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EXISTENCE OF NONTRIVIAL PERIODIC SOLUTIONS FOR A CLASS OF p–HAMILTONIAN SYSTEMS

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Abstract. In this paper, we investigate a class of p-Hamiltonian systems. By means of the Mountain Pass Lemma, we obtain the existence of one nontrivial periodic solution under some new conditions.

1. Introduction and main result

We consider the following p-Hamiltonian systems

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' + A(t)|u|^{p-2}u = \nabla F(t,u), \quad a.e.\ t \in [0,T], \\ u(T)-u(0) = u'(T)-u'(0) = 0, \end{cases} \tag{1.1}$$

where $p>1$, $T>0$, $N \geqslant 1$, $u(t) = (u_1(t),\ u_2(t),\ \cdots, u_N(t))$, $A(t) := (a_{ij}(t))_{N\times N} \in C([0,T];\mathbb{R}^{N\times N})$ is a symmetric matrix and T-periodic in t, a_{ij} is continuous. Furthermore, there exists a positive constant $\underline{\lambda}$ such that $(A(t)|x|^{p-2}x,x) \geqslant \underline{\lambda}|x|^p$ for all $x \in \mathbb{R}^N$ and $t \in [0,T]$. $\nabla F(t,x) := (\partial F/\partial x_1, \partial F/\partial x_2, \cdots, \partial F/\partial x_N)$, $F : [0,T]\times\mathbb{R}^N \to \mathbb{R}$ is T-periodic in t and satisfies the following assumption:

(A) $F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$, continuously differentiable in x for $a.e.\ t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+,\mathbb{R}^+)$, $b \in L^1(0,T;\mathbb{R}^+)$ such that

$$|F(t,x)| \leqslant a(|x|)b(t), \quad |\nabla F(t,x)| \leqslant a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and $a.e.\ t \in [0,T]$.

If $A(t) = 0$, the system (1.1) becomes

$$\begin{cases} -(|u'|^{p-2}u')' = \nabla F(t,u), \qquad a.e.\ t \in [0,T], \\ u(T)-u(0) = u'(T)-u'(0) = 0. \end{cases} \tag{1.2}$$

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Many monographs regard it as the vector p-Hamiltonian systems, there have been many existence and multiplicity results of solutions for problem (1.2), see [8, 11, 13, 22, 25, 27, 31] and references therein. In [8], considering nonlinear periodic systems driven by the vector p-Laplacian, Jebelean and Papageorgiou proved the existence and multiplicity of solutions for system (1.2). In [25], by using minimax methods of critical point theory, Xu and Tang got the existence of periodic solutions for system (1.2). Taking account of an improved inequality in [23], Zhang and Tang [28] also obtained an estimation of periodic solution for system (1.2). Moreover, under a new condition, they proved in [29] the existence of a nonconstant solution for system (1.2) by the linking theorem. In [11], Li, Agarwal and Tang got some existence theorems of infinitely many periodic solutions for system (1.2) by minimax methods in critical point theory. In [13], Ma and Zhang obtained a sequence of distinct periodic solutions for system (1.2) and got the following result.

THEOREM 1. (see [13, Theorem 2]) *Suppose that F satisfies assumption* (A) *and the following conditions:*

(Z_1) $F(t,x) \geqslant 0$ *for all* $(t,x) \in [0,T]\times\mathbb{R}^N$.

(Z_2) $\lim\limits_{|x|\to 0}\frac{F(t,x)}{|x|^p} = 0 < \liminf\limits_{|x|\to\infty}\frac{F(t,x)}{|x|^p}$ *uniformly for a.e.* $t\in[0,T]$.

(Z_3) $\limsup\limits_{|x|\to+\infty}\frac{F(t,x)}{|x|^p} \leqslant M_0 < +\infty$ *uniformly for some* $M_0>0$ *and a.e.* $t\in[0,T]$.

(Z_4) *There exists* $\gamma\in L^1(0,T;\mathbb{R}^+)$ *such that*

$$(\nabla F(t,x),x) - pF(t,x) \geqslant \gamma(t) \quad \textit{for all } x\in\mathbb{R}^N \textit{ and a.e. } t\in[0,T].$$

(Z_5) $\lim\limits_{|x|\to\infty}[(\nabla F(t,x),x)-pF(t,x)] = +\infty$ *for a.e.* $t\in[0,T]$.

Then system (1.2) *has a sequence of distinct periodic solutions with period* k_jT *satisfying* $k_j\in\mathbb{N}$ *and* $k_j\to\infty$ *as* $j\to\infty$.

In [10], Li et al proved the existence of a nonconstant T-periodic solution for system (1.2).

THEOREM 2. (see [10, Theorem 1.4]) *Suppose that* $F(t,x)$ *satisfies assumption* (A) *and the following conditions:*

(I_1) $F(t,x)\geqslant 0$ *for all* $(t,x)\in[0,T]\times\mathbb{R}^N$.

(I_2) $\lim\limits_{|x|\to 0}\frac{F(t,x)}{|x|^p} = 0$ *uniformly for a.e.* $t\in[0,T]$.

(I_3) *There exist constants* $\mu_2>p$ *and* $L_0>0$ *and a function* $W\in L^1(0,T;\mathbb{R})$ *such that, for all* $x\in\mathbb{R}^N$ *with* $|x|\geqslant L_0$,

$$\mu_2F(t,x) - (\nabla F(t,x),x) \leqslant W(t)|x|^p \quad \textit{for a.e. } t\in[0,T]$$

and

$$\limsup_{|x|\to\infty}\frac{\mu_2F(t,x)-(\nabla F(t,x),x)}{|x|^p} \leqslant 0 \quad \textit{uniformly for a.e. } t\in[0,T].$$

(I_4) *There exists* $\Omega \subset [0,T]$ *with meas* $\Omega > 0$ *such that*

$$\liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^p} > 0 \quad \text{uniformly for a.e. } t \in \Omega.$$

Then system (1.2) *possesses a nonconstant* T*-periodic solution.*

For $p = 2$ and $A(t) = 0$, system (1.1) degenerates to the following widely well-known non-autonomous second order Hamiltonian systems

$$\begin{cases} \ddot{u}(t) + \nabla F(t,u(t)) = 0, & a.e.\ t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases} \tag{1.3}$$

Many existence results have been obtained for (1.3) via variational methods, such as [4, 12, 14, 16, 18, 19, 20, 21, 24, 26, 30, 32] and references therein. In 1978, Rabinowitz [15] proved the existence of nonconstant periodic solutions for system (1.3) under the following condition: there exist $\mu > 2$ and $L > 0$ satisfying

$$0 < \mu F(t,x) \leqslant (\nabla F(t,x),x) \quad \text{for all } |x| \geqslant L \text{ and } a.e.\ t \in [0,T].$$

For more than forty years, the above hypothesis has been extensively used in the literature, see [3, 5, 6] and their references. In [7], using a new condition, Fei proved the existence of nonconstant periodic solutions for system (1.3). Employing a new saddle point theorem and using linking methods, Schechter studied in [17] the existence of periodic solutions for system (1.3).

For $p = 2$ and $A(t) \neq 0$, Ke and Liao proved in [9] that there is a nontrivial periodic solution for system (1.1) by introducing a new growth condition and using the Mountain Pass Lemma under the (C) condition. In this paper, we investigate the existence of periodic solutions for p-Hamiltonian systems (1.1). By applying the Mountain Pass Lemma in critical point theory, we obtain the following result for problem (1.1).

THEOREM 3. *Assume that* F *satisfies assumption* (A) *and the following conditions:*

(H_1) $\limsup_{|x|\to 0} \frac{F(t,x)}{|x|^p} = 0$ *uniformly in a.e.* $t \in [0,T]$.

(H_2) *There exist a constant* $r_0 > 0$ *and a function* $\theta(x)$ *such that*

$$0 < (p + \theta(x))F(t,x) \leqslant (\nabla F(t,x),x)$$

for $|x| \geqslant r_0$ *and a.e.* $t \in [0,T]$, *where* $\theta(x) : \{x \in \mathbb{R}^N : |x| \geqslant r_0\} \to \mathbb{R}$ *is continuous and satisfies the following assumptions:*

(i) $\theta(x) > 0$ *for all* $|x| \geqslant r_0$.

(ii) $\lim_{|x|\to+\infty} \theta(x)|x|^p = +\infty$.

(iii) *There is* $x^0 \in \mathbb{R}^N$ *with* $|x^0| = 1$ *satisfying*

$$\lim_{r \to +\infty} \int_{r_0}^{r} \frac{\theta(sx^0)}{s} ds = +\infty.$$

Then system (1.1) *has a nontrivial periodic solution.*

REMARK 1. There are functions $F(t,x)$ satisfying the assumptions of Theorem 3 and not satisfying the conditions of Theorem 1 and Theorem 2. For example, set

$$F(t,x) = \begin{cases} |x|^p \ln |x|, & |x| \geqslant 3, \\ |x|^p \ln 3, & 2 \leqslant |x| \leqslant 3, \\ |x|^p \ln(1+|x|), & |x| \leqslant 2. \end{cases}$$

Choosing $\theta(x) = 1/\ln|x|$, $r_0 = 3$ and $x^0 = (1,0,\cdots,0) \in \mathbb{R}^N$, it is easy to verify that $F(t,x)$ satisfies the conditions of Theorem 3. However, $F(t,x)$ does not satisfy condition (Z_3) of Theorem 1 and (I_3) of Theorem 2. Moreover, our result extends Theorem 1.1 in [9].

2. Proof of main result

In this section, the Sobolev space $W_T^{1,p}$ is defined as

$$W_T^{1,p} = \left\{ u : [0,T] \to \mathbb{R}^N \,\middle|\, \begin{array}{l} u \text{ is absolutely continuous} \\ u(0) = u(T) \text{ and } u' \in L^p(0,T;\mathbb{R}^N) \end{array} \right\},$$

and endowed with the norm

$$\|u\|_A = \left(\int_0^T |u'(t)|^p dt + \int_0^T (A(t)|u(t)|^{p-2}u(t), u(t))dt \right)^{1/p}.$$

Observe that

$$\begin{aligned} (A(t)|x|^{p-2}x, x) &= |x|^{p-2} \sum_{i,j=1}^{N} a_{ij}(t) x_i x_j \\ &\leqslant |x|^{p-2} \sum_{i,j=1}^{N} |a_{ij}(t)||x_i||x_j| \\ &\leqslant \left(\sum_{i,j=1}^{N} \|a_{ij}(t)\|_\infty \right) |x|^p, \end{aligned}$$

where $\|\cdot\|_\infty = \sup_{t\in[0,T]} |\cdot|$ and $|\cdot|$ is the usual norm on $\mathbb{R}^N$. Hence there exists a constant $\bar{\lambda} \geqslant \sum_{i,j=1}^{N} \|a_{ij}(t)\|_\infty$ such that $(A(t)|x|^{p-2}x, x) \leqslant \bar{\lambda}|x|^p$ for all $x \in \mathbb{R}^N$. So we have

$\underline{\lambda}|x|^p \leqslant (A(t)|x|^{p-2}x,x) \leqslant \bar{\lambda}|x|^p$ and deduce

$$\min\{1,\underline{\lambda}\}\|u\|^p \leqslant \|u\|_A^p \leqslant \max\{1,\bar{\lambda}\}\|u\|^p,$$

where

$$\|u\| = \left(\int_0^T |u(t)|^p dt + \int_0^T |u'(t)|^p dt\right)^{1/p}.$$

Then the norms $\|\cdot\|$ and $\|\cdot\|_A$ are equivalent. Besides, it can be seen from the assumption (A) that the functional Φ, given by

$$\Phi(u) = \frac{1}{p}\int_0^T |u'(t)|^p dt + \frac{1}{p}\int_0^T (A(t)|u(t)|^{p-2}u(t),u(t))dt - \int_0^T F(t,u(t))dt,$$

is continuously differentiable on $W_T^{1,p}$. Furthermore, we have

$$\begin{aligned}\langle\Phi'(u),v\rangle = &\int_0^T ((|u'(t)|^{p-2}u'(t),v'(t)) + (A(t)|u(t)|^{p-2}u(t),v(t)))dt \\ &- \int_0^T (\nabla F(t,u(t)),v(t))dt\end{aligned}$$

for all $u,v \in W_T^{1,p}$. Moreover, it is known that the weak solutions of system (1.1) are exactly the critical points of Φ in $W_T^{1,p}$ (see [14]).

We need the Mountain Pass Lemma under the (PS) condition, and refer to [1] for more details.

THEOREM 4. (see [1, Mountain Pass Lemma]) *Let $(X,\|\cdot\|_X)$ be a Banach space, and let $\Phi \in C^1(X,\mathbb{R})$ satisfy the (PS) condition. Suppose that $\Phi(0) = 0$ and*

(A_1) *There exist positive constants ρ and α such that $\Phi(u) \geqslant \alpha > 0$ for all $u \in X$ with $\|u\|_X = \rho$.*

(A_2) *There exists $e \in X$ with $\|e\|_X > \rho$ such that $\Phi(e) < 0$.*

Then Φ possesses a critical value $c \geqslant \alpha$ given by

$$c := \inf_{\gamma\in\Gamma}\sup_{s\in[0,1]} \Phi(\gamma(s)),$$

where

$$\Gamma := \{\gamma \in C([0,1],X) | \gamma(0) = 0, \gamma(1) = e\}.$$

In [2], a deformation lemma is proved under a weaker condition than the usual (PS) condition. So, due to [2], we can replace the (PS) condition of the Mountain Pass Lemma with the (C) condition, which will be explained later, and the lemma is still true.

Proof. We divide the proof into three steps.

Step 1. We show that there exist positive numbers ρ and α such that $\inf_{\|u\|_A=\rho} \Phi(u) \geqslant \alpha > 0$. Note that $W_T^{1,p} \hookrightarrow C_B$, where C_B is the space of the continuous functions on $[0,T]$, with the norm $\|u\|_\infty = \sup_{t\in[0,T]} |u(t)|$, and $|\cdot|$ is the usual norm on $\mathbb{R}^N$. Thus by Sobolev's inequality (see [14, proposition 1.1]), there is $M > 0$ such that

$$\|u\|_\infty \leqslant M\|u\|_A \tag{2.1}$$

for all $u \in W_T^{1,p}$. From (H_1), we have for any $\varepsilon \in (0, 1/(2pM^pT))$ that there exists a constant $\delta > 0$ such that

$$|F(t,x)| \leqslant \varepsilon |x|^p$$

for $|x| < \delta$ and $a.e.\ t \in [0,T]$. Taking $\rho \in (0, \delta/(2M))$, by (2.1), we have

$$\begin{aligned}
\Phi(u) &= \frac{1}{p}\int_0^T |u'(t)|^p dt + \frac{1}{p}\int_0^T (A(t)|u(t)|^{p-2}u(t), u(t))dt - \int_0^T F(t,u(t))dt \\
&= \frac{1}{p}\|u\|_A^p - \int_0^T F(t,u(t))dt \\
&\geqslant \frac{1}{p}\|u\|_A^p - \varepsilon\int_0^T |u(t)|^p dt \\
&\geqslant \left(\frac{1}{p} - \varepsilon M^p T\right)\|u\|_A^p \\
&\geqslant \frac{\rho^p}{2p}
\end{aligned}$$

for $u \in W_T^{1,p}$ with $\|u\|_A = \rho$. Setting $\alpha := \rho^p/2p > 0$, one has $\inf_{\|u\|_A=\rho} \Phi(u) \geqslant \alpha > 0$. Then the step 1 is finished.

Step 2. We show that under assumptions (A) and (H_2) there exists $u_0 \in W_T^{1,p}$ with $\|u_0\|_A > \rho$ such that $\Phi(u_0) < 0$. When $s \geqslant r_0$, from assumptions (A) and (H_2), one has

$$\int_{r_0}^s \frac{(\nabla F(t,\tau x^0), x^0)}{F(t,\tau x^0)} d\tau \geqslant \int_{r_0}^s \frac{p + \theta(\tau x^0)}{\tau} d\tau.$$

Then, we obtain that

$$\ln \frac{F(t,sx^0)}{F(t,r_0x^0)} \geqslant \ln \frac{s^p}{r_0^p} + \int_{r_0}^s \frac{\theta(\tau x^0)}{\tau} d\tau.$$

So, we get

$$F(t,sx^0) \geqslant \frac{F(t,r_0x^0)}{r_0^p} \cdot e^{\int_{r_0}^s \frac{\theta(\tau x^0)}{\tau}d\tau} \cdot s^p$$

for $s \geqslant r_0$ and $a.e.\ t \in [0,T]$. In addition, by assumption (H_2), it follows that

$$\lim_{s\to+\infty} e^{\int_{r_0}^s \frac{\theta(\tau x^0)}{\tau}d\tau} = +\infty. \tag{2.2}$$

Then we have

$$\begin{aligned}\Phi(sx^0) &= \frac{1}{p}\|sx^0\|_A^p - \int_0^T F(t,sx^0)dt \\ &\leqslant \frac{\bar{\lambda}^p s^p}{p} - \int_0^T \frac{F(t,r_0x^0)}{r_0^p}\cdot e^{\int_{r_0}^s \frac{\theta(\tau x^0)}{\tau}d\tau}\cdot s^p dt \\ &= \left(\frac{\bar{\lambda}^p}{p} - \int_0^T \frac{F(t,r_0x^0)}{r_0^p}\cdot e^{\int_{r_0}^s \frac{\theta(\tau x^0)}{\tau}d\tau} dt\right)s^p \\ &= \left(\frac{\bar{\lambda}^p}{p} - \frac{1}{r_0^p}e^{\int_{r_0}^s \frac{\theta(\tau x^0)}{\tau}d\tau}\int_0^T F(t,r_0x^0)dt\right)s^p\end{aligned}$$

for $s \geqslant r_0$ and $a.e.\ t \in [0,T]$. From (2.2), one has

$$e^{\int_{r_0}^s \frac{\theta(\tau x^0)}{\tau}d\tau} > \frac{2\bar{\lambda}^p r_0^p}{p\int_0^T F(t,r_0x^0)dt}$$

for s large enough. So, it implies that

$$\Phi(sx_0) < -\frac{\bar{\lambda}^p s^p}{p} < 0$$

for s large enough. Then the step 2 is finished.

Step 3. We show that under the assumptions (A), (H_1) and (H_2), Φ satisfies the (C) condition. That is, for every constant c and sequence $\{u_n\} \subset W_T^{1,p}$, $\{u_n\}$ has a convergent subsequence if

$$\|\Phi'(u_n)\|(1+\|u_n\|) \to 0 \text{ and } \Phi(u_n) \to c \quad \text{as } n \to \infty. \tag{2.3}$$

In order to show that Φ satisfies the (C) condition, by a standard argument (see [3]), it suffices to prove that every sequence $\{u_n\}$ satisfying (2.3) is bounded. If $\inf\limits_{|x|\geqslant r_0} \theta(x) > 0$, the proof is trivial.

Assume now that $\inf\limits_{|x|\geqslant r_0} \theta(x) = 0$. We show that if $\{a_n\} \subset \{x \in \mathbb{R}^N : |x| \geqslant r_0\}$ is a sequence such that $\lim\limits_{n\to\infty} \theta(a_n) = 0$, then

$$\lim_{n\to\infty} |a_n| = +\infty. \tag{2.4}$$

Otherwise, there exists a bounded subsequence of $\{a_n\}$, still denoted by $\{a_n\}$. Passing to a subsequence if necessary, we can suppose that there exists $a_0 \in \{x \in \mathbb{R}^N : |x| \geqslant r_0\}$ satisfying

$$\lim_{n\to\infty} a_n = a_0. \tag{2.5}$$

From (i) of assumption (H_2), (2.5), and the continuity of θ, it follows that

$$0 = \lim_{n\to\infty} \theta(a_n) = \theta(\lim_{n\to\infty} a_n) = \theta(a_0) > 0,$$

a contradiction. So, we have

$$\lim_{n\to\infty} |a_n| = +\infty.$$

Suppose that $\{u_n\}$ is unbounded. Then, if necessary, after going to a subsequence, we can assume that

$$\xi_n := \|u_n\|_A \to +\infty \text{ as } n \to \infty. \tag{2.6}$$

Let $v_n = u_n/\|u_n\|_A$. Then one has $\|v_n\|_A = 1$ and $u_n = \|u_n\|_A v_n = \xi_n v_n$. Using (2.1), we have

$$\|v_n\|_\infty \leqslant M.$$

Choose $x_{\xi_n} \in \{x \in \mathbb{R}^N : r_0 \leqslant |x| \leqslant \xi_n M\}$, such that

$$\theta(x_{\xi_n}) = \min_{r_0 \leqslant |x| \leqslant \xi_n M} \theta(x). \tag{2.7}$$

Then, we have $\xi_n \geqslant |x_{\xi_n}|/M$. Furthermore, we get from (2.6), (2.7) and $\inf_{|x|\geqslant r_0} \theta(x) = 0$ that

$$\lim_{n\to\infty} \theta(x_{\xi_n}) = 0.$$

Thus, one sees from (2.4) that

$$\lim_{n\to+\infty} |x_{\xi_n}| = +\infty. \tag{2.8}$$

Set

$$\Omega_n^- = \{t \in [0,T] : |\xi_n v_n(t)| < r_0\}, \quad \Omega_n^+ = \{t \in [0,T] : |\xi_n v_n(t)| \geqslant r_0\}.$$

Then, one obtains from (2.3) that

$$\begin{aligned} o(1) &= |\langle \Phi'(u_n), u_n \rangle| \\ &= \left| \|u_n\|_A^p - \int_0^T (\nabla F(t, u_n(t)), u_n(t)) dt \right| \\ &= \left| \xi_n^p - \int_{\Omega_n^-} (\nabla F(t, \xi_n v_n(t)), \xi_n v_n(t)) dt - \int_{\Omega_n^+} (\nabla F(t, \xi_n v_n(t)), \xi_n v_n(t)) dt \right|, \end{aligned}$$

with $o(1) \to 0$ as $n \to \infty$. So, we get

$$\int_{\Omega_n^+} (\nabla F(t, \xi_n v_n), \xi_n v_n) dt \leqslant \xi_n^p + \left| \int_{\Omega_n^-} (\nabla F(t, \xi_n v_n), \xi_n v_n) dt \right| + o(1). \tag{2.9}$$

In addition, for $a.e.\ t \in \Omega_n^+$, we have, according to (H_2), that

$$0 < (p + \theta(x_{\xi_n})) F(t, \xi_n v_n) \leqslant (\nabla F(t, \xi_n v_n), \xi_n v_n).$$

Hence, we get

$$\int_{\Omega_n^+} F(t, \xi_n v_n(t)) dt \leqslant \frac{1}{p + \theta(x_{\xi_n})} \int_{\Omega_n^+} (\nabla F(t, \xi_n v_n(t)), \xi_n v_n(t)) dt. \tag{2.10}$$

Besides, from assumption (A) , there exists $\beta > 0$ such that

$$\left|\int_{\Omega_n^-}(\nabla F(t,\xi_n v_n(t)),\xi_n v_n(t))dt\right|,\ \left|\int_{\Omega_n^-}F(t,\xi_n v_n(t))dt\right| \leqslant \beta. \tag{2.11}$$

We deduce from (2.9), (2.10) and (2.11) that

$$\begin{aligned}
\Phi(u_n) &= \Phi(\xi_n v_n)\\
&= \frac{1}{p}\xi_n^p - \int_{\Omega_n^+}F(t,\xi_n v_n(t))dt - \int_{\Omega_n^-}F(t,\xi_n v_n(t))dt\\
&\geqslant \frac{1}{p}\xi_n^p - \beta - \frac{1}{p+\theta(x_{\xi_n})}\int_{\Omega_n^+}(\nabla F(t,\xi_n v_n(t)),\xi_n v_n(t))dt\\
&\geqslant \frac{1}{p}\xi_n^p - \beta - \frac{1}{p+\theta(x_{\xi_n})}\left(\xi_n^p + \left|\int_{\Omega_n^-}(\nabla F(t,\xi_n v_n),\xi_n v_n)dt\right| + o(1)\right)\\
&\geqslant \frac{\theta(x_{\xi_n})|\xi_n|^p}{p(p+\theta(x_{\xi_n}))} - \beta - \frac{\beta+o(1)}{p+\theta(x_{\xi_n})}\\
&\geqslant \frac{\theta(x_{\xi_n})|x_{\xi_n}|^p}{M^p p(p+\theta(x_{\xi_n}))} - \frac{(p+1)\beta+o(1)}{p}.
\end{aligned}$$

Owing to (2.8), we have $|x_{\xi_n}| \to +\infty$ as $n \to +\infty$. So, from (ii) of assumption (H_2), we obtain that $\Phi(u_n) \to +\infty$ as $n \to \infty$, which contradicts (2.3). Hence, $\{u_n\}$ is bounded in $W_T^{1,p}$.

Now, it is effortless to verify that $\Phi(0) = 0$. Uniting the above three steps and Theorem 4, we get the existence of a nontrivial critical point of Φ which is the nontrivial periodic solution to system (1.1). □

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