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ENTIRE SOLUTIONS FOR SEVERAL SYSTEMS OF NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, by using the Nevanlinna theory and the Hadamard factorization theory of meromorphic functions, we obtain the existence and the forms of the finite order transcendental entire solutions of several systems of nonlinear differential equations.

1. Introduction

In 1637, Fermat stated the conjecture (which is known as Fermat's last theorem) that the equation $x^m + y^m = 1$ cannot have positive rational solutions if $m > 2$. Since then, the equation has been a subject of intense and often heated discussions. In 1995, Wiles [23, 24] confirmed the profound conjecture. However, several research directions derived from this conjecture are still a popular issue of concern to many mathematicians today, and one of these issues is discussing the solutions of Fermat-type functional and differential equations (see e.g. [11] and [14]).

In this paper, we mainly concern the existence and expression forms of entire solutions of the Fermat-type differential equations which are clearly related to the functional equations

$$f^m(z) + g^n(z) = 1 \quad (z \in \mathbb{C},\ m,n \in \mathbb{N}_+). \tag{1.1}$$

The entire solutions of equation (1.1) were completely analyzed by Montel [17], Cartan [2], Iyer [12] and Gross [7, 8]. For the convenience of the reader, one can list the related results as follows:

THEOREM 1. *The solutions $f(z)$ and $g(z)$ for equation (1.1) are characterized as follows:*

(1) *If $m = n = 2$, then the entire solutions are $f(z) = \cos(h(z))$ and $g(z) = \sin(h(z))$, where $h(z)$ is any entire function, and the meromorphic solutions are $f(z) = \dfrac{1-\alpha^2(z)}{1+\alpha^2(z)}$ and $g(z) = \dfrac{2\alpha(z)}{1+\alpha^2(z)}$, where $\alpha(z)$ is any meromorphic function.*

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(2) *If* $\frac{1}{m}+\frac{1}{n}<1$, *then the entire solutions* $f(z)$ *and* $g(z)$ *must be both constant.*

(3) *If* $m=n=3$, *then the meromorphic solutions are* $f(z)=\frac{\sqrt{3}+\wp'(h(z))}{2\sqrt{3}\wp(h(z))}$ *and* $g(z)=\frac{\eta(\sqrt{3}-\wp'(h(z)))}{2\sqrt{3}\wp(h(z))}$, *where* $h(z)$ *is any entire function,* $\eta^3=1$ *and* $\wp(z)$ *is denoted as the Weierstrass* $\wp$*-function that satisfies* $\wp'^2(z)=4\wp^3(z)-1$ *under appropriate periods.*

(4) *If* $m=n\geqslant 4$, *then the meromorphic solutions* $f(z)$ *and* $g(z)$ *must be both constant.*

In recent years, replying on the rapid development of Nevanlinna theory in meromorphic function with one and several variables, there were lots of references focusing on the solutions of the Fermat-type equations in the case the function $f(z)$ has a special relationship with $g(z)$.

In 2004, Yang and Li [25] considered the differential equations

$$f^2(z)+L^2(f(z))=a(z), \tag{1.2}$$

where $L(f(z))=\sum_{k=0}^{n}b_k(z)f^{(k)}(z)$ $(n\in\mathbb{N}_+)$ is a linear differential polynomial in $f(z)$, and $a(z),b_0(z),b_1(z),\cdots,b_{n-1}(z)$ are polynomials and $b_n(z)$ is a nonzero constant. Precisely, they obtained the following result.

THEOREM 2. ([25]) *If* $a(z)\not\equiv 0$, *then the transcendental meromorphic solution of the equation* (1.2) *must have the form*

$$f(z)=\frac{P(z)\mathrm{e}^{R(z)}+Q(z)\mathrm{e}^{-R(z)}}{2},$$

where $P(z),Q(z),R(z)$ *are polynomials, and* $P(z)Q(z)=a(z)$. *If furthermore all* $b_k(z)$ $(k=0,1,\cdots,n-1)$ *are constants, then* $\deg P(z)+\deg Q(z)\leqslant n-1$. *Moreover,* $R(z)=\lambda$ *is a nonzero constant which satisfies the following equations*

$$\sum_{k=0}^{n}b_k(z)\lambda^k=-\mathrm{i},\quad \sum_{k=j}^{n}b_k(z)\binom{k}{j}\lambda^{k-j}=0,\ \ j=1,2,\cdots,p,$$

and

$$\sum_{k=0}^{n}b_k(z)(-\lambda)^k=\mathrm{i},\quad \sum_{k=j}^{n}b_k(z)\binom{k}{j}(-\lambda)^{k-j}=0,\ \ j=1,2,\cdots,q,$$

where p *and* q *are the degree of* $P(z)$ *and* $Q(z)$.

It is easily seen from Theorem 2 that the equation $f^2(z)+f'^2(z)=1$ has only transcendental entire solutions with the form $f(z)=\frac{A}{2}\mathrm{e}^{Bz}+\frac{1}{2A}\mathrm{e}^{-Bz}$, where A,B are nonzero constants.

Saleeby [21] made some elementary observations on right factors of meromorphic function to describe complex analytic solutions to the quadratic trinomial functional equation $f^2(z)+2\omega f(z)g(z)+g^2(z)=1$, $\omega^2\neq 1$ and obtained a result associate with the partial differential equations $u_x^2(x,y)+2\omega u_x(x,y)u_y(x,y)+u_y^2(x,y)=1$. Later, Liu and Yang [15] studied the existence and form of solutions of the following quadratic trinomial differential equation

$$f^2(z)+2\omega f(z)f'(z)+f'^2(z)=1, \tag{1.3}$$

and proved

THEOREM 3. ([15]) *If $\omega\in\mathbb{C}$ and $\omega^2\neq 0,1$, then equation (1.3) has no transcendental meromorphic solutions.*

In 2019, Han and Lü [9] considered the equation (1.1) when $g(z)=f'(z)$ and 1 is replaced by $\mathrm{e}^{\alpha z+\beta}(\alpha,\beta\in\mathbb{C})$ and showed

THEOREM 4. ([9]) *The meromorphic solutions of equation*

$$f^2(z)+f'^2(z)=\mathrm{e}^{\alpha z+\beta}$$

must be entire functions. Either $\alpha=0$ and the general solutions of equation are $f(z)=\mathrm{e}^{\frac{\beta}{2}}\sin(z+b)$, or $f(z)=d\mathrm{e}^{\frac{\alpha z+\beta}{2}}$. Here $b,d\in\mathbb{C}$ and $d^2(4+\alpha^2)=4$.

More generally, Luo, Xu and Hu [16] investigated the quadratic trinomial functional equations

$$f^2(z)+2\omega f(z)f'(z)+f'^2(z)=\mathrm{e}^{h(z)}, \tag{1.4}$$

and proved

THEOREM 5. ([16]) *Let $\omega\in\mathbb{C}$ and $\omega^2\neq 0,1$, $h(z)$ is a nonconstant polynomial, and $f(z)$ is a transcendental entire solution of finite order for equation (1.4). Then $h(z)$ must be of the form $h(z)=az+b$, where a and b are constants.*

A trivial verification shows (1.2) and (1.4) can be rewritten as

$$\{f(z)+\mathrm{i}L(f(z))\}\{f(z)-\mathrm{i}L(f(z))\}=a(z)$$

and

$$\{f(z)+\lambda_1 f'(z)\}\{f(z)+\lambda_2 f'(z)\}=\mathrm{e}^{g(z)},$$

respectively, where λ_1,λ_2 are constants satisfy $\lambda_1+\lambda_2=2\omega$, $\lambda_1\lambda_2=1$. Motivated by the above observations, it is natural to raise the following question.

QUESTION 1. How to describe the entire solutions for the equation of more general form as

$$(af(z)+bf'(z))^m(cf(z)+df'(z))^n=\mathrm{e}^{h(z)}, \tag{1.5}$$

where $m,n\in\mathbb{N}_+$, $a,b,c,d\in\mathbb{C}$ are constants and h is a entire function?

In addition, the investigations of the solutions for systems involving these general quadratic equations in $\mathbb{C}$ will be a novel work. Henceforth, we consider six classes of different systems of general nonlinear differential equations in $\mathbb{C}$ as follows:

$$\begin{cases}(af+bf')^m(cg+dg')^n=\mathrm{e}^h,\\(ag+bg')^m(cf+df')^n=\mathrm{e}^h,\end{cases}\tag{1.6}$$

$$\begin{cases}(af+bf')^m(cg+dg')^n=\mathrm{e}^h,\\(ag+bg')^n(cf+df')^m=\mathrm{e}^h,\end{cases}\tag{1.7}$$

$$\begin{cases}(af+bg')^m(cf'+dg)^n=\mathrm{e}^h,\\(af'+bg)^m(cf+dg')^n=\mathrm{e}^h,\end{cases}\tag{1.8}$$

$$\begin{cases}(af+bg')^m(cf'+dg)^n=\mathrm{e}^h,\\(af'+bg)^n(cf+dg')^m=\mathrm{e}^h,\end{cases}\tag{1.9}$$

$$\begin{cases}(af+bg)^m(cf'+dg')^n=\mathrm{e}^h,\\(af'+bg')^m(cf+dg)^n=\mathrm{e}^h,\end{cases}\tag{1.10}$$

and

$$\begin{cases}(af+bg)^m(cf'+dg')^n=\mathrm{e}^h,\\(af'+bg')^n(cf+dg)^m=\mathrm{e}^h.\end{cases}\tag{1.11}$$

The above discussions lead us to raise a natural question.

QUESTION 2. What can be said about the existence and forms of the solutions of the above systems of nonlinear differential equations in $\mathbb{C}$?

This paper focuses on describing transcendental solutions for various systems of nonlinear differential equations. In fact, we obtain the description of entire solutions of finite order for the systems (1.6), (1.7), (1.8), (1.9), (1.10) and (1.11), when $m,n\in\mathbb{N}_+$, $a,b,c,d\in\mathbb{C}-\{0\}$ satisfying that $ad\neq bc$ and h is a nonconstant entire solutions in $\mathbb{C}$. Next, we will give our main results and proofs in three sections.

2. Systems of ODEs (1.6) and (1.7)

The following theorems are concerned with the systems (1.6) and (1.7).

THEOREM 6. *Let $m,n\in\mathbb{N}_+$, $a,b,c,d\in\mathbb{C}-\{0\}$ satisfying that $ad\neq bc$ and h be a nonconstant entire solutions in $\mathbb{C}$, then (f,g) is the pair of entire solutions of finite order for system (1.6) in $\mathbb{C}$ if and only if the following assertions hold:*

(i)

$$\begin{cases}h(z)=\left(m\dfrac{a-c\mathrm{e}^A}{d\mathrm{e}^A-b}+n\dfrac{a-c\mathrm{e}^B}{d\mathrm{e}^B-b}\right)z+m(A+C_1)+nC_2,\\ f(z)=\dfrac{d\mathrm{e}^A-b}{ad-bc}\mathrm{e}^{\frac{a-c\mathrm{e}^A}{d\mathrm{e}^A-b}z+C_1},\\ g(z)=\dfrac{d\mathrm{e}^B-b}{ad-bc}\mathrm{e}^{\frac{a-c\mathrm{e}^B}{d\mathrm{e}^B-b}z+C_2},\end{cases}$$

where A, B, C_1, C_2 *are constants satisfying that* $(m-n)\left(\frac{a-c\mathrm{e}^A}{d\mathrm{e}^A-b}-\frac{a-c\mathrm{e}^B}{d\mathrm{e}^B-b}\right)=0$ *and* $(m-n)(C_1-C_2)=m(B-A)$*;*

(ii)

$$\begin{cases} h(z)=\left(n\frac{a-c\mathrm{e}^B}{d\mathrm{e}^B-b}-m\frac{c}{d}\right)z+mC_1+nC_3, \\ f(z)=\frac{d\mathrm{e}^{-\frac{c}{d}z+C_1}-b\mathrm{e}^{-\frac{a}{b}z+C_2}}{ad-bc}, \\ g(z)=\frac{d\mathrm{e}^B-b}{ad-bc}\mathrm{e}^{\frac{a-c\mathrm{e}^B}{d\mathrm{e}^B-b}z+C_3}, \end{cases}$$

where B, C_1, C_2, C_3 *are constants satisfying that* $n\frac{a}{b}-m\frac{c}{d}+(n-m)\frac{a-c\mathrm{e}^B}{d\mathrm{e}^B-b}=0$ *and* $m(C_1-C_3)+n(C_3-C_2)=mB$*;*

(iii)

$$\begin{cases} h(z)=\left(m\frac{a-c\mathrm{e}^A}{d\mathrm{e}^A-b}-n\frac{a}{b}\right)z+m(A+C_3)+nC_2, \\ f(z)=\frac{d\mathrm{e}^A-b}{ad-bc}\mathrm{e}^{\frac{a-c\mathrm{e}^A}{d\mathrm{e}^A-b}z+C_3}, \\ g(z)=\frac{d\mathrm{e}^{-\frac{c}{d}z+C_1}-b\mathrm{e}^{-\frac{a}{b}z+C_2}}{ad-bc}, \end{cases}$$

where A, C_1, C_2, C_3 *are constants satisfying that* $m\frac{c}{d}-n\frac{a}{b}+(m-n)\frac{a-c\mathrm{e}^A}{d\mathrm{e}^A-b}=0$ *and* $m(C_1-C_3)+n(C_3-C_2)=mA$*;*

(iv)

$$\begin{cases} h(z)=-\left(m\frac{c}{d}+n\frac{a}{b}\right)z+mC_1+nC_4, \\ f(z)=\frac{d\mathrm{e}^{-\frac{c}{d}z+C_1}-b\mathrm{e}^{-\frac{a}{b}z+C_2}}{ad-bc}, \\ g(z)=\frac{d\mathrm{e}^{-\frac{c}{d}z+C_3}-b\mathrm{e}^{-\frac{a}{b}z+C_4}}{ad-bc}, \end{cases}$$

where C_1, C_2, C_3, C_4 *are constants satisfying that* $m(C_1-C_3)+n(C_4-C_2)=0$.

THEOREM 7. *Let* $m, n \in \mathbb{N}_+$, $a, b, c, d \in \mathbb{C}-\{0\}$ *satisfying that* $ad \neq bc$ *and* h *be a nonconstant entire solutions in* $\mathbb{C}$*, then* (f, g) *is the pair of entire solutions of finite order for system* (1.7) *in* $\mathbb{C}$ *if and only if the following assertions hold:*

(i)

$$\begin{cases} h(z)=\left(m\frac{a-c\mathrm{e}^A}{d\mathrm{e}^A-b}+n\frac{a-c\mathrm{e}^B}{d\mathrm{e}^B-b}\right)z+m(A+C_1)+nC_2, \\ f(z)=\frac{d\mathrm{e}^A-b}{ad-bc}\mathrm{e}^{\frac{a-c\mathrm{e}^A}{d\mathrm{e}^A-b}z+C_1}, \\ g(z)=\frac{d\mathrm{e}^B-b}{ad-bc}\mathrm{e}^{\frac{a-c\mathrm{e}^B}{d\mathrm{e}^B-b}z+C_2}, \end{cases}$$

where A, B, C_1, C_2 *are constants satisfying that* $mA=nB$*;*

(ii) $m = n$ *and*

$$\begin{cases} h(z) = -m\left(\frac{c}{d}+\frac{a}{b}\right)z + m(C_1+C_4), \\ f(z) = \dfrac{d\mathrm{e}^{-\frac{c}{d}z+C_1} - b\mathrm{e}^{-\frac{a}{b}z+C_2}}{ad-bc}, \\ g(z) = \dfrac{d\mathrm{e}^{-\frac{c}{d}z+C_3} - b\mathrm{e}^{-\frac{a}{b}z+C_4}}{ad-bc}, \end{cases}$$

where C_1, C_2, C_3, C_4 *are constants satisfying that* $C_1 - C_2 - C_3 + C_4 = 0$.

REMARK 1. We can answer Question 1 by taking $g = f$ in (1.6) or (1.7).

REMARK 2. If set $g = f$, $m = n = 1$ and $c = \frac{1}{a}$, $d = \frac{1}{b}$, $\omega = \frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right)$ in (1.6) or (1.7), we then deduce equation (1.4). So both Theorem 2.1 and Theorem 2.2 include Theorem 1.5.

The following lemmas play the key roles in proving our results.

LEMMA 1. ([19, 22]) *If* f *is an entire function in* $\mathbb{C}^n$ *satisfying that its counting function of zeros* $n(r,f)$ *is of finite order and* $f(0) \neq 0$, *then there exist a canonical function* g *in* $\mathbb{C}^n$ *and a function* h *in* $\mathbb{C}^n$ *such that* $f = g\mathrm{e}^h$. *For the special case* $n = 1$, g *is the canonical product of Weierstrass.*

LEMMA 2. ([18]) *If* f *and* g *are entire functions and* $f(g)$ *is an entire function of finite order, then there are only two possible cases: either*

(1) *The internal function* g *is a polynomial and the external function* f *is of finite order; or else*

(2) *The internal function* g *is not a polynomial but a function of finite order, and the external function* f *is of zero order.*

LEMMA 3. ([10]) *Suppose that* $a_0, a_1, \cdots, a_n$ $(n \geqslant 1)$ *are meromorphic functions in* $\mathbb{C}^m$ *and* $b_0, b_1, \cdots, b_n$ *are entire functions in* $\mathbb{C}^m$ *such that* $b_i - b_j$ *are not constants for* $0 \leqslant i < j \leqslant n$. *If* $\sum_{k=0}^{n} a_k \mathrm{e}^{b_k} \equiv 0$ *and* $T(r,a_k) = o(T(r))$ $(k = 0,1,\cdots,n)$ *as* $r \to +\infty$ *outside of a possible exceptional set of finite linear measure, where* $T(r) = \min_{0 \leqslant i < j \leqslant n} T(r, \mathrm{e}^{b_i - b_j})$, *then* $a_k \equiv 0$ $(k = 0,1,\cdots,n)$.

Proof of Theorems 6 and 7. The sufficiency is obvious, and we only give proof of the necessity.

We consider the more general system of equations:

$$\begin{cases} (af+bf')^m(cg+dg')^n = \mathrm{e}^h, \\ (ag+bg')^s(cf+df')^t = \mathrm{e}^h, \end{cases} \tag{2.1}$$

where $m,n,s,t \in \mathbb{N}_+$, $a,b,c,d \in \mathbb{C} - \{0\}$ satisfying that $ad \neq bc$ and h is a nonconstant entire solutions in $\mathbb{C}$. The system of ODEs (2.1) includes the systems (1.6) and (1.7) as special cases.

Let (f,g) be a pair of transcendental entire solutions of system (2.1). It is easy to see from (2.1) that $af+bf'$, $cg+dg'$, $ag+bg'$, $cf+df'$ have no any zero and pole. Thus, by Lemmas 1 and 2, there exist four polynomials $\alpha_1,\alpha_2,\beta_1,\beta_2$ in $\mathbb{C}$ such that

$$af+bf'=\mathrm{e}^{\alpha_1},\quad cg+dg'=\mathrm{e}^{\beta_1},$$
$$ag+bg'=\mathrm{e}^{\alpha_2},\quad cf+df'=\mathrm{e}^{\beta_2}.$$

This immediately leads to

$$m\alpha_1+n\beta_1=s\alpha_2+t\beta_2=h. \tag{2.2}$$

Noting that $ad\neq bc$, and solving the above systems, we obtain

$$f=\frac{d\mathrm{e}^{\alpha_1}-b\mathrm{e}^{\beta_2}}{ad-bc}, \tag{2.3}$$

$$f'=\frac{a\mathrm{e}^{\beta_2}-c\mathrm{e}^{\alpha_1}}{ad-bc}, \tag{2.4}$$

$$g=\frac{d\mathrm{e}^{\alpha_2}-b\mathrm{e}^{\beta_1}}{ad-bc}, \tag{2.5}$$

$$g'=\frac{a\mathrm{e}^{\beta_1}-c\mathrm{e}^{\alpha_2}}{ad-bc}. \tag{2.6}$$

By taking the derivative of both sides of (2.3) and (2.5) and combining (2.4) and (2.6), we get

$$(c+d\alpha_1')\mathrm{e}^{\alpha_1}=(a+b\beta_2')\mathrm{e}^{\beta_2}, \tag{2.7}$$

$$(c+d\alpha_2')\mathrm{e}^{\alpha_2}=(a+b\beta_1')\mathrm{e}^{\beta_1}. \tag{2.8}$$

Now, four cases are discussed below.

Case 1. Both $\alpha_1-\beta_2$ and $\alpha_2-\beta_1$ are constants.

Suppose $\alpha_1-\beta_2=A$, $\alpha_2-\beta_1=B$, then $\alpha_1'=\beta_2'$, $\alpha_2'=\beta_1'$, therefore (2.7) and (2.8) become

$$(d\mathrm{e}^{A}-b)\alpha_1'=a-c\mathrm{e}^{A}, \tag{2.9}$$

$$(d\mathrm{e}^{B}-b)\alpha_2'=a-c\mathrm{e}^{B}. \tag{2.10}$$

If $d\mathrm{e}^{A}-b=0$, we have $a-c\mathrm{e}^{A}=0$ by (2.9). And we then obtain $ad=bc$, which is a contradiction with the assumption. Therefore, $d\mathrm{e}^{A}-b\neq 0$. Similarly, we have $d\mathrm{e}^{B}-b\neq 0$ from (2.10). Thus, it follows from (2.9) and (2.10) that

$$\alpha_1'=\beta_2'=\frac{a-c\mathrm{e}^{A}}{d\mathrm{e}^{A}-b},\quad \alpha_2'=\beta_1'=\frac{a-c\mathrm{e}^{B}}{d\mathrm{e}^{B}-b}.$$

Hence

$$\begin{aligned}\beta_2(z)&=\frac{a-c\mathrm{e}^{A}}{d\mathrm{e}^{A}-b}z+C_1,\alpha_1(z)=\frac{a-c\mathrm{e}^{A}}{d\mathrm{e}^{A}-b}z+C_1+A,\\ \beta_1(z)&=\frac{a-c\mathrm{e}^{B}}{d\mathrm{e}^{B}-b}z+C_2,\alpha_2(z)=\frac{a-c\mathrm{e}^{B}}{d\mathrm{e}^{B}-b}z+C_2+B,\end{aligned} \tag{2.11}$$

where C_1, C_2 are constants. Thus, it yields from (2.2) that

$$\begin{aligned} h(z) &= \left(m\frac{a-ce^A}{de^A-b} + n\frac{a-ce^B}{de^B-b} \right) z + m(A+C_1) + nC_2 \\ &= \left(s\frac{a-ce^B}{de^B-b} + t\frac{a-ce^A}{de^A-b} \right) z + s(B+C_2) + tC_1, \end{aligned}$$

which implies that

$$\begin{aligned} (m-t)\frac{a-ce^A}{de^A-b} + (n-s)\frac{a-ce^B}{de^B-b} &= 0, \\ (m-t)C_1 + (n-s)C_2 &= sB - mA. \end{aligned}$$

Substituting (2.11) into (2.3) and (2.5), we have

$$\begin{aligned} f(z) &= \frac{de^{\alpha_1(z)} - be^{\beta_2(z)}}{ad-bc} = \frac{de^A-b}{ad-bc} e^{\frac{a-ce^A}{de^A-b}z+C_1}, \\ g(z) &= \frac{de^{\alpha_2(z)} - be^{\beta_1(z)}}{ad-bc} = \frac{de^B-b}{ad-bc} e^{\frac{a-ce^B}{de^B-b}z+C_2}. \end{aligned}$$

Case 2. $\alpha_2 - \beta_1$ is constant and $\alpha_1 - \beta_2$ is not constant.

Suppose $\alpha_2 - \beta_1 = B$, then $\alpha_2' = \beta_1'$, and because α_1 and β_2 are both polynomials, according to Lemma 3 for (2.9) we can know

$$\begin{aligned} c + d\alpha_1' = 0, a + b\beta_2' &= 0, \\ (de^B - b)\alpha_2' &= a - ce^B. \end{aligned}$$

Similar to the discussion in Case 1, we get

$$\begin{aligned} \alpha_1(z) &= -\frac{c}{d}z + C_1, \quad \beta_2(z) = -\frac{a}{b}z + C_2, \\ \beta_1(z) &= \frac{a-ce^B}{de^B-b}z + C_3, \quad \alpha_2(z) = \frac{a-ce^B}{de^B-b}z + C_3 + B, \end{aligned} \tag{2.12}$$

where C_1, C_2, C_3 are constants. Thus, it yields from (2.2) that

$$\begin{aligned} h(z) &= \left(n\frac{a-ce^B}{de^B-b} - m\frac{c}{d} \right) z + mC_1 + nC_3 \\ &= \left(s\frac{a-ce^B}{de^B-b} - t\frac{a}{b} \right) z + s(B+C_3) + tC_2, \end{aligned}$$

which implies that

$$\begin{aligned} t\frac{a}{b} - m\frac{c}{d} + (n-s)\frac{a-ce^B}{de^B-b} &= 0, \\ mC_1 - tC_2 + (n-s)C_3 &= sB. \end{aligned}$$

Substituting (2.12) into (2.3) and (2.5), we have

$$f(z)=\frac{d\mathrm{e}^{\alpha_1(z)}-b\mathrm{e}^{\beta_2(z)}}{ad-bc}=\frac{d\mathrm{e}^{-\frac{c}{d}z+C_1}-b\mathrm{e}^{-\frac{a}{b}z+C_2}}{ad-bc},$$
$$g(z)=\frac{d\mathrm{e}^{\alpha_2(z)}-b\mathrm{e}^{\beta_1(z)}}{ad-bc}=\frac{d\mathrm{e}^{B}-b}{ad-bc}\mathrm{e}^{\frac{a-c\mathrm{e}^{B}}{d\mathrm{e}^{B}-b}z+C_3}.$$

Case 3. $\alpha_1-\beta_2$ is constant and $\alpha_2-\beta_1$ is not constant.

Suppose $\alpha_1-\beta_2=A$, similar to the discussion in Case 2, we get

$$\begin{aligned}&\alpha_2(z)=-\frac{c}{d}z+C_1,\quad \beta_1(z)=-\frac{a}{b}z+C_2,\\ &\beta_2(z)=\frac{a-c\mathrm{e}^{A}}{d\mathrm{e}^{A}-b}z+C_3,\quad \alpha_1(z)=\frac{a-c\mathrm{e}^{A}}{d\mathrm{e}^{A}-b}z+C_3+A,\end{aligned}\tag{2.13}$$

where C_1,C_2,C_3 are constants. Thus, it yields from (2.2) that

$$\begin{aligned}h(z)&=\left(m\frac{a-c\mathrm{e}^{A}}{d\mathrm{e}^{A}-b}-n\frac{a}{b}\right)z+m(A+C_3)+nC_2\\ &=\left(t\frac{a-c\mathrm{e}^{A}}{d\mathrm{e}^{A}-b}-s\frac{c}{d}\right)z+sC_1+tC_3,\end{aligned}$$

which implies that

$$\begin{aligned}&s\frac{c}{d}-n\frac{a}{b}+(m-t)\frac{a-c\mathrm{e}^{A}}{d\mathrm{e}^{A}-b}=0,\\ &sC_1-nC_2+(t-m)C_3=mA.\end{aligned}$$

In view of (2.3), (2.5) and (2.13), we have

$$f(z)=\frac{d\mathrm{e}^{\alpha_1(z)}-b\mathrm{e}^{\beta_2(z)}}{ad-bc}=\frac{d\mathrm{e}^{A}-b}{ad-bc}\mathrm{e}^{\frac{a-c\mathrm{e}^{A}}{d\mathrm{e}^{A}-b}z+C_3},$$
$$g(z)=\frac{d\mathrm{e}^{\alpha_2(z)}-b\mathrm{e}^{\beta_1(z)}}{ad-bc}=\frac{d\mathrm{e}^{-\frac{c}{d}z+C_1}-b\mathrm{e}^{-\frac{a}{b}z+C_2}}{ad-bc}.$$

Case 4. Neither $\alpha_1-\beta_2$ nor $\alpha_2-\beta_1$ are constants.

Since α_1, α_2, β_1 and β_2 are both polynomials, according to Lemma 3 for (2.9) and (2.10) we can obtain

$$c+d\alpha_1'=0,\quad a+b\beta_2'=0,\quad c+d\alpha_2'=0,\quad a+b\beta_1'=0.$$

Thus

$$\begin{aligned}&\alpha_1(z)=-\frac{c}{d}z+C_1,\quad \beta_2(z)=-\frac{a}{b}z+C_2,\\ &\alpha_2(z)=-\frac{c}{d}z+C_3,\quad \beta_1(z)=-\frac{a}{b}z+C_4,\end{aligned}\tag{2.14}$$

where C_1, C_2, C_3, C_4 are constants. We then have from (2.2) that

$$\begin{aligned} h(z) &= -\left(m\frac{c}{d}+n\frac{a}{b}\right)z+mC_1+nC_4 \\ &= -\left(s\frac{c}{d}+t\frac{a}{b}\right)z+sC_3+tC_2, \end{aligned}$$

which gives that

$$(m-s)\frac{c}{d}+(n-t)\frac{a}{b}=0,$$
$$mC_1-tC_2-sC_3+nC_4=0.$$

Substituting (2.14) into (2.3) and (2.5), we have

$$\begin{aligned} f(z) &= \frac{de^{\alpha_1(z)}-be^{\beta_2(z)}}{ad-bc} = \frac{de^{-\frac{c}{d}z+C_1}-be^{-\frac{a}{b}z+C_2}}{ad-bc}, \\ g(z) &= \frac{de^{\alpha_2(z)}-be^{\beta_1(z)}}{ad-bc} = \frac{de^{-\frac{c}{d}z+C_3}-be^{-\frac{a}{b}z+C_4}}{ad-bc}. \end{aligned}$$

The proof above works for the case $(s,t)=(m,n)$ and $(s,t)=(n,m)$ at the same time. Therefore, this completes the proof of Theorems 6 and 7. □

3. Systems of ODEs (1.8) and (1.9)

The following theorems deal with entire solutions of finite order for systems (1.8) and (1.9).

THEOREM 8. *Let $m,n\in\mathbb{N}_+$, $a,b,c,d\in\mathbb{C}-\{0\}$ satisfying that $ad\neq bc$ and h be a nonconstant entire solutions in $\mathbb{C}$, then (f,g) is the pair of entire solutions of finite order for system (1.8) in $\mathbb{C}$ if and only if the following assertions hold:*

(i)

$$\begin{cases} h(z)=\dfrac{(m+n)(ae^{\frac{m}{n}A+B}-ce^{A})}{ae^{B}-c}z+C, \\ f(z)=\dfrac{de^{\frac{n(A-B)}{m+n}}-be^{\frac{m(B-A)}{m+n}+\frac{m}{n}A}}{ad-bc}e^{\frac{ae^{\frac{m}{n}A+B}-ce^{A}}{ae^{B}-c}z+\frac{C}{m+n}}, \\ g(z)=\dfrac{ae^{\frac{m(B-A)}{m+n}}-ce^{-\frac{mA+nB}{m+n}}}{ad-bc}e^{\frac{ae^{\frac{m}{n}A+B}-ce^{A}}{ae^{B}-c}z+\frac{C}{m+n}}, \end{cases}$$

where A,B,C are constants satisfying that $c\neq ae^{B}$;

(ii)

$$\begin{cases} h(z)=\dfrac{(m+n)(be^{B}-d)}{be^{\frac{m}{n}A+B}-de^{A}}z+C, \\ f(z)=\dfrac{de^{\frac{n(A-B)}{m+n}}-be^{\frac{m(B-A)}{m+n}+\frac{m}{n}A}}{ad-bc}e^{\frac{be^{B}-d}{be^{\frac{m}{n}A+B}-de^{A}}z+\frac{C}{m+n}}, \\ g(z)=\dfrac{ae^{\frac{m(B-A)}{m+n}}-ce^{-\frac{mA+nB}{m+n}}}{ad-bc}e^{\frac{be^{B}-d}{be^{\frac{m}{n}A+B}-de^{A}}z+\frac{C}{m+n}}, \end{cases}$$

where A,B,C are constants satisfying that $d \neq b\mathrm{e}^{\frac{m}{n}A+B-A}$;

(iii)

$$\begin{cases} h(z) = \left(m\mathrm{e}^{A} + n\mathrm{e}^{\frac{m}{n}A}\right)z + m(A+C_1) + nC_2, \\ f(z) = \dfrac{d\mathrm{e}^{\mathrm{e}^{A}z+C_1+A} - b\mathrm{e}^{\mathrm{e}^{\frac{m}{n}A}z+C_2+\frac{m}{n}A}}{ad-bc}, \\ g(z) = \dfrac{a\mathrm{e}^{\mathrm{e}^{\frac{m}{n}A}z+C_2} - c\mathrm{e}^{\mathrm{e}^{A}z+C_1}}{ad-bc}, \end{cases}$$

where A,C_1,C_2 are constants satisfying that $\mathrm{e}^{2A} = \mathrm{e}^{\frac{2m}{n}A} = 1$ and $\mathrm{e}^{\left(\frac{m}{n}-1\right)A} \neq 1$.

THEOREM 9. *Let $m,n \in \mathbb{N}_+$, $a,b,c,d \in \mathbb{C}-\{0\}$ satisfying that $ad \neq bc$ and h be a nonconstant entire solutions in $\mathbb{C}$, then (f,g) is the pair of entire solutions of finite order for system (1.9) in $\mathbb{C}$ if and only if the following assertions hold:*

(i)

$$\begin{cases} h(z) = \left(m\mathrm{e}^{B} + n\mathrm{e}^{A}\right)z + m(B+C_2) + n(A+C_1), \\ f(z) = \dfrac{d\mathrm{e}^{\mathrm{e}^{B}z+C_2+B} - b\mathrm{e}^{\mathrm{e}^{A}z+C_1}}{ad-bc}, \\ g(z) = \dfrac{a\mathrm{e}^{\mathrm{e}^{A}z+C_1+A} - c\mathrm{e}^{\mathrm{e}^{B}z+C_2}}{ad-bc}, \end{cases}$$

where A,B,C_1,C_2 are constants satisfying that $\mathrm{e}^{2A} = \mathrm{e}^{2B} = 1$, $\mathrm{e}^{A+B} \neq 1$, $(m-n)(\mathrm{e}^{A} - \mathrm{e}^{B}) = 0$ and $(m-n)(C_1-C_2) = mB+nA$;

(ii)

$$\begin{cases} h(z) = \dfrac{(m+n)(d\mathrm{e}^{\frac{m}{n}A} - b)\mathrm{e}^{B}}{d\mathrm{e}^{A} - b}z + C, \\ f(z) = \dfrac{d\mathrm{e}^{\frac{n(A-B)}{m+n}} - b\mathrm{e}^{-\frac{mA+nB}{m+n}}}{ad-bc}\mathrm{e}^{\frac{(d\mathrm{e}^{\frac{m}{n}A}-b)\mathrm{e}^{B}}{d\mathrm{e}^{A}-b}z+\frac{C}{m+n}}, \\ g(z) = \dfrac{a\mathrm{e}^{\frac{m(B-A)}{m+n}} - c\mathrm{e}^{\frac{m(B-A)}{m+n}+\frac{m}{n}A}}{ad-bc}\mathrm{e}^{\frac{(d\mathrm{e}^{\frac{m}{n}A}-b)\mathrm{e}^{B}}{d\mathrm{e}^{A}-b}z+\frac{C}{m+n}}, \end{cases}$$

where A,B,C are constants satisfying that $d\mathrm{e}^{A} \neq b$;

(iii)

$$\begin{cases} h(z) = \dfrac{(m+n)(c\mathrm{e}^{A} - a)}{(c\mathrm{e}^{\frac{m}{n}A} - a)\mathrm{e}^{B}}z + C, \\ f(z) = \dfrac{d\mathrm{e}^{\frac{n(A-B)}{m+n}} - b\mathrm{e}^{-\frac{mA+nB}{m+n}}}{ad-bc}\mathrm{e}^{\frac{c\mathrm{e}^{A}-a}{(c\mathrm{e}^{\frac{m}{n}A}-a)\mathrm{e}^{B}}z+\frac{C}{m+n}}, \\ g(z) = \dfrac{a\mathrm{e}^{\frac{m(B-A)}{m+n}} - c\mathrm{e}^{\frac{m(B-A)}{m+n}+\frac{m}{n}A}}{ad-bc}\mathrm{e}^{\frac{c\mathrm{e}^{A}-a}{(c\mathrm{e}^{\frac{m}{n}A}-a)\mathrm{e}^{B}}z+\frac{C}{m+n}}, \end{cases}$$

where A,B,C are constants satisfying that $c\mathrm{e}^{\frac{m}{n}A} \neq a$.

The proofs of these two theorems are given below.

Proof of Theorem 8. The sufficiency is obvious, and we only give proof of the necessity.

Let (f,g) be a pair of transcendental entire solutions of system (1.8). It is easy to see from (1.8) that $af+bg'$, $cf'+dg$, $af'+bg$, $cf+dg'$ have no any zero and pole. Thus, by Lemmas 1 and 2, there exist four polynomials $\alpha_1,\alpha_2,\beta_1,\beta_2$ in $\mathbb{C}$ such that

$$af+bg'=\mathrm{e}^{\alpha_1},\quad cf'+dg=\mathrm{e}^{\beta_1},$$
$$af'+bg=\mathrm{e}^{\alpha_2},\quad cf+dg'=\mathrm{e}^{\beta_2}.$$

This immediately leads to

$$m\alpha_1+n\beta_1=m\alpha_2+n\beta_2=h. \tag{3.1}$$

Noting that $ad\neq bc$, and solving the above systems, we obtain

$$f=\frac{d\mathrm{e}^{\alpha_1}-b\mathrm{e}^{\beta_2}}{ad-bc}, \tag{3.2}$$

$$f'=\frac{d\mathrm{e}^{\alpha_2}-b\mathrm{e}^{\beta_1}}{ad-bc}, \tag{3.3}$$

$$g=\frac{a\mathrm{e}^{\beta_1}-c\mathrm{e}^{\alpha_2}}{ad-bc}, \tag{3.4}$$

$$g'=\frac{a\mathrm{e}^{\beta_2}-c\mathrm{e}^{\alpha_1}}{ad-bc}. \tag{3.5}$$

By taking the derivative of both sides of (3.2) and (3.4) and combining (3.3) and (3.5), we can get

$$d(\alpha_1'\mathrm{e}^{\alpha_1}-\mathrm{e}^{\alpha_2})=b(\beta_2'\mathrm{e}^{\beta_2}-\mathrm{e}^{\beta_1}), \tag{3.6}$$

$$a(\beta_1'\mathrm{e}^{\beta_1}-\mathrm{e}^{\beta_2})=c(\alpha_2'\mathrm{e}^{\alpha_2}-\mathrm{e}^{\alpha_1}). \tag{3.7}$$

We next consider two different cases in the following.

Case 1. $\alpha_1-\alpha_2$ is not constant.

According to (3.1) we know that $\beta_2-\beta_1=\frac{m}{n}(\alpha_1-\alpha_2)$ is not a constant either.

Case 1.1. $\beta_1-\alpha_2$ is constant.

Suppose $\beta_1-\alpha_2=A$, then $\beta_2-\alpha_2=\beta_2-\beta_1+A$ is not a constant, and there is $\alpha_2'=\beta_1'$, therefore (3.6) and (3.7) become

$$d\alpha_1'\mathrm{e}^{\alpha_1}-b\beta_2'\mathrm{e}^{\beta_2}=(d-b\mathrm{e}^{A})\mathrm{e}^{\alpha_2}, \tag{3.8}$$

$$c\mathrm{e}^{\alpha_1}-a\mathrm{e}^{\beta_2}=(c-a\mathrm{e}^{A})\beta_1'\mathrm{e}^{\alpha_2}. \tag{3.9}$$

Since $a,b,c,d\neq 0$, according to Lemma 3 for (3.9), $\alpha_1-\beta_2$ must be a constant, set $\alpha_1-\beta_2=B$, now (3.8) and (3.9) become

$$(d\mathrm{e}^{B}-b)\beta_2'\mathrm{e}^{\beta_2}=(d\mathrm{e}^{-A}-b)\mathrm{e}^{\beta_1},$$
$$(c\mathrm{e}^{B}-a)\mathrm{e}^{\beta_2}=(c\mathrm{e}^{-A}-a)\beta_1'\mathrm{e}^{\beta_1}.$$

Since $\beta_1-\beta_2$ is not a constant, using Lemma 3 again we can get

$$(d\mathrm{e}^{B}-b)\beta_2' = d\mathrm{e}^{-A}-b = c\mathrm{e}^{B}-a = (c\mathrm{e}^{-A}-a)\beta_1' = 0.$$

Considering $ad\neq bc$, there must be $\beta_1'=\beta_2'=0$, which shows that $\alpha_1,\alpha_2,\beta_1,\beta_2$ are all constants, combining with (3.1), we can get that h is also a constant, which contradict with the conditions of the Theorem.

Case 1.2. $\beta_1-\alpha_2$ is not constant.

Since $a,b,c,d\neq 0$, using Lemma 3 for (3.6) and (3.7), we know that at least one of $\alpha_1-\beta_1$, $\alpha_1-\beta_2$ and $\alpha_2-\beta_2$ is a constant.

Case 1.2.1. $\alpha_2-\beta_2$ is constant.

Set $\alpha_2-\beta_2=A$, then neither $\alpha_1-\beta_1=\dfrac{m+n}{n}(\alpha_1-\alpha_2)+A$ nor $\alpha_1-\beta_2=\alpha_1-\alpha_2+A$ are constants, now (3.6) and (3.7) become

$$d\alpha_1'\mathrm{e}^{\alpha_1}+b\mathrm{e}^{\beta_1} = (d\mathrm{e}^{A}+b\beta_2')\mathrm{e}^{\beta_2},$$
$$(c\alpha_2'\mathrm{e}^{A}+a)\mathrm{e}^{\beta_2} = c\mathrm{e}^{\alpha_1}+a\beta_1'\mathrm{e}^{\beta_1}.$$

Again, we can get $b=c=0$ by Lemma 3, which contradict with the conditions of the Theorem.

Case 1.2.2. $\alpha_1-\beta_2$ is constant.

Set $\alpha_1-\beta_2=A$, then $\alpha_2-\beta_2=\alpha_2-\alpha_1+A$ is not constant, now (3.6) and (3.7) become

$$(d\alpha_1'\mathrm{e}^{A}-b\beta_2')\mathrm{e}^{\beta_2} = d\mathrm{e}^{\alpha_2}-b\mathrm{e}^{\beta_1},$$
$$(c\mathrm{e}^{A}-a)\mathrm{e}^{\beta_2} = c\alpha_2'\mathrm{e}^{\alpha_2}-a\beta_1'\mathrm{e}^{\beta_1}.$$

Using Lemma 3 again, we can get $b=d=0$, which contradict with the conditions of the Theorem.

Case 1.2.3. $\alpha_1-\beta_1$ is constant.

Set $\alpha_1-\beta_1=A$, then $\alpha_2-\beta_1=\alpha_2-\alpha_1+A$ is not constant, now (3.6) and (3.7) become

$$(b+d\alpha_1'\mathrm{e}^{A})\mathrm{e}^{\beta_1} = b\beta_2'\mathrm{e}^{\beta_2}+d\mathrm{e}^{\alpha_2},$$
$$(a\beta_1'+c\mathrm{e}^{A})\mathrm{e}^{\beta_1} = c\alpha_2'\mathrm{e}^{\alpha_2}+a\mathrm{e}^{\beta_2}.$$

According to Case 1.2.1, we know that $\alpha_2-\beta_2$ will not be a constant. Using Lemma 3, we can get $a=d=0$, which contradict with the conditions of the Theorem.

Case 2. $\alpha_1-\alpha_2$ is constant.

Set $\alpha_1-\alpha_2=A$, then there are $\beta_2-\beta_1=\dfrac{m}{n}A$, $\beta_2-\alpha_2=\beta_1-\alpha_2+\dfrac{m}{n}A$ and $\alpha_1'=\alpha_2'$, $\beta_1'=\beta_2'$, now (3.6) and (3.7) become

$$b(\beta_1'\mathrm{e}^{\frac{m}{n}A}-1)\mathrm{e}^{\beta_1} = d(\alpha_1'\mathrm{e}^{A}-1)\mathrm{e}^{\alpha_2}, \tag{3.10}$$
$$a(\beta_1'-\mathrm{e}^{\frac{m}{n}A})\mathrm{e}^{\beta_1} = c(\alpha_1'-\mathrm{e}^{A})\mathrm{e}^{\alpha_2}. \tag{3.11}$$

Case 2.1. $\beta_1-\alpha_2$ is constant.

Set $\beta_1-\alpha_2=B$, then according to (3.1) it can be concluded that

$$\begin{aligned} \alpha_1&=\frac{h}{m+n}+\frac{n(A-B)}{m+n}, \quad \alpha_2=\frac{h}{m+n}-\frac{mA+nB}{m+n},\\ \beta_1&=\frac{h}{m+n}+\frac{m(B-A)}{m+n}, \quad \beta_2=\frac{h}{m+n}+\frac{m(B-A)}{m+n}+\frac{m}{n}A. \end{aligned} \tag{3.12}$$

Substituting these into (3.10) and (3.11), we obtain

$$\begin{aligned} (be^{\frac{m}{n}A+B}-de^{A})h'&=(m+n)(be^{B}-d),\\ (ae^{B}-c)h'&=(m+n)(ae^{\frac{m}{n}A+B}-ce^{A}). \end{aligned}$$

Obviously, $be^{\frac{m}{n}A+B}-de^{A}=0$ and $ae^{B}-c=0$ can not hold simultaneously. Otherwise it will result in $ad=bc$, contradiction.

Case 2.1.1. $ae^{B}-c\neq 0$.

Now we have $h'=\frac{(m+n)(ae^{\frac{m}{n}A+B}-ce^{A})}{ae^{B}-c}$, thus $h(z)=\frac{(m+n)(ae^{\frac{m}{n}A+B}-ce^{A})}{ae^{B}-c}z+C$, combining with (3.12), (3.2) and (3.4), we can get

$$\begin{aligned} f(z)&=\frac{de^{\alpha_1(z)}-be^{\beta_2(z)}}{ad-bc}=\frac{de^{\frac{n(A-B)}{m+n}}-be^{\frac{m(B-A)}{m+n}+\frac{m}{n}A}}{ad-bc}e^{\frac{ae^{\frac{m}{n}A+B}-ce^{A}}{ae^{B}-c}z+\frac{C}{m+n}},\\ g(z)&=\frac{ae^{\beta_1(z)}-ce^{\alpha_2(z)}}{ad-bc}=\frac{ae^{\frac{m(B-A)}{m+n}}-ce^{-\frac{mA+nB}{m+n}}}{ad-bc}e^{\frac{ae^{\frac{m}{n}A+B}-ce^{A}}{ae^{B}-c}z+\frac{C}{m+n}}, \end{aligned}$$

where A,B,C are constants satisfying that $c\neq ae^{B}$. This is Theorems 8 (i).

Case 2.1.2. $be^{\frac{m}{n}A+B}-de^{A}\neq 0$.

Now we have $h'=\frac{(m+n)(be^{B}-d)}{be^{\frac{m}{n}A+B}-de^{A}}$, thus $h(z)=\frac{(m+n)(be^{B}-d)}{be^{\frac{m}{n}A+B}-de^{A}}z+C$, combining with (3.12), (3.2) and (3.4), we can get

$$\begin{aligned} f(z)&=\frac{de^{\alpha_1(z)}-be^{\beta_2(z)}}{ad-bc}=\frac{de^{\frac{n(A-B)}{m+n}}-be^{\frac{m(B-A)}{m+n}+\frac{m}{n}A}}{ad-bc}e^{\frac{be^{B}-d}{be^{\frac{m}{n}A+B}-de^{A}}z+\frac{C}{m+n}},\\ g(z)&=\frac{ae^{\beta_1(z)}-ce^{\alpha_2(z)}}{ad-bc}=\frac{ae^{\frac{m(B-A)}{m+n}}-ce^{-\frac{mA+nB}{m+n}}}{ad-bc}e^{\frac{be^{B}-d}{be^{\frac{m}{n}A+B}-de^{A}}z+\frac{C}{m+n}}, \end{aligned}$$

where A,B,C are constants satisfying that $d\neq be^{\frac{m}{n}A+B-A}$. Thus, we get (ii) in Theorems 8.

Case 2.2. $\beta_1-\alpha_2$ is not constant.

According to Lemma 3, we can get

$$\alpha_1'=\alpha_2'=e^{A}=e^{-A}, \quad \beta_1'=\beta_2'=e^{\frac{m}{n}A}=e^{-\frac{m}{n}A}.$$

Considering $\beta_1-\alpha_2$ is not constant, there is $e^{2A}=e^{\frac{2m}{n}A}=1$, $e^{(\frac{m}{n}-1)A}\neq 1$, therefore

$$\alpha_2(z)=e^A z+C_1, \quad \alpha_1(z)=e^A z+C_1+A$$
$$\beta_1(z)=e^{\frac{m}{n}A}z+C_2, \quad \beta_2(z)=e^{\frac{m}{n}A}z+C_2+\frac{m}{n}A,$$

where A,C_1,C_2 are constants. In view of (3.1), (3.2) and (3.4), we have

$$h(z)=m\alpha_1(z)+n\beta_1(z)=\left(me^A+ne^{\frac{m}{n}A}\right)z+m(A+C_1)+nC_2,$$
$$f(z)=\frac{de^{\alpha_1(z)}-be^{\beta_2(z)}}{ad-bc}=\frac{de^{e^A z+C_1+A}-be^{e^{\frac{m}{n}A}z+C_2+\frac{m}{n}A}}{ad-bc},$$
$$g(z)=\frac{ae^{\beta_1(z)}-ce^{\alpha_2(z)}}{ad-bc}=\frac{ae^{e^{\frac{m}{n}A}z+C_2}-ce^{e^A z+C_1}}{ad-bc}.$$

Thus, we complete the proof of Theorems 8 (iii). □

The proof of Theorem 9 is roughly the same as the proof of Theorem 8, we also only give proof of the necessity and will briefly describe the similar parts.

Proof of Theorem 9. Following the same reasoning, we can still get (3.2), (3.4), (3.6) and (3.7), but (3.1) becomes

$$m\alpha_1+n\beta_1=n\alpha_2+m\beta_2=h. \tag{3.13}$$

Case 1. $\alpha_1-\beta_2$ is not constant.
According to (3.13) we know that $\alpha_2-\beta_1=\frac{m}{n}(\alpha_1-\beta_2)$ is not a constant either.

Case 1.1. $\beta_1-\beta_2$ is constant.
Suppose $\beta_1-\beta_2=A$, then $\alpha_2-\beta_2=\alpha_2-\beta_1+A$ is not a constant, and there is $\beta_1'=\beta_2'$, therefore (3.6) and (3.7) become

$$d\alpha_1'e^{\alpha_1}-de^{\alpha_2}=b(\beta_2'-e^A)e^{\beta_2}, \tag{3.14}$$
$$ce^{\alpha_1}-c\alpha_2'e^{\alpha_2}=a(1-\beta_1'e^A)e^{\beta_2}. \tag{3.15}$$

Since $a,b,c,d\neq 0$, according to Lemma 3 for (3.15), $\alpha_1-\alpha_2$ must be a constant, set $\alpha_1-\alpha_2=B$, now (3.14) and (3.15) become

$$d(\alpha_1'e^B-1)e^{\alpha_2}=b(\beta_2'-e^A)e^{\beta_2},$$
$$c(e^B-\alpha_2')e^{\alpha_2}=a(1-\beta_1'e^A)e^{\beta_2}.$$

Since $\alpha_2-\beta_2$ is not a constant, using Lemma 3 again we can get

$$d(\alpha_1'e^B-1)=b(\beta_2'-e^A)=c(e^B-\alpha_2')=a(1-\beta_1'e^A)=0,$$

Considering $ad\neq bc$, there must be

$$\alpha_1'=e^{-B}, \quad \alpha_2'=e^B, \quad \beta_1'=e^{-A}, \quad \beta_2'=e^A.$$

According to $\beta_1-\beta_2=A, \alpha_1-\alpha_2=B$ and $\alpha_1-\beta_2$ is not constant, we can get

$$\begin{gathered} e^{2A}=e^{2B}=1, \quad e^{A+B}\neq 1, \\ \alpha_2(z)=e^{B}z+C_2, \quad \alpha_1(z)=e^{B}z+C_2+B, \\ \beta_2(z)=e^{A}z+C_1, \quad \beta_1(z)=e^{A}z+C_1+A, \end{gathered}$$

where A,B,C_1,C_2 are constants, substituting these into (3.13), (3.2) and (3.4)

$$\begin{aligned} h(z)&=m\alpha_1(z)+n\beta_1(z)=\left(me^{B}+ne^{A}\right)z+m(B+C_2)+n(A+C_1) \\ &=n\alpha_2(z)+m\beta_2(z)=\left(ne^{B}+me^{A}\right)z+mC_1+nC_2, \\ f(z)&=\frac{de^{\alpha_1(z)}-be^{\beta_2(z)}}{ad-bc}=\frac{de^{e^{B}z+C_2+B}-be^{e^{A}z+C_1}}{ad-bc}, \\ g(z)&=\frac{ae^{\beta_1(z)}-ce^{\alpha_2(z)}}{ad-bc}=\frac{ae^{e^{A}z+C_1+A}-ce^{e^{B}z+C_2}}{ad-bc}, \end{aligned}$$

and there are $(m-n)\left(e^{A}-e^{B}\right)=0$ and $(m-n)(C_1-C_2)=mB+nA$. This completes the proof of Theorems 9 (i).

Case 1.2. $\beta_1-\beta_2$ is not constant.

Since $a,b,c,d\neq 0$, using Lemma 3 for (3.6) and (3.7), we know that at least one of $\alpha_1-\beta_1$, $\alpha_1-\alpha_2$ and $\alpha_2-\beta_2$ is a constant.

Case 1.2.1. $\alpha_1-\alpha_2$ is constant.

Set $\alpha_1-\alpha_2=A$, then $\beta_2-\alpha_2=\beta_2-\alpha_1+A$ is not constant, now (3.6) and (3.7) become

$$\begin{gathered} d(\alpha_1'e^{A}-1)e^{\alpha_2}=b(\beta_2'e^{\beta_2}-e^{\beta_1}), \\ a(\beta_1'e^{\beta_1}-e^{\beta_2})=c(\alpha_2'-e^{A})e^{\alpha_2}. \end{gathered}$$

Using Lemma 3 again, we can get $a=b=c=d=0$, which contradict with the conditions of the Theorem.

Case 1.2.2. $\alpha_2-\beta_2$ is constant.

Set $\alpha_2-\beta_2=A$, then $\alpha_1-\beta_1=\dfrac{m+n}{m}(\alpha_2-\beta_1)-A$ is not constant, now (3.6) and (3.7) become

$$\begin{gathered} d\alpha_1'e^{\alpha_1}+be^{\beta_1}=(de^{A}+b\beta_2')e^{\beta_2}, \\ (c\alpha_2'e^{A}+a)e^{\beta_2}=ce^{\alpha_1}+a\beta_1'e^{\beta_1}. \end{gathered}$$

Using Lemma 3 again, we can get $b=c=0$, which contradict with the conditions of the Theorem.

Case 1.2.3. $\alpha_1-\beta_1$ is constant.

Set $\alpha_1-\beta_1=A$, then $\alpha_2-\beta_2=\frac{m+n}{n}(\alpha_1-\beta_2)-A$ is not constant, now (3.6) and (3.7) become

$$(b+d\alpha_1'e^A)e^{\beta_1}=b\beta_2'e^{\beta_2}+de^{\alpha_2},$$
$$(a\beta_1'+ce^A)e^{\beta_1}=c\alpha_2'e^{\alpha_2}+ae^{\beta_2}.$$

Using Lemma 3 again, we can get $a=d=0$, which contradict with the conditions of the Theorem.

Case 2. $\alpha_1-\beta_2$ is constant.

Set $\alpha_1-\beta_2=A$, then there are $\alpha_2-\beta_1=\frac{m}{n}A$, $\alpha_2-\beta_2=\beta_1-\beta_2+\frac{m}{n}A$ and $\alpha_1'=\beta_2'$, $\alpha_2'=\beta_1'$, now (3.6) and (3.7) become

$$(de^{\frac{m}{n}A}-b)e^{\beta_1}=(de^A-b)\beta_2'e^{\beta_2}, \tag{3.16}$$
$$(a-ce^{\frac{m}{n}A})\beta_1'e^{\beta_1}=(a-ce^A)e^{\beta_2}. \tag{3.17}$$

Case 2.1. $\beta_1-\beta_2$ is constant.

Set $\beta_1-\beta_2=B$, then according to (3.13) it can be concluded that

$$\begin{aligned} \alpha_1&=\frac{h}{m+n}+\frac{n(A-B)}{m+n}, \quad \alpha_2=\frac{h}{m+n}+\frac{m(B-A)}{m+n}+\frac{m}{n}A,\\ \beta_1&=\frac{h}{m+n}+\frac{m(B-A)}{m+n}, \quad \beta_2=\frac{h}{m+n}-\frac{mA+nB}{m+n}. \end{aligned} \tag{3.18}$$

Substituting these into (3.16) and (3.17), we obtain

$$(de^A-b)h'=(m+n)(de^{\frac{m}{n}A}-b)e^B,$$
$$(ce^{\frac{m}{n}A}-a)e^Bh'=(m+n)(ce^A-a).$$

Obviously, de^A-b and $ce^{\frac{m}{n}A}-a$ can not hold simultaneously.

Case 2.1.1. $de^A-b\neq 0$.

Now we have $h'=\frac{(m+n)(de^{\frac{m}{n}A}-b)e^B}{de^A-b}$, thus $h(z)=\frac{(m+n)(de^{\frac{m}{n}A}-b)e^B}{de^A-b}z+C$, combining with (3.18), (3.2) and (3.4), we can get

$$f(z)=\frac{de^{\alpha_1(z)}-be^{\beta_2(z)}}{ad-bc}=\frac{de^{\frac{n(A-B)}{m+n}}-be^{-\frac{mA+nB}{m+n}}}{ad-bc}e^{\frac{(de^{\frac{m}{n}A}-b)e^B}{de^A-b}z+\frac{C}{m+n}},$$
$$g(z)=\frac{ae^{\beta_1(z)}-ce^{\alpha_2(z)}}{ad-bc}=\frac{ae^{\frac{m(B-A)}{m+n}}-ce^{\frac{m(B-A)}{m+n}+\frac{m}{n}A}}{ad-bc}e^{\frac{(de^{\frac{m}{n}A}-b)e^B}{de^A-b}z+\frac{C}{m+n}},$$

where A,B,C are constants satisfying that $de^A\neq b$. This is Theorems 9 (ii).

Case 2.1.2. $ce^{\frac{m}{n}A} - a \neq 0$.

Now we have $h' = \dfrac{(m+n)(ce^A - a)}{(ce^{\frac{m}{n}A} - a)e^B}$, thus $h(z) = \dfrac{(m+n)(ce^A - a)}{(ce^{\frac{m}{n}A} - a)e^B} z + C$, combining with (3.18), (3.2) and (3.4), we can get

$$f(z) = \frac{de^{\alpha_1(z)} - be^{\beta_2(z)}}{ad - bc} = \frac{de^{\frac{n(A-B)}{m+n}} - be^{-\frac{mA+nB}{m+n}}}{ad - bc} e^{\frac{ce^A - a}{(ce^{\frac{m}{n}A} - a)e^B} z + \frac{C}{m+n}},$$

$$g(z) = \frac{ae^{\beta_1(z)} - ce^{\alpha_2(z)}}{ad - bc} = \frac{ae^{\frac{m(B-A)}{m+n}} - ce^{\frac{m(B-A)}{m+n} + \frac{m}{n}A}}{ad - bc} e^{\frac{ce^A - a}{(ce^{\frac{m}{n}A} - a)e^B} z + \frac{C}{m+n}},$$

where A, B, C are constants satisfying that $ce^{\frac{m}{n}A} \neq a$. This is Theorems 9 (iii).

Case 2.2. $\beta_1 - \beta_2$ is not constant.

According to Lemma 3, we can get

$$de^{\frac{m}{n}A} - b = (de^A - b)\beta_2' = (a - ce^{\frac{m}{n}A})\beta_1' = a - ce^A = 0.$$

Considering $ad \neq bc$, there must be $\beta_1' = \beta_2' = 0$, which shows that $\alpha_1, \alpha_2, \beta_1, \beta_2$ are all constants, combining with (3.13), we can get that h is also a constant, which contradict with the conditions of the Theorem. □

4. Systems of ODEs (1.10) and (1.11)

In this section, we give two theorems about the last two systems (1.10) and (1.11) and their proofs.

THEOREM 10. *Let $m, n \in \mathbb{N}_+$, $a, b, c, d \in \mathbb{C} - \{0\}$ satisfying that $ad \neq bc$ and h be a nonconstant entire solutions in $\mathbb{C}$, then (f, g) is the pair of entire solutions of finite order for system (1.10) in $\mathbb{C}$ if and only if:*

$$\begin{cases} h(z) = \left(me^A + ne^B\right) z + mC_1 + n(B + C_2), \\ f(z) = \dfrac{de^{e^A z + C_1} - be^{e^B z + C_2}}{ad - bc}, \\ g(z) = \dfrac{ae^{e^B z + C_2} - ce^{e^A z + C_1}}{ad - bc}, \end{cases}$$

where A, B, C_1, C_2 are constants satisfying that $mA = nB$.

THEOREM 11. *Let $m, n \in \mathbb{N}_+$, $a, b, c, d \in \mathbb{C} - \{0\}$ satisfying that $ad \neq bc$ and h be a nonconstant entire solutions in $\mathbb{C}$, then (f, g) is the pair of entire solutions of finite order for system (1.11) in $\mathbb{C}$ if and only if:*

$$\begin{cases} h(z) = \left(me^A + ne^B\right) z + mC_1 + n(B + C_2), \\ f(z) = \dfrac{de^{e^A z + C_1} - be^{e^B z + C_2}}{ad - bc}, \\ g(z) = \dfrac{ae^{e^B z + C_2} - ce^{e^A z + C_1}}{ad - bc}, \end{cases}$$

where A, B, C_1, C_2 *are constants satisfying that* $(m-n)\left(e^A - e^B\right) = 0$ *and* $(m-n)(C_1 - C_2) = n(A-B)$.

Proof of Theorems 10 *and* 11. The sufficiency is obvious, and we only give proof of the necessity.

We also consider the more general system of equations:

$$\begin{cases} (af+bg)^m(cf'+dg')^n = e^h, \\ (af'+bg')^s(cf+dg)^t = e^h, \end{cases} \tag{4.1}$$

where $m, n, s, t \in \mathbb{N}_+$, $a, b, c, d \in \mathbb{C} - \{0\}$ satisfying that $ad \neq bc$ and h is a nonconstant entire solutions in $\mathbb{C}$. The system of ODEs (4.1) includes the systems (1.10) and (1.11) as special cases.

Let (f, g) be a pair of transcendental entire solutions of system (4.1). It is easy to see from (4.1) that $af+bg, cf'+dg', af'+bg', cf+dg$ have no any zero and pole. Thus, by Lemmas 1 and 2, there exist four polynomials $\alpha_1, \alpha_2, \beta_1, \beta_2$ in $\mathbb{C}$ such that

$$af+bg = e^{\alpha_1}, \tag{4.2}$$

$$cf'+dg' = e^{\beta_1}, \tag{4.3}$$

$$af'+bg' = e^{\alpha_2}, \tag{4.4}$$

$$cf+dg = e^{\beta_2}. \tag{4.5}$$

This immediately leads to

$$m\alpha_1 + n\beta_1 = s\alpha_2 + t\beta_2 = h. \tag{4.6}$$

Noting that $ad \neq bc$, and solving the above systems, we obtain

$$f = \frac{de^{\alpha_1} - be^{\beta_2}}{ad-bc}, \tag{4.7}$$

$$f' = \frac{de^{\alpha_2} - be^{\beta_1}}{ad-bc},$$

$$g = \frac{ae^{\beta_2} - ce^{\alpha_1}}{ad-bc}, \tag{4.8}$$

$$g' = \frac{ae^{\beta_1} - ce^{\alpha_2}}{ad-bc}.$$

By taking the derivative of both sides of (4.2) and (4.5) and combining (4.4) and (4.3), we get

$$\alpha_1' = e^{\alpha_2 - \alpha_1}, \tag{4.9}$$

$$\beta_2' = e^{\beta_1 - \beta_2}. \tag{4.10}$$

If $\alpha_2 - \alpha_1$ is not constant, then the left side of (4.9) is a polynomial, but the right side of (4.9) is a transcendental function, which is a contradiction. Thus $\alpha_2 - \alpha_1$ is a constant, set $\alpha_2-\alpha_1=A$. And similarly $\beta_1-\beta_2$ is also a constant, set $\beta_1-\beta_2=B$. So we have

$$\alpha_1'=\alpha_2'=\mathrm{e}^A,\quad \beta_1'=\beta_2'=\mathrm{e}^B,$$

therefore

$$\begin{aligned}\alpha_1(z)&=\mathrm{e}^Az+C_1,\quad \alpha_2(z)=\mathrm{e}^Az+C_1+A,\\ \beta_2(z)&=\mathrm{e}^Bz+C_2,\quad \beta_1(z)=\mathrm{e}^Bz+C_2+B,\end{aligned}\tag{4.11}$$

where C_1,C_2 are constants. Substituting (4.11) into (4.6), we obtain

$$\begin{aligned}h(z)&=\left(m\mathrm{e}^A+n\mathrm{e}^B\right)z+mC_1+n(B+C_2)\\ &=\left(s\mathrm{e}^A+t\mathrm{e}^B\right)z+s(A+C_1)+tC_2,\end{aligned}$$

which implies that $(m-s)\mathrm{e}^A+(n-t)\mathrm{e}^B=0$ and $(m-s)C_1+(n-t)C_2=sA-nB$, then in view of (4.7), (4.9) and (4.11), we have

$$\begin{aligned}f(z)&=\frac{d\mathrm{e}^{\alpha_1(z)}-b\mathrm{e}^{\beta_2(z)}}{ad-bc}=\frac{d\mathrm{e}^{\mathrm{e}^Az+C_1}-b\mathrm{e}^{\mathrm{e}^Bz+C_2}}{ad-bc},\\ g(z)&=\frac{a\mathrm{e}^{\beta_2(z)}-c\mathrm{e}^{\alpha_1(z)}}{ad-bc}=\frac{a\mathrm{e}^{\mathrm{e}^Bz+C_2}-c\mathrm{e}^{\mathrm{e}^Az+C_1}}{ad-bc}.\end{aligned}$$

The proof above works for the case $(s,t)=(m,n)$ and $(s,t)=(n,m)$ at the same time. Therefore, this completes the proof of Theorems 10 and 11. □

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