)\right)dt$$

$$= \int_0^1\int_0^1 \left(u_t(x,y,0) - \int_0^T p_2(t)u(x,y,t)dt\right)dxdy - \left(h_2'(0) - \int_0^T p_1(t)h_2(t)dt\right)$$

$$= \int_0^1\int_0^1 \psi(x,y)dxdy - \left(h_2'(0) - \int_0^T p_1(t)h_2(t)dt\right) = 0. \tag{21}$$

Since, by virtue of Lemma 1, problem (20), (21) has only a trivial solution, then

$$\int_0^1\int_0^1 u(x,y,t)dxdy - h_2(t) = 0,\ 0 \leqslant t \leqslant T;$$

i.e., the condition (6) is satisfied. □

3. Classical solvability of inverse boundary-value problem

We seek the first component of classical solution of the problem (1)–(4), (9), (10) in the form

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} u_{k,n}(t)\cos\lambda_k x \sin\gamma_n y, \tag{22}$$

where

$$\lambda_k = \frac{\pi}{2}(2k-1),\ \gamma_n = \frac{\pi}{2}(2n-1),\ k,n = 1,2,\ldots,$$

$$u_{k,n}(t) = 4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x \sin\gamma_n y dxdy,\ k,n = 1,2,\ldots.$$

Applying the method of separation of variables to determine the desired coefficients $u_{k,n}(t)$ $(k,n = 1,2,\ldots)$ of the function $u(x,y,t)$ from (1), (2), we obtain:

$$u_{k,n}''(t) + (\lambda_k^4 + \gamma_n^4 - \beta(\lambda_k^2 + \gamma_n^2))u_{k,n}(t)$$

$$= F_{k,n}(t;u,a,b),\ k,n = 1,2,\ldots;\ 0 \leqslant t \leqslant T, \tag{23}$$

$$\begin{aligned} u_{k,n}(0) &= \int_0^T p_1(t)u_{k,n}(t)dt + \varphi_{k,n}, \\ u_{k,n}'(0) &= \int_0^T p_2(t)u_{k,n}(t)dt + \psi_{k,n},\ k,n = 1,2,\ldots, \end{aligned} \tag{24}$$

where

$$F_{k,n}(t;u,a,b) = f_{k,n}(t) + a(t)u_{k,n}(t) + b(t)g_{k,n}(t),\ k,n = 1,2,\ldots,$$

$$f_{k,n}(t)=4\int_0^1\int_0^1 f(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy,\ k,n=1,2,\ldots,$$

$$g_{k,n}(t)=4\int_0^1\int_0^1 g(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy,\ k,n=1,2,\ldots,$$

$$\varphi_{k,n}=4\int_0^1\int_0^1 \varphi(x,y)\cos\lambda_k x\sin\gamma_n y dxdy,\ k,n=1,2,\ldots,$$

$$\psi_{k,n}=4\int_0^1\int_0^1 \psi(x,y)\cos\lambda_k x\sin\gamma_n y dxdy,\ k,n=1,2,\ldots.$$

Assume that $0<\beta<\frac{\pi^2}{4}$. Then solving the problem (23), (24) gives

$$u_{k,n}(t)=\left(\varphi_{k,n}+\int_0^T p_1(t)u_{k,n}(t)dt\right)\cos\beta_{k,n}t$$

$$+\frac{1}{\beta_{k,n}}\left(\psi_{k,n}+\int_0^T p_2(t)u_{k,n}(t)dt\right)\sin\beta_{k,n}t$$

$$+\frac{1}{\beta_{k,n}}\int_0^t F_{k,n}(\tau;u,a,b)\sin\beta_{k,n}(t-\tau)d\tau,\ k,n=1,2,\ldots;\ 0\leqslant t\leqslant T, \tag{25}$$

where

$$\beta_{k,n}=\sqrt{\lambda_k^4+2\lambda_k^2\gamma_n^2+\gamma_n^4-\beta(\lambda_k^2+\gamma_n^2)}.$$

Substituting the expression of $u_{k,n}(t)$ $(k,n=1,2,\ldots)$ into (22), we find

$$u(x,y,t)=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left\{\left(\varphi_{k,n}+\int_0^T p_1(t)u_{k,n}(t)dt\right)\cos\beta_{k,n}t\right.$$

$$+\frac{1}{\beta_{k,n}}\left(\psi_{k,n}+\int_0^T p_2(t)u_{k,n}(t)dt\right)\sin\beta_{k,n}t$$

$$\left.+\frac{1}{\beta_{k,n}}\int_0^t F_k(\tau;u,a,b)\sin\beta_{k,n}(t-\tau)d\tau\right\}\cos\lambda_k x\sin\gamma_n y. \tag{26}$$

Now from (9) and (10), taking into account (22), respectively, we get:

$$\begin{aligned} a(t)h_1(t)+b(t)g(0,1,t) &= h_1''(t)-f(0,1,t) \\ &+\sum_{k=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{n+1}(2\lambda_k^2\gamma_n^2+\beta_{k,n}^2)u_{k,n}(t),\ 0\leqslant t\leqslant T, \end{aligned} \tag{27}$$

$$\begin{aligned} a(t)h_2(t)+b(t)\int_0^1\int_0^1 g(x,y,t)dxdy &= h_2''(t)-\int_0^1\int_0^1 f(x,y,t)dxdy \\ &+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{k+1}\left(2\lambda_k\gamma_n+\frac{\beta_{k,n}^2}{\lambda_k\gamma_n}\right)u_{k,n}(t),\ 0\leqslant t\leqslant T. \end{aligned} \tag{28}$$

Let us suppose that

$$h(t)\equiv h_1(t)\int_0^1\int_0^1 g(x,y,t)dxdy-h_2(t)g(0,1,t)\neq 0,\ 0\leqslant t\leqslant T. \tag{29}$$

Then from (27) and (28), we find:

$$\begin{aligned} a(t) = [h(t)]^{-1}\Bigg\{ &(h_2''(t)-f(0,1,t))\int_0^1\int_0^1 g(x,y,t)dxdy \\ &-\left(h_2''(t)-\int_0^1\int_0^1 g(x,y,t)dxdy\right)g(0,1,t) \\ &+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left((-1)^{n+1}\int_0^1\int_0^1 g(x,y,t)dxdy-(-1)^{k+1}\frac{g(0,1,t)}{\lambda_k\gamma_n}\right) \\ &\times(2\lambda_k^2\gamma_n^2+\beta_{k,n}^2)u_{k,n}(t)\Bigg\},\ 0\leqslant t\leqslant T, \end{aligned} \tag{30}$$

$$\begin{aligned} b(t) = [h(t)]^{-1}\Bigg\{ &\left(h_2''(t)-\int_0^1\int_0^1 f(x,y,t)dxdy\right)h_1(t)-(h_1''(t)-f(0,1,t))h_2(t) \\ &+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left((-1)^{k+1}\frac{h_1(t)}{\lambda_k\gamma_n}-(-1)^{n+1}h_2(t)\right) \\ &\times(2\lambda_k^2\gamma_n^2+\beta_{k,n}^2)u_{k,n}(t)\Bigg\},\ 0\leqslant t\leqslant T. \end{aligned} \tag{31}$$

The following expressions for the second and third components of the solution

$\{u(x,y,t),a(t),b(t)\}$ to problem (1)–(4), (9), (10)

$$\begin{aligned}
a(t) = [h(t)]^{-1}\Bigg\{ & (h_1''(t) - f(0,1,t)) \int_0^1\int_0^1 g(x,y,t)dxdy \\
& - \left(h_2''(t) - \int_0^1\int_0^1 f(x,y,t)dxdy \right) g(0,1,t) \\
& + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \left((-1)^{n+1} \int_0^1\int_0^1 g(x,y,t)dxdy - (-1)^{k+1}\frac{g(0,1,t)}{\lambda_k\gamma_n} \right)(2\lambda_k^2\gamma_n^2 + \beta_{k,n}^2) \\
& \times \Bigg[\left(\varphi_{k,n} + \int_0^T p_1(t)u_{k,n}(t)dt \right)\cos\beta_{k,n}t \\
& + \frac{1}{\beta_{k,n}} \left(\psi_{k,n} + \int_0^T p_2(t)u_{k,n}(t)dt \right)\sin\beta_{k,n}t \\
& + \frac{1}{\beta_{k,n}} \int_0^t F_{k,n}(\tau;u,a,b)\sin\beta_{k,n}(t-\tau)d\tau \Bigg] \Bigg\},
\end{aligned} \tag{32}$$

$$\begin{aligned}
b(t) = [h(t)]^{-1}\Bigg\{ & \left(h_2''(t) - \int_0^1\int_0^1 f(x,y,t)dxdy \right) h_1(t) \\
& - (h_1''(t) - f(0,1,t))h_2(t) \\
& + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \left((-1)^{k+1}\frac{h_1(t)}{\lambda_k\gamma_n} - (-1)^{n+1}h_2(t) \right)(2\lambda_k^2\gamma_n^2 + \beta_{k,n}^2) \\
& \times \Bigg[\left(\varphi_{k,n} + \int_0^T p_1(t)u_{k,n}(t)dt \right)\cos\beta_{k,n}t \\
& + \frac{1}{\beta_{k,n}} \left(\psi_{k,n} + \int_0^T p_2(t)u_{k,n}(t)dt \right)\sin\beta_{k,n}t \\
& + \frac{1}{\beta_{k,n}} \int_0^t F_{k,n}(\tau;u,a,b)\sin\beta_{k,n}(t-\tau)d\tau \Bigg] \Bigg\},
\end{aligned} \tag{33}$$

respectively, were obtained by substituting (25) into (30) and (31).

Thus the solution of problem (1)–(4), (9), (10) was reduced to the solution of systems (26), (32), (33) with respect to unknown functions $u(x,y,t), a(t)$, and $b(t)$.

The following lemma plays an important role in studying the uniqueness of the solution to problem (1)–(4), (9), (10).

LEMMA 2. *If* $\{u(x,y,t),a(t),b(t)\}$ *is any solution to problem* (1)–(4), (9), (10), *then the functions*

$$u_{k,n}(t)=4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy,\ k,n=1,2,\ldots$$

satisfy the system (25) *on an interval* $[0,T]$.

Proof. Let $\{u(x,y,t),a(t),b(t)\}$ be any solution of the problem (1)–(4), (9), (10). Then multiplying both sides of the Eq. (1) by the special functions $4\cos\lambda_k x\sin\gamma_n y$ $(k,n=1,2,\ldots)$, integrating with respect to x and y over the interval $[0,1]$, and using the relations

$$4\int_0^1\int_0^1 u_{tt}(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy$$

$$=\frac{d^2}{dt^2}\left(4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\right)=u''_{k,n}(t),\ k,n=1,2,\ldots,$$

$$4\int_0^1\int_0^1 u_{xx}(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy$$

$$=-\lambda_k^2\left(4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\right)=-\lambda_k^2 u_{k,n}(t),\ k,n=1,2,\ldots,$$

$$4\int_0^1\int_0^1 u_{yy}(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy$$

$$-\gamma_n^2\left(4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\right)=-\gamma_k^2 u_{k,n}(t),\ k,n=1,2,\ldots,$$

$$4\int_0^1\int_0^1 u_{xxyy}(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy$$

$$=\lambda_k^2\gamma_n^2\left(4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\right)=\lambda_k^2\gamma_n^2 u_{k,n}(t),\ k,n=1,2,\ldots,$$

$$4\int_0^1\int_0^1 u_{xxxx}(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy$$

$$=\lambda_k^4\left(4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\right)=\lambda_k^4 u_{k,n}(t),\ k,n=1,2,\ldots,$$

$$4\int_0^1\int_0^1 u_{yyyy}(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy$$

$$=\gamma_n^4\left(4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\right)=\gamma_k^4 u_{k,n}(t),\ k,n=1,2,\ldots,$$

we obtain that the Equation (23) is satisfied.

In like manner, it follows from (2) that condition (24) is also satisfied.

Thus, the system of functions $u_{k,n}(t)\ (k,n=1,2,\ldots)$ is a solution of problem (23), (24). Hence it follows directly that the functions $u_{k,n}(t)\ (k,n=1,2,\ldots)$ also satisfy the system (25) on $[0,T]$. □

Obviously, if $u_{k,n}(t)=4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\ \ (k,n=1,2,\ldots)$ is a solution to system (25), then the triple $\{u(x,y,t),a(t),b(t)\}$ of functions

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}u_{k,n}(t)\cos\lambda_k x\sin\gamma_n y,$$

$a(t)$, and $b(t)$ is also a solution to system (26), (32), (33).

It follows from the Lemma 2 that

COROLLARY 1. *Assume that the system* (26)*,* (32)*,* (33) *has a unique solution. Then the problem* (1)–(4)*,* (9)*,* (10) *has at most one solution, i.e., if the problem* (1)–(4)*,* (9)*,* (10) *has a solution, then it is unique.*

Let us consider the functional space $B_{2,T}^5$ that is introduced in the study of [11], where $B_{2,T}^5$ denotes a set of all functions of the form

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}u_{k,n}(t)\cos\lambda_k x\sin\gamma_n y,$$

considered in D_T. Moreover, the functions $u_{k,n}(t)\ (k,n=1,2,\ldots)$ contained in last sum are continuously differentiable on $[0,T]$ and

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^5\left\|u_{k,n}(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}<+\infty,$$

where

$$\mu_{k,n}^5=(\lambda_k^4+\gamma_n^4)\sqrt{\lambda_k^2+\gamma_n^2}.$$

Let E_T^5 denote the space consisting of the topological product $B_{2,T}^5\times C[0,T]\times C[0,T]$, which is the norm of the element $z=\{u,a,b\}$ defined by the formula

$$\|z\|_{E_T^5}=\|u(x,y,t)\|_{B_{2,T}^5}+\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}.$$

It is clear that the spaces $B_{2,T}^5$ and E_T^5 are Banach spaces.

Let us now consider the operator

$$\Phi(u,a,b) = \{\Phi_1(u,a,b), \Phi_2(u,a,b), \Phi_3(u,a,b)\},$$

in the space E_T^5, where

$$\Phi_1(u,a,b) = \tilde{u}(x,y,t) \equiv \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \tilde{u}_{k,n}(t)\cos\lambda_k x \sin\gamma_n y,$$

$$\Phi_2(u,a,b) = \tilde{a}(t), \Phi_2(u,a,b) = \tilde{b}(t),$$

and the functions $\tilde{u}_{k,n}(t)$ $(k=1,2,\ldots)$, $\tilde{a}(t)$, and $\tilde{b}(t)$ are equal to the right-hand sides of (23), (30), and (31), respectively.

It is easy to see that

$$\mu_{k,n}^5 \leqslant (\lambda_k^4 + \gamma_k^4)(\lambda_k + \gamma_n) = \lambda_k^5 + \lambda_k^4\gamma_n + \gamma_k^4\lambda_k + \gamma_k^5.$$

Taking into account this relation, we obtain

$$\begin{aligned}
&\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} (\mu_{k,n}^5 \left\|\tilde{u}_{k,n}(t)\right\|_{C[0,T]})^2\right\}^{\frac{1}{2}} \\
&\leqslant \sqrt{23}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} (\lambda_k^5 \left|\varphi_{k,n}\right|)^2\right)^{\frac{1}{2}} + \sqrt{23}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} (\lambda_k^4\gamma_n \left|\varphi_{k,n}\right|)^2\right)^{\frac{1}{2}} \\
&+\sqrt{23}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} (\lambda_K\gamma_n^4 \left|\varphi_{k,n}\right|)^2\right)^{\frac{1}{2}} + \sqrt{23}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\gamma_n^5 \left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}} \\
&+2\sqrt{23}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} (\lambda_k^3 \left|\psi_{k,n}\right|)^2\right)^{\frac{1}{2}} + 2\sqrt{23}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} (\lambda_k^2\gamma_n \left|\psi_{k,n}\right|)^2\right)^{\frac{1}{2}} \\
&+2\sqrt{23}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} (\lambda_k\gamma_n^2 \left|\psi_{k,n}\right|)^2\right)^{\frac{1}{2}} + 2\sqrt{23}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} (\gamma_n^3 \left|\psi_{k,n}\right|)^2\right)^{\frac{1}{2}} \\
&+\sqrt{23}T(\|p_1(t)\|_{C[0,T]} + \|p_2(t)\|_{C[0,T]})\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} (\mu_{k,n}^5 \left\|u_{k,n}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}} \\
&+2\sqrt{23T}\left(\left(\int_0^T \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} (\lambda_k^3 \left|f_{k,n}(\tau)\right|)^2 d\tau\right)^{\frac{1}{2}} + \left(\int_0^T \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} (\lambda_k^2\gamma_n \left|f_{k,n}(\tau)\right|)^2 d\tau\right)^{\frac{1}{2}}\right. \\
&\left.+\left(\int_0^T \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} (\lambda_k\gamma_n^2 \left|f_{k,n}(\tau)\right|)^2 d\tau\right)^{\frac{1}{2}} + \left(\int_0^T \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} (\gamma_n^3 \left|f_{k,n}(\tau)\right|)^2 d\tau\right)^{\frac{1}{2}}\right)
\end{aligned}$$

$$
+\sqrt{23}T\,\|a(t)\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^{5}\,\|u_{k,n}(t)\|_{C[0,T]})^{2}\right)^{\frac{1}{2}}
$$

$$
+2\sqrt{23T}\,\|b(t)\|_{C[0,T]}\left(\left(\int_{0}^{T}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_{k}^{3}\,|g_{k,n}(\tau)|)^{2}d\tau\right)^{\frac{1}{2}}\right.
$$

$$
+\left(\int_{0}^{T}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_{k}^{2}\gamma_{n}\,|g_{k,n}(\tau)|)^{2}d\tau\right)^{\frac{1}{2}}+\left(\int_{0}^{T}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_{k}\gamma_{n}^{2}\,|g_{k,n}(\tau)|)^{2}d\tau\right)^{\frac{1}{2}}
$$

$$
\left.+\left(\int_{0}^{T}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_{n}^{3}\,|g_{k,n}(\tau)|)^{2}d\tau\right)^{\frac{1}{2}}\right), \tag{34}
$$

$$
\|\tilde{a}(t)\|_{C[0,T]}
$$

$$
\leqslant \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|(h_1''(t)-f(0,1,t))\int_{0}^{1}\int_{0}^{1}g(x,y,t)dxdy\right.\right.
$$

$$
\left.-\left(h_2''(t)-\int_{0}^{1}\int_{0}^{1}f(x,y,t)dxdy\right)g(0,1,t)\right\|_{C[0,T]}
$$

$$
+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}}\left\|\left|\int_{0}^{1}\int_{0}^{1}g(x,y,t)dxdy\right|+|g(0,1,t)|\right\|_{C[0,T]}
$$

$$
\times\left[\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_{k}^{5}\,|\varphi_{k,n}|)^{2}\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_{k}^{4}\gamma_{n}\,|\varphi_{k,n}|)^{2}\right)^{\frac{1}{2}}\right.
$$

$$
+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_{k}\gamma_{n}^{4}\,|\varphi_{k,n}|)^{2}\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_{n}^{5}\,|\varphi_{k,n}|)^{2}\right)^{\frac{1}{2}}
$$

$$
+2\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_{k}^{3}\,|\psi_{k,n}|)^{2}\right)^{\frac{1}{2}}+2\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_{k}^{2}\gamma_{n}\,|\psi_{k,n}|)^{2}\right)^{\frac{1}{2}}
$$

$$
+2\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_{k}\gamma_{n}^{2}\,|\psi_{k,n}|)^{2}\right)^{\frac{1}{2}}+2\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_{n}^{3}\,|\psi_{k,n}|)^{2}\right)^{\frac{1}{2}}
$$

$$
+T(\|p_1(t)\|_{C[0,T]}+\|p_2(t)\|_{C[0,T]})\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^{5}\,\|u_{k,n}(t)\|_{C[0,T]})^{2}\right)^{\frac{1}{2}}
$$

$$+2\sqrt{T}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^3\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\gamma_n\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right.$$

$$\left.+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k\gamma_n^2\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^3\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right)$$

$$+T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^5\left\|u_{k,n}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{T}\left\|b(t)\right\|_{C[0,T]}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^3\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right.$$

$$+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\gamma_n\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k\gamma_n^2\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}$$

$$\left.\left.\left.+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^3\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right)\right]\right\}, \tag{35}$$

$$\begin{aligned}
&\left\|\tilde{b}(t)\right\|_{C[0,T]}\\
\leqslant\ &\left\|[h(t)]^{-1}\right\|_{C[0,T]}\\
&\times\left\{\left\|\left(h_2''(t)-\int_0^1\int_0^1 f(x,y,t)dxdy\right)h_1(t)-(h_1''(t)-f(0,1,t))h_2(t)\right\|_{C[0,T]}\right.\\
&+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_k^{-2}\right)^{\frac{1}{2}}\left\||h_1(t)|+|h_2(t)|\right\|_{C[0,T]}\left[\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^5\left|\varphi_{k,n}\right|)^2\right)^{\frac{1}{2}}\right.\\
&+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^4\gamma_n\left|\varphi_{k,n}\right|)^2\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k\gamma_n^4\left|\varphi_{k,n}\right|)^2\right)^{\frac{1}{2}}\\
&+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^5\left|\varphi_{k,n}\right|)^2\right)^{\frac{1}{2}}+2\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^3\left|\psi_{k,n}\right|)^2\right)^{\frac{1}{2}}\\
&+2\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\gamma_n\left|\psi_{k,n}\right|)^2\right)^{\frac{1}{2}}+2\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k\gamma_n^2\left|\psi_{k,n}\right|)^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$+2\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^3\left|\psi_{k,n}\right|)^2\right)^{\frac{1}{2}}$$

$$+T(\|p_1(t)\|_{C[0,T]}+\|p_2(t)\|_{C[0,T]})\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^5\left\|u_{k,n}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{T}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^3\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\gamma_n\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right.$$

$$\left.+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k\gamma_n^2\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^3\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right)$$

$$+T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^5\left\|u_{k,n}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{T}\left\|b(t)\right\|_{C[0,T]}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^3\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right.$$

$$+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\gamma_n\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k\gamma_n^2\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}$$

$$\left.\left.\left.+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^3\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right)\right]\right\}. \tag{36}$$

We impose the following conditions on the data of problem (1)–(4), (9), (10):

C_1) $\varphi(x,y),\varphi_x(x,y),\varphi_{xx}(x,y),\varphi_y(x,y),\varphi_{xy}(x,y),\varphi_{yy}(x,y),$
$\varphi_{xxx}(x,y),\varphi_{xxy}(x,y),\varphi_{xyy}(x,y),\varphi_{yyy}(x,y),$
$\varphi_{xxxx}(x,y),\varphi_{xxxy}(x,y),\varphi_{xxyy}(x,y),\varphi_{xyyy}(x,y),\varphi_{yyyy}(x,y)\in C(\bar{Q}_{xy}),$
$\varphi_{xxxxy}(x,y),\varphi_{xyyyy}(x,y),\varphi_{xxxxx}(x,y),\varphi_{yyyyy}(x,y)\in L_2(Q_{xy}),$
$\varphi_x(0,y)=\varphi(1,y)=\varphi_{xx}(1,y)=\varphi_{xxx}(0,y)=\varphi_{xxxx}(1,y)=0,\ 0\leqslant y\leqslant 1,$
$\varphi(x,0)=\varphi_y(x,1)=\varphi_{yy}(x,0)=\varphi_{yyy}(x,1)=\varphi_{yyyy}(x,0)=0,\ 0\leqslant x\leqslant 1.$

C_2) $\psi(x,y),\psi_x(x,y),\psi_y(x,y),\psi_{xx}(x,y),\psi_{xy}(x,y),\psi_{yy}(x,y)\in C(\bar{Q}_{xy}),$
$\psi_{xxx}(x,y),\psi_{xyy}(x,y),\psi_{xxy}(x,y),\psi_{yyy}(x,y)\in L_2(Q_{xy}),$
$\psi_x(0,y)=\psi(1,y)=\psi_{xx}(1,y)=0,0\leqslant y\leqslant 1,$
$\psi(x,0)=\psi_y(x,1)=\psi_{yy}(x,0)=0,0\leqslant x\leqslant 1.$

C_3) $f(x,y,t),f_x(x,y,t),f_y(x,y,t),f_{xx}(x,y,t),f_{xy}(x,y,t),f_{yy}(x,y,t)\in C(D_T),$
$f_{xxx}(x,y,t),f_{xxy}(x,y,t),f_{xyy}(x,y,t),f_{yyy}(x,y,t)\in L_2(D_T),$

$f_x(0,y,t) = f(1,y,t) = f_{xx}(1,y,t) = 0, 0 \leqslant y \leqslant 1, 0 \leqslant t \leqslant T,$
$f(x,0,t) = f_y(x,1,t) = f_{yy}(x,0,t) = 0, 0 \leqslant x \leqslant 1, 0 \leqslant t \leqslant T.$

C_4) $g(x,y,t), g_x(x,y,t), g_y(x,y,t), g_{xx}(x,y,t), g_{xy}(x,y,t), g_{yy}(x,y,t) \in C(D_T),$
$g_{xxx}(x,y,t), g_{xxy}(x,y,t), g_{xyy}(x,y,t), g_{yyy}(x,y,t) \in L_2(D_T),$
$g_x(0,y,t) = g(1,y,t) = g_{xx}(1,y,t) = 0, 0 \leqslant y \leqslant 1, 0 \leqslant t \leqslant T,$
$g(x,0,t) = g_y(x,1,t) = g_{yy}(x,0,t) = 0, 0 \leqslant x \leqslant 1, 0 \leqslant t \leqslant T.$

C_5) $0 < \beta < \frac{\pi^2}{4}$, $h_1(t), h_2(t) \in C^2[0,T]$, and
$h(t) \equiv h_1(t) \int_0^1 \int_0^1 g(x,y,t)dxdy - h_2(t)g(0,1,t) \neq 0, \ 0 \leqslant t \leqslant T.$

Then, from (34)–(36), respectively, we obtain

$$\begin{aligned} \|u(x,y,t)\|_{B^5_{2,T}} &= \left\{ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\mu_{k,n}^5 \|u_{k,n}(t)\|_{C[0,T]})^2 \right\}^{\frac{1}{2}} \\ &\leqslant A_1(T) + B_1(T) \|a(t)\|_{C[0,T]} \|u(x,y,t)\|_{B^5_{2,T}} \\ &\quad + C_1(T) \|b(t)\|_{C[0,T]} + D_1(T) \|u(x,y,t)\|_{B^5_{2,T}}, \end{aligned} \tag{37}$$

$$\begin{aligned} \|\tilde{a}(t)\|_{C[0,T]} &\leqslant A_2(T) + B_2(T) \|a(t)\|_{C[0,T]} \|u(x,y,t)\|_{B^5_{2,T}} \\ &\quad + C_2(T) \|b(t)\|_{C[0,T]} + D_2(T) \|u(x,y,t)\|_{B^5_{2,T}}, \end{aligned} \tag{38}$$

$$\begin{aligned} \|\tilde{b}(t)\|_{C[0,T]} &\leqslant A_3(T) + B_3(T) \|a(t)\|_{C[0,T]} \|u(x,y,t)\|_{B^5_{2,T}} \\ &\quad + C_3(T) \|b(t)\|_{C[0,T]} + D_3(T) \|u(x,y,t)\|_{B^5_{2,T}}, \end{aligned} \tag{39}$$

where

$$\begin{aligned} A_1(T) &= \sqrt{23} \|\varphi_{xxxxx}(x,y)\|_{L_2(Q_{xy})} + \sqrt{23} \|\varphi_{xxxyy}(x,y)\|_{L_2(Q_{xy})} \\ &\quad + \sqrt{23} \|\varphi_{xxyyy}(x,y)\|_{L_2(Q_{xy})} + \sqrt{23} \|\varphi_{yyy}(x,y)\|_{L_2(Q_{xy})} \\ &\quad + 2\sqrt{23} \|\psi_{xxx}(x,y)\|_{L_2(Q_{xy})} + 2\sqrt{23} \|\psi_{xxy}(x,y)\|_{L_2(Q_{xy})} \\ &\quad + 2\sqrt{23} \|\psi_{xyy}(x,y)\|_{L_2(Q_{xy})} + 2\sqrt{23} \|\varphi_{yyy}(x,y)\|_{L_2(Q_{xy})} \\ &\quad + 2\sqrt{23T} (\|f_{xxx}(x,y,t)\|_{L_2(D_T)} + \|f_{xyy}(x,y,t)\|_{L_2(D_T)} \\ &\quad + \|f_{xxy}(x,y,t)\|_{L_2(D_T)} + \|f_{yyy}(x,y,t)\|_{L_2(D_T)}), \\ B_1(T) &= \sqrt{23}T, \\ C_1(T) &= 2\sqrt{23T} (\|g_{xxx}(x,y,t)\|_{L_2(D_T)} + \|g_{xyy}(x,y,t)\|_{L_2(D_T)} \\ &\quad + \|g_{xxy}(x,y,t)\|_{L_2(D_T)} + \|g_{yyy}(x,y,t)\|_{L_2(D_T)}), \\ D_1(T) &= \sqrt{23}T (\|p_1(t)\|_{C[0,T]} + \|p_2(t)\|_{C[0,T]}), \end{aligned}$$

$$
\begin{aligned}
A_2(T) = {} & \left\|[h(t)]^{-1}\right\|_{C[0,T]} \\
& \times\left\{\left\|\left(h_1''(t)-f(0,1,t)\right)\int_0^1\int_0^1 g(x,y,t)dxdy\right.\right. \\
& \left.-\left(h_2''(t)-\int_0^1\int_0^1 f(x,y,t)dxdy\right)g(0,1,t)\right\|_{C[0,T]} \\
& +\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}}\left\|\left|\int_0^1\int_0^1 g(x,y,t)dxdy\right|+|g(0,1,t)|\right\|_{C[0,T]} \\
& \times[\|\varphi_{xxxxx}(x,y)\|_{L_2(Q_{xy})}+\left\|\varphi_{xxxyy}(x,y)\right\|_{L_2(Q_{xy})}+\left\|\varphi_{xyyyy}(x,y)\right\|_{L_2(Q_{xy})} \\
& +\left\|\varphi_{yyyyy}(x,y)\right\|_{L_2(Q_{xy})}+2\left\|\psi_{xxx}(x,y)\right\|_{L_2(Q_{xy})}+2\left\|\psi_{xxy}(x,y)\right\|_{L_2(Q_{xy})} \\
& +2\left\|\psi_{xyy}(x,y)\right\|_{L_2(Q_{xy})}+2\left\|\varphi_{yyy}(x,y)\right\|_{L_2(Q_{xy})} \\
& +2\sqrt{T}(\|f_{xxx}(x,y,t)\|_{L_2(D_T)}+\left\|f_{xyy}(x,y,t)\right\|_{L_2(D_T)} \\
& \left.+\left\|f_{xxy}(x,y,t)\right\|_{L_2(D_T)}+\left\|f_{yyy}(x,y,t)\right\|_{L_2(D_T)}]\right\}, \\
B_2(T) = {} & \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}} \\
& \times\left\|\left|\int_0^1\int_0^1 g(x,y,t)dxdy\right|+|g(0,1,t)|\right\|_{C[0,T]} T, \\
C_2(T) = {} & \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}} \\
& \times\left\|\left|\int_0^1\int_0^1 g(x,y,t)dxdy\right|+|g(0,1,t)|\right\|_{C[0,T]} \\
& \times 2\sqrt{T}(\|g_{xxx}(x,y,t)\|_{L_2(D_T)}+\left\|g_{xyy}(x,y,t)\right\|_{L_2(D_T)} \\
& +\left\|g_{xxy}(x,y,t)\right\|_{L_2(D_T)}+\left\|g_{yyy}(x,y,t)\right\|_{L_2(D_T)}), \\
D_2(T) = {} & \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}} \\
& \times\left\|\left|\int_0^1\int_0^1 g(x,y,t)dxdy\right|+|g(0,1,t)|\right\|_{C[0,T]} \\
& \times T(\|p_1(t)\|_{C[0,T]}+\|p_2(t)\|_{C[0,T]}),
\end{aligned}
$$

$$
\begin{aligned}
A_3(T) = {} & \left\|[h(t)]^{-1}\right\|_{C[0,T]} \\
& \times \left\{ \left\| \left(h_2''(t) - \int_0^1 \int_0^1 f(x,y,t)\,dxdy \right) h_1(t) - (h_1''(t) - f(0,1,t)) h_2(t) \right\|_{C[0,T]} \right. \\
& + \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu_k^{-2} \right)^{\frac{1}{2}} \| |h_1(t)| + |h_2(t)| \|_{C[0,T]} \left[\|\varphi_{xxxxx}(x,y)\|_{L_2(Q_{xy})} \right. \\
& + \left\|\varphi_{xxxyy}(x,y)\right\|_{L_2(Q_{xy})} + \left\|\varphi_{xxyyy}(x,y)\right\|_{L_2(Q_{xy})} + \left\|\varphi_{yyyyy}(x,y)\right\|_{L_2(Q_{xy})} \\
& + 2\|\psi_{xxx}(x,y)\|_{L_2(Q_{xy})} + 2\left\|\psi_{xxy}(x,y)\right\|_{L_2(Q_{xy})} \\
& + 2\left\|\psi_{xyy}(x,y)\right\|_{L_2(Q_{xy})} + 2\left\|\varphi_{yyy}(x,y)\right\|_{L_2(Q_{xy})} \\
& + 2\sqrt{T}(\|f_{xxx}(x,y,t)\|_{L_2(D_T)} + \left\|f_{xyy}(x,y,t)\right\|_{L_2(D_T)} \\
& \left. \left. + \left\|f_{xxy}(x,y,t)\right\|_{L_2(D_T)} + \left\|f_{yyy}(x,y,t)\right\|_{L_2(D_T)})\right] \right\},
\end{aligned}
$$

$$
B_3(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]} \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu_{k,n}^{-2} \right)^{\frac{1}{2}} \| |h_1(t)| + |h_2(t)| \|_{C[0,T]} T,
$$

$$
\begin{aligned}
C_3(T) = {} & \left\|[h(t)]^{-1}\right\|_{C[0,T]} \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu_{k,n}^{-2} \right)^{\frac{1}{2}} \| |h_1(t)| + |h_2(t)| \|_{C[0,T]} \\
& \times 2\sqrt{T}(\|g_{xxx}(x,y,t)\|_{L_2(D_T)} + \left\|g_{xyy}(x,y,t)\right\|_{L_2(D_T)} \\
& + \left\|g_{xxy}(x,y,t)\right\|_{L_2(D_T)} + \left\|g_{yyy}(x,y,t)\right\|_{L_2(D_T)}),
\end{aligned}
$$

$$
\begin{aligned}
D_3(T) = {} & \left\|[h(t)]^{-1}\right\|_{C[0,T]} \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu_{k,n}^{-2} \right)^{\frac{1}{2}} \| |h_1(t)| + |h_2(t)| \|_{C[0,T]} \\
& \times T(\|p_1(t)\|_{C[0,T]} + \|p_2(t)\|_{C[0,T]}).
\end{aligned}
$$

From inequalities (36)–(39), we conclude

$$
\begin{aligned}
& \|\tilde{u}(x,y,t)\|_{B_{2,T}^{3,2}} + \|\tilde{a}(t)\|_{C[0,T]} + \left\|\tilde{b}(t)\right\|_{C[0,T]} \\
\leqslant {} & A(T) + B(T)\|a(t)\|_{C[0,T]} \|u(x,y,t)\|_{B_{2,T}^{5}} \\
& + C(T)\|b(t)\|_{C[0,T]} + D(T)\|u(x,y,t)\|_{B_{2,T}^{5}},
\end{aligned}
\tag{40}
$$

where

$$
\begin{aligned}
& A(T) = A_1(T) + A_2(T) + A_3(T), B(T) = B_1(T) + B_2(T) + B_3(T), \\
& C(T) = C_1(T) + C_2(T) + C_3(T), D(T) = D_1(T) + D_2(T) + D_3(T).
\end{aligned}
\tag{41}
$$

So, we can prove the following theorem.

THEOREM 2. *Let the conditions* $C_1)$–$C_5)$ *and the condition*

$$(A(T)+2)(B(T)(A(T)+2)+C(T)+D(T))<1, \tag{42}$$

be fulfilled. Then, problem (1)–(4)*,* (9)*,* (10) *has a unique solution in the ball* $K = K_R(\|z\|_{E_T^5} \leqslant R = A(T)+2)$ *of the space* E_T^5*.*

REMARK 1. Inequality (42) is satisfied for sufficiently small values of T.

Proof. Let's consider the operator equation

$$z = \Phi z, \tag{43}$$

in the space E_T^5, where $z = \{u,a,b\}$. The components $\Phi_i(u,a,b)$ $(i=1,2,3)$ of operator $\Phi(u,a,b)$ defined by the right side of equations (26), (32), (33), respectively. Now, consider the operator $\Phi(u,a,b)$ in the ball $K = K_R(\|z\|_{E_T^5} \leqslant R = A(T)+2)$ of the space E_T^5.

Analogously to (40) we obtain that for any $z, z_1, z_2 \in K_R$ the following estimates hold:

$$\begin{aligned} \|\Phi z\|_{E_T^5} \leqslant\, & A(T)+B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,y,t)\|_{B_{2,T}^5} \\ & +C(T)\,\|b(t)\|_{C[0,T]}+D(T)\,\|u(x,y,t)\|_{B_{2,T}^5} \\ \leqslant\, & A(T)+B(T)(A(T)+2)^2+C(T)(A(T)+2) \\ & +D(T)(A(T)+2), \end{aligned} \tag{44}$$

$$\begin{aligned} \|\Phi z_1-\Phi z_2\|_{E_T^5} \leqslant\, & B(T)R(\|a_1(t)-a_2(t)\|_{C[0,T]}+\|u_1(x,y,t)-u_2(x,y,t)\|_{B_{2,T}^5}) \\ & +C(T)\,\|b_1(t)-b_2(t)\|_{C[0,T]} \\ & +D(T)\,\|u_1(x,y,t)-u_2(x,y,t)\|_{B_{2,T}^5}. \end{aligned} \tag{45}$$

Then by (42), from estimates (44) and (45) it is clear that the operator Φ acts in a ball $K = K_R$ and satisfy the assertion of the contraction mapping principle. Therefore the operator Φ has a unique fixed point $\{u,a,b\}$ in the ball $K = K_R$, which is a unique solution of equation (43); i.e. $\{u,a,b\}$ is a unique solution of the system (26), (32), (33) in the ball $K = K_R$.

Thus, we obtain that the function $u(x,y,t)$ as an element of the space $B_{2,T}^5$ is continuous and has continuous derivatives $u_x(x,y,t)$, $u_{xx}(x,y,t)$, $u_y(x,y,t)$, $u_{xy}(x,y,t)$, $u_{yy}(x,y,t)$, $u_{xxx}(x,y,t)$, $u_{xxy}(x,y,t)$, $u_{xyy}(x,y,t)$, $u_{yyy}(x,y,t)$, $u_{xxxx}(x,y,t)$, $u_{xxxy}(x,y,t)$, $u_{xxyy}(x,y,t)$, $u_{xyyy}(x,y,t)$, and $u_{yyyy}(x,y,t)$ in D_T.

From the equation (2) it is easy to see that

$$u''_{k,n}(t)+(\lambda_k^4+\gamma_n^4-\beta(\lambda_k^2+\gamma_n^2))u_{k,n}(t)=F_{k,n}(t;u,a,b),\ k,n=1,2,\ldots,\ 0\leqslant t\leqslant T,$$

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}\left\|u''_{k,n}(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}$$
$$\leqslant \sqrt{5}[\|u(x,y,t)\|_{B^5_{2,T}}$$
$$+\left\|\|f_x(x,y,t)+f_y(x,y,t)\|_{C[0,T]}\right\|_{L_2(Q_{xy})}$$
$$+\left\|\|a(t)(u_x(x,y,t)+u_y(x,y,t))\|_{C[0,T]}\right\|_{L_2(Q_{xy})}$$
$$+\left\|\|b(t)\,(g_x(x,y,t)+g_y(x,y,t))\|_{C[0,T]}\right\|_{L_2(Q_{xy})}\Big].$$

Thus it follows that $u_{tt}(x,y,t)$ is continuous in D_T.

It is not hard to verify that equation (1) and conditions (2)–(4), (9), (10) are satisfied in the usual sense. Thus, the solution of the problem (1)–(4), (9), (10) is a triple $\{u(x,y,t),a(t),b(t)\}$. By virtue of the Lemma 2, it is unique in the ball $K=K_R$. □

In summary, from Theorem 1 and Theorem 2, straightforward implies the unique solvability of the original problem (1)–(6).

THEOREM 3. *Suppose that all assumptions of Theorem 2, the compatibility conditions (11), (12), and the inequality*

$$\left(T\,\|p_2(t)\|_{C[0,T]}+\|p_1(t)\|_{C[0,T]}+\frac{T}{2}(A(T)+2)\right)T<1,$$

holds. Then problem (1)–(6) has a unique classical solution in the ball $K=K_R(\|z\|_{E^5_T}\leqslant R=A(T)+2)$ *of the space* E^5_T.

Conclusion

In this paper, we have studied the classical solvability of the inverse boundary-value problem for the linearized equation of motion of a homogeneous beam with pinned ends. To study the solvability of the considered problem, we first performed a transformation from the original problem to some auxiliary equivalent problem with trivial boundary conditions. Then, using the Fourier method and contraction mappings principle, the existence and uniqueness theorem for the auxiliary problem is proved. Furthermore, based on the equivalence of these problems, we establish an existence and uniqueness theorem for the classical solution of the original inverse boundary-value problem.

Acknowledgement. The authors sincerely thank the anonymous reviewers for their careful reading of our manuscript and their many insightful comments and suggestions.

REFERENCES

[1] KH. E. ABBASOVA, Y. T. MEHRALIYEV, AND E. I. AZIZBAYOV, *Inverse boundary-value problem for linearized equation of motion of a homogeneous elastic beam*, Int. J. Appl. Math. **33** (2020), 157–170.

[2] E. AZIZBAYOV AND Y. MEHRALIYEV, *A boundary value problem for equation of homogeneous bar with periodic conditions*, Amer. J. Appl. Math. Stat. **3** (2015), 252–256.

[3] E. AZIZBAYOV AND Y. MEHRALIYEV, *A time-nonlocal boundary value problem for equation of homogeneous bar motion*, Bull. Kyiv National Univ. Ser. Math. Mekh. **27** (2012), 20–23.

[4] B. S. BARDIN AND S. D. FURTA, *Local Theory of Existence of the Periodic Wave Movements of an Infinite Beam on not Linearly Elastic Basis*, Mir, Moscow, 1998, (in Russian).

[5] J. R. CANNON AND Y. P. LIN, *An inverse problem of finding a parameter in a semi-linear heat equation*, J. Math. Anal. Appl. **145** (1990), 470–484.

[6] T. GOY, M. NEGRYCH AND I. SAVKA, *On nonlocal boundary value problem for the equation of motion of a homogeneous elastic beam with pinned-pinned ends*, Carpathian Math. Publ. **10** (2018), 105–113.

[7] M. I. IVANCHOV, *Inverse Problem for Equations of Parabolic Type*, VNTL Publishers, LVIV, 2003.

[8] V. K. IVANOV, V. V. VASIN, AND V. P. TANANA, *Theory of Linear Ill-posed Problems and its Applications*, De Gruyter, Berlin, 2002.

[9] YU. A. IZYUMOV AND V. N. SYROMYATNIKOV, *Phase Transitions and Crystal Symmetry*, Springer, New York, 1990.

[10] N. B. KERIMOV AND M. I. ISMAILOV, *An inverse coefficient problem for the heat equation in the case of nonlocal boundary conditions*, J. Math. Anal. Appl. **396** (2012), 546–554.

[11] K. I. KHUDAVERDIYEV AND A. A. VELIYEV, *Investigation of One-Dimensional Mixed Problem for a Class of Pseudohyperbolic Equations of Third Order with Nonlinear Operator Right Side*, Chashyoghly, Baku, 2010, (in Russian).

[12] D. V. KOSTIN, *About one analysis scheme for the two-mode deflections of a weakly inhomogeneous elastic beam*, Dokl. Akad. Nauk **418** (2008), 295–299, (in Russian).

[13] A. I. KOZHANOV, AND T. N. SHIPINA, *Inverse problems of finding the lowest coefficient in the elliptic equation*, J. Sib. Fed. Univ. **14** (2021), 528–542.

[14] A. I. KOZHANOV, *Composite Type Equations and Inverse Problems*, Utrecht: VSP, 1999.

[15] M. M. LAVRENTIEV, *Inverse Problems of Mathematical Physics*, Utrecht: VSP, 2003.

[16] D. LESNIC, *Inverse Problems with Applications in Science and Engineering*, Chapman and Hall/CRC, London, 2021.

[17] YA. T. MEGRALIEV AND F. KH. ALIZADE, *Inverse boundary value problem for a Boussinesq type equation of fourth order with nonlocal time integral conditions of the second kind*, Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki **26** (2016), 503–514, (in Russian).

[18] Y. T. MEHRALIYEV AND E. I. AZIZBAYOV, *On a nonlinear inverse boundary value problem for the linearized equation of motion of a homogeneous elastic beam with periodic and integral conditions*, Proceedings of the 7th International Conference on Control and Optimization with Industrial Applications **2** (2020), 266–268.

[19] Y. T. MEHRALIYEV AND E. I. AZIZBAYOV, *Simultaneous determination of time-dependent coefficients in the linearized equation of motion of a homogeneous beam*, Proceedings of the 8th International Conference on Control and Optimization with Industrial Applications **2** (2022), 336–338.

[20] Y. T. MEHRALIYEV AND F. KANCA, *An inverse boundary value problem for a second order elliptic equation in a rectangle*, Math. Model. Anal. **19** (2014), 241–256.

[21] Y. T. MEHRALIYEV, HE YANG, AND E. I. AZIZBAYOV, *Recovery of the unknown coefficients in a two-dimensional hyperbolic equation*, Math. Methods Appl. Sci. **46** (2022), 1723–1739.

[22] YU. O. MITROPOLSKY AND B. I. MOSEENKOV, *Investigation of Oscillations in Systems with Distributed Parameters*, publishing house of Kyiv University, Kyiv, 1961, (in Ukrainian).

[23] G. K. NAMAZOV, *Inverse Problems of the Theory of Equations of Mathematical Physics*, Maarif, Baku, 1984, (in Russian).

[24] A. I. PRILEPKO, D. G. ORLOVSKY, AND I. A. VASIN, *Methods for Solving Inverse Problems in Mathematical Physics*, Marcel Dekker, New York, 2000.

[25] A. G. RAMM, *Inverse Problems*, Springer, New York, 2005.

[26] V. G. ROMANOV, *Investigation Methods for Inverse Problems*, De Gruyter, Berlin, 2002.

[27] K. B. SABITOV, *Fluctuations of a beam with clamped ends*, J. Samara State Tech. Univ. Ser. Phys. Math. Sci. **19** (2015), 311–324, (in Russian).
[28] J. M. T. THOMPSON, *A General Theory of Elastic Stability*, John Wiley & Sons, Hoboken, 1973.
[29] A. N. TIKHONOV, *On stability of inverse problems*, Dokl. Akad. Nauk SSSR **39** (1943), 195–198, (in Russian).
[30] V. Z. VLASOV AND N. N. LEONT'EV, *Beams, Plates and Covers on the Elastic Basis*, Mir, Moscow, 1960, (in Russian).

(Received December 1, 2023)

Elvin I. Azizbayov
Department of Management of Intelligent Systems
Academy of Public Administration under the
President of the Republic of Azerbaijan
Baku AZ1001, Azerbaijan
and
Department of Mathematics
Khazar University
Baku AZ1096, Azerbaijan
e-mail: eazizbayov@dia.edu.az

Afag I. Hasanova
Department of General Economics
Western Caspian University
Baku AZ1001, Azerbaijan
e-mail: afaghasanova@mail.ru

Yashar T. Mehraliyev
Department of Differential and Integral Equations
Baku State University
Baku AZ1148, Azerbaijan
e-mail: yashar_aze@mail.ru