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CONTINUOUS DEPENDENCE OF SOLUTIONS FOR A VISCOELASTIC PSEUDO–PARABOLIC EQUATION OF CARRIER TYPE

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Abstract. This paper explores an initial boundary value problem for a viscoelastic nonlinear pseudo-parabolic equation of Carrier type. The existence and uniqueness of solutions are established by the linear approximation and the Faedo-Galerkin method. Under appropriately sufficient conditions, the continuous dependence of solutions on the relaxation functions, and the nonlinear components in the problem are also studied.

1. Introduction

In this paper, we consider the following initial boundary value problem for a class of nonlinear viscoelastic pseudo-parabolic equation

$$u_t - \frac{1}{x}\frac{\partial}{\partial x}\left(\mu(x,t,\|u(t)\|_0^2)xu_x\right) + \alpha(t)Au_t - \int_0^t g(t-s)Au(s)ds = f(x,t,u,u_x),\ 1<x<R,\ 0<t<T, \tag{1.1}$$

$$u(1,t) = u(R,t) = 0, \tag{1.2}$$

$$u(x,0) = \tilde{u}_0(x), \tag{1.3}$$

where $R>1$ is a given constant and μ, α, f, g, $\tilde{u}_0$ are given functions, the linear operator A is defined by $Au \equiv -(u_{xx} + \frac{1}{x}u_x)$, and the integral quantity $\|u(t)\|_0^2 = \int_1^R xu^2(x,t)dx$ is known as Carrier term.

It is well known that pseudo-parabolic equations have arisen in sciences such as hydrodynamics, thermodynamics, filtration theory, and are used for describing a variety of important physical processes such as the unidirectional propagation of nonlinear, dispersive, long waves [5], the aggregation of population [22], the unsteady property of second-grade or third-grade fluid flows [1], [2], [14] and so on. In this context, the

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pseudo-parabolic equation (1.1) is inspired by the mathematical model in the work of Hayat et.al. [13] describing unsteady flows of second-grade fluid in a circular cylinder

$$\begin{cases} w_t = \left(\nu + \alpha \frac{\partial}{\partial t}\right)\left(w_{rr} + \frac{1}{r} w_r\right) - Nw, \quad 0 < r < a, \quad t > 0, \\ w(a,t) = W, \quad t > 0, \\ w(r,0) = 0, \quad 0 \leqslant r < a, \end{cases} \tag{1.4}$$

where $w(r,t)$ stands for the velocity along the z-axis, ν is the kinematic viscosity, α is the material parameter, N is the imposed magnetic field, W is the constant velocity at $r = a$ and a is the radius of the cylinder. Later, some extending investigations of (1.4) have been mentioned in literature. For example, in [19], the authors proved some results of the local existence and exponential decay of solutions for the following viscoelastic pseudo-parabolic problem

$$\begin{cases} u_t - \left(\mu(t) + \alpha(t)\frac{\partial}{\partial t}\right) Au + \int_0^t g(t-s)Au(s)ds \\ \qquad\qquad = f(x,t,u), \quad 1 < x < R, \quad 0 < t < T, \\ u_x(1,t) - \zeta u(1,t) = u(R,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \end{cases} \tag{1.5}$$

where $R > 1$, $\zeta \geqslant 0$ are given constants, $\mu(t)$, $\alpha(t)$, f, g, $\tilde{u}_0$ are given functions and $u = u(x,t)$ is unknown function. After that, the results shown in [19] have been extended by Ngoc et.al. in [17] and [18], in which they have proved the global existence, uniqueness, finite time blow-up, and general decay of solutions for the problem (1.5). Further, we introduce several other results of the problem (1.5) related to the long-time behavior of solutions for the pseudo-parabolic equations with "(η, T)-periodic" condition and "$(N+1)$-points condition in time" respectively shown in [20] and [21].

Studying pseudo-parabolic equations in multi-dimensional cases, some results such as stability, global existence, and finite time blow-up for the following viscoelastic pseudo-parabolic equations

$$u_t - k\Delta u_t - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = f(u), \tag{1.6}$$

have been considered. The potential well method is one of the favorite methods commonly applied to studying the equation (1.6). For example, under considering the equation (1.6) with $f(u) = |u|^{p-2}u$ and by using the Galerkin method and an improved potential well method depending on time t, Sun et.al. [24] proved the global existence and the finite time blow-up of solutions with low initial energy level $J(u_0) \leqslant d$ (depth of potential well). Moreover, the authors also obtained the upper bounds for the blow-up time of solutions at arbitrary energy level by Levine's concavity method. When the source $f(u)$ of the equation (1.6) is variable-exponent type, precisely $f(u) = |u|^{p(x)-2}u$, Messaoudi and Talahmeh [16] derived an estimate of the blow-up time of solutions with initial data at arbitrary energy levels. In addition, we also refer to other recent results of local/global existence, decay, and blow-up of solutions for the pseudo-parabolic equations related to (1.6), see [6], [7], [8], [26]–[29].

In the above works, the authors have essentially paid attention to results of local/global existence, uniqueness, blow-up property, and large-time behavior of solutions, however, there have been few studies of continuous dependence of solutions on data mentioned. After the earlier works of Douglis [9] and Fritz [10], the topic of continuous dependence of solutions for partial differential equations on data has received much attention. Indeed, Gür and Güleç [11] proved the continuous dependence of solutions on the parameters α and β to the initial boundary value problem for a strongly damped nonlinear wave equation

$$\begin{cases} u_{tt} - \Delta u + \beta |u_t|^2 u_t = \alpha \Delta u_t, \;\; x \in \Omega, \;\; t > 0, \\ u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), \;\; x \in \Omega, \\ u = 0, x \in \partial\Omega, \;\; t > 0. \end{cases} \tag{1.7}$$

In another work, Benilan and Crandall [4] considered the continuous dependence of solutions for the following Cauchy problem on nonlinearities

$$\begin{cases} u_t - \Delta \phi(u) = 0, \;\; \text{in} \;\; \mathbb{R}^n \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \;\; x \in \mathbb{R}^n. \end{cases} \tag{1.8}$$

The authors defined the continuous dependence on data of solutions in the sense

$$\|u_m(t) - u_\infty(t)\|_{L^1(\mathbb{R}^n)} \to 0, \;\; \text{as} \;\; \phi_m \to \phi_\infty, \;\; \text{with} \;\; \phi_m \;\; \text{instead of} \;\; \phi,$$

where $\phi_m : \mathbb{R} \to \mathbb{R}$ are continuous and nondecreasing functions, $\phi_m(0) = 0$, and u_m are solutions of (1.8). Also, the readers can refer to some similar results recently shown in [3], [12], [25], in which the continuous dependence of solutions under the effects of small perturbations of parameters has been considered.

Inspired by the works mentioned above, in this paper, we provide a new contribution to the continuous dependence of solutions on data for pseudo-parabolic equations. More precisely, it is related to the relaxation functions g, α, and the nonlinear components μ, f of the problem (1.1)–(1.3). This result is presented in Section 3 (see Theorem 3.1), and summarized as follows. Suppose that $u = u(f, \mu, \alpha, g)$ and $u_j = u(f_j, \mu_j, \alpha_j, g_j)$ are the solutions of the problem (1.1)–(1.3) respectively depending on the datum (f, μ, α, g) and $(f_j, \mu_j, \alpha_j, g_j)$ sastified

$$\begin{aligned} &\sup_{M>0} \|f_j - f\|_{C^1(\Omega_M)} + \sup_{M>0} \|\mu_j - \mu\|_{C^2(\hat{\Omega}_M)} \\ &+ \|\alpha_j - \alpha\|_{C^1([0,T^*])} + \|g_j - g\|_{H^1(0,T^*)} \;\; \to 0, \;\; \text{as} \;\; j \to \infty, \end{aligned}$$

where T^* is a fixed positive constant. Then u_j strongly converges to u in $W_1(T)$ as $j \to \infty$. Our paper is organized as follows. Section 2 presents some preliminaries and results of the existence and uniqueness of the problem (1.1)–(1.3) proved by the linear approximation and the Faedo-Galerkin method. In Section 3, we show the continuous dependence of solutions for the problem (1.1)–(1.3) on the relaxation functions $g(t)$, $\alpha(t)$, and the nonlinear terms μ, f. Finally, in Section 4, we summarize the obtained main results of our paper, and propose some potential studies in the future.

2. Existence and uniqueness

Throughout this paper, we set $\Omega = (1,R)$ and use $L^2 = L^2(\Omega)$ to denote the Lebesgue space with the inner product defined by $(u,v) = \int_1^R u(x)v(x)dx$, L^2-norm of a function $u \in L^2$ is denoted by $\|u\| = \sqrt{(u,u)}$. We use $H^m = H^m(\Omega)$ to denote the Sobolev spaces with the norm $\|u\|_{H^m} = \left(\sum_{i=0}^m \left\|D^i u\right\|^2\right)^{1/2}$.

Moreover, we also introduce three weighted scalar products

$$
\begin{aligned}
&\langle u,v\rangle = \int_1^R xu(x)v(x)dx,\ u,v \in L^2, \\
&\langle u,v\rangle_1 = \langle u,v\rangle + \langle u_x,v_x\rangle,\ u,v \in H^1, \\
&\langle u,v\rangle_2 = \langle u,v\rangle + \langle u_x,v_x\rangle + \langle u_{xx},v_{xx}\rangle,\ \ u,\ v \in H^2,
\end{aligned} \tag{2.1}
$$

then $L^2,\ H^1,\ H^2$ are the Hilbert spaces with respect to the above scalar products. We denote $\|u\|_0 = \sqrt{\langle u,u\rangle},\ u \in L^2;\ \|u\|_1 = \sqrt{\langle u,u\rangle_1},\ u \in H^1;\ \|u\|_2 = \sqrt{\langle u,u\rangle_2},\ u \in H^2$.

Put

$$
H_0^1 = \left\{v \in H^1 : v(1) = v(R) = 0\right\}. \tag{2.2}
$$

The symmetric bilinear form $a(\cdot,\cdot)$ is defined by

$$
a(u,w) = \langle u_x,v_x\rangle, \quad \text{for all} \quad u,\ v \in H_0^1. \tag{2.3}
$$

Then, we have the following lemmas.

LEMMA 2.1. *The imbeddings $H^1 \hookrightarrow C^0(\bar{\Omega})$ is compact and*

$$
\|v\|_{C^0(\overline{\Omega})} \leqslant \gamma_0 \sqrt{\|v\|_0^2 + \|v_x\|_0^2}, \quad \textit{for all} \ \ v \in H^1, \tag{2.4}
$$

where

$$
\gamma_0 = \sqrt{\frac{R}{2(R-1)} + \sqrt{1 + \left(\frac{R}{2(R-1)}\right)^2}}. \tag{2.5}
$$

LEMMA 2.2. *The imbeddings $H_0^1 \hookrightarrow C^0(\bar{\Omega})$ is compact and*

$$
\begin{aligned}
&(i) \quad \|v\|_{C^0(\overline{\Omega})} \leqslant \sqrt{R-1}\,\|v_x\|_0, \quad \textit{for all} \quad v \in H_0^1, \\
&(ii) \quad \|v\|_0 \leqslant \frac{\sqrt{2R}(R-1)}{2}\,\|v_x\|_0 \quad \textit{for all} \quad v \in H_0^1.
\end{aligned} \tag{2.6}
$$

LEMMA 2.3. *The symmetric bilinear form $a(\cdot,\cdot)$ is continuous on $H_0^1 \times H_0^1$ and coercive on H_0^1.*

Moreover, we also have

$$
\begin{aligned}
&(i) \quad |a(u,v)| \leqslant \|u_x\|_0\,\|v_x\|_0, \\
&(ii) \quad a(v,v) \geqslant \|v_x\|_0^2,
\end{aligned} \tag{2.7}
$$

for all $u,\ v \in H_0^1$.

LEMMA 2.4. *There exists the Hilbert orthonormal base* $\{w_j\}$ *of* L^2 *consisting of the eigenfunctions* w_j *corresponding to the eigenvalue* λ_j *such that*

$$\begin{cases} 0<\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \lambda_{j+1} \leqslant \cdots, \ \lim_{j\to+\infty} \lambda_j = +\infty, \\ a(w_j, v) = \lambda_j \langle w_j, v\rangle \quad \text{for all} \quad v \in H_0^1, \ j = 1,2,\cdots. \end{cases} \tag{2.8}$$

Furthermore, the sequence $\{w_j/\sqrt{\lambda_j}\}$ *is also the Hilbert orthonormal base of* H_0^1 *with respect to the scalar product* $a(\cdot,\cdot)$.

On the other hand, we have w_j *satisfying the following boundary value problem*

$$\begin{cases} Aw_j \equiv -\left(w_{jxx} + \frac{1}{x} w_{jx}\right) = -\frac{1}{x}\frac{\partial}{\partial x}(x w_{jx}) = \lambda_j w_j, \quad \text{in} \quad (1,R), \\ w_j(1) = w_j(R) = 0, \quad w_j \in C^\infty([1,R]). \end{cases} \tag{2.9}$$

The proof of Lemma 2.4 can be found in [[23], p. 87, Theorem 7.7], with $H = L^2$ and $a(\cdot,\cdot)$ as defined by (2.3).

LEMMA 2.5. *The operator* $A : H_0^1 \to H^{-1} \equiv \left(H_0^1\right)'$ *in* (2.9) *is uniquely defined by Lax-Milgram's lemma, i.e.,*

$$a(u,v) = \langle Au, v\rangle, \quad \text{for all} \ u, \ v \in H_0^1. \tag{2.10}$$

REMARK 2.1. The sequence $\left\{w_j/\sqrt{\lambda_j + \lambda_j^2}\right\}$ is also the Hilbert orthonormal base of $H^2 \cap H_0^1$ with respect to the scalar product $(u,v) \longmapsto a(u,v) + \langle Au, Av\rangle$.

LEMMA 2.6. *On* $H^2 \cap H_0^1$, *the two norms* $u \mapsto \|u\|_{H^2\cap H_0^1} = \left(\|u_x\|_0^2 + \|Au\|_0^2\right)^{1/2}$ *and* $u \mapsto \|u\|_\# = \left(\|u_x\|_0^2 + \|u_{xx}\|_0^2\right)^{1/2}$ *are equivalent. More specifically, there exists a positive constant* $\bar{a}_1$ *such that*

$$\bar{a}_1 \|u\|_\# \leqslant \|u\|_{H^2\cap H_0^1} \leqslant \sqrt{3}\, \|u\|_\#, \ \forall u \in H^2 \cap H_0^1. \tag{2.11}$$

Proof. $u \in H^2 \cap H_0^1$, we have

$$\begin{aligned} \|Au\|_0^2 &= \int_1^R x\,|Au(x)|^2\,dx = \int_1^R x\left(u_{xx} + \frac{1}{x}u_x\right)^2 dx \\ &= \int_1^R \left(x u_{xx}^2 + 2u_{xx}u_x + \frac{1}{x}u_x^2\right) dx = \|u_{xx}\|_0^2 + 2\left\langle u_{xx}, \frac{1}{x}u_x\right\rangle + \left\|\frac{1}{x}u_x\right\|_0^2. \end{aligned}$$

(i) For $0 < \varepsilon < 1$, choose $k > \frac{1}{\varepsilon}$ and set $k_1 = \min\left\{1-\varepsilon, k - \frac{1}{\varepsilon}\right\}$, we have

$$2\left\langle u_{xx}, \frac{1}{x}u_x\right\rangle \leqslant 2\,\|u_{xx}\|_0 \left\|\frac{1}{x}u_x\right\|_0 \leqslant \varepsilon\,\|u_{xx}\|_0^2 + \frac{1}{\varepsilon}\,\|u_x\|_0^2;$$

$$\begin{aligned}\|Au\|_0^2 &\geqslant \|u_{xx}\|_0^2 + 2\left\langle u_{xx}, \frac{1}{x}u_x\right\rangle \geqslant \|u_{xx}\|_0^2 - \varepsilon\|u_{xx}\|_0^2 - \frac{1}{\varepsilon}\|u_x\|_0^2 \\ &= (1-\varepsilon)\|u_{xx}\|_0^2 - \frac{1}{\varepsilon}\|u_x\|_0^2.\end{aligned}$$

We deduce that

$$\begin{aligned}(k+1)\left(\|u_x\|_0^2 + \|Au\|_0^2\right) &\geqslant k\|u_x\|_0^2 + \|Au\|_0^2 \\ &\geqslant k\|u_x\|_0^2 + (1-\varepsilon)\|u_{xx}\|_0^2 - \frac{1}{\varepsilon}\|u_x\|_0^2 \\ &= (1-\varepsilon)\|u_{xx}\|_0^2 + \left(k - \frac{1}{\varepsilon}\right)\|u_x\|_0^2 \\ &\geqslant k_1\left(\|u_{xx}\|_0^2 + \|u_x\|_0^2\right) = k_1\|u\|_{\#}^2.\end{aligned}$$

Thus

$$\|u\|_{H^2\cap H_0^1}^2 = \|u_x\|_0^2 + \|Au\|_0^2 \geqslant \frac{k_1}{k+1}\|u\|_{\#}^2 = \bar{a}_1^2\|u\|_{\#}^2.$$

(ii) We have

$$\begin{aligned}\|Au\|_0^2 &\leqslant \|u_{xx}\|_0^2 + \|u_{xx}\|_0^2 + \left\|\frac{1}{x}u_x\right\|_0^2 + \left\|\frac{1}{x}u_x\right\|_0^2 \\ &\leqslant 2\|u_{xx}\|_0^2 + 2\left\|\frac{1}{x}u_x\right\|_0^2 \leqslant 2\|u_{xx}\|_0^2 + 2\|u_x\|_0^2.\end{aligned}$$

Therefore

$$\begin{aligned}\|u\|_{H^2\cap H_0^1}^2 &= \|u_x\|_0^2 + \|Au\|_0^2 \leqslant 2\|u_{xx}\|_0^2 + 3\|u_x\|_0^2 \\ &\leqslant 3\left(\|u_{xx}\|_0^2 + \|u_x\|_0^2\right) = 3\|u\|_{\#}^2.\end{aligned}$$

Lemma 2.6 is proved. □

The notation $\|\cdot\|_X$ is the norm in the Banach space X, and X' is the dual space of X. We denote by $L^p(0,T;X)$, $1 \leqslant p \leqslant \infty$ for the Banach space of functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p\, dt\right)^{1/p} < \infty \text{ for } 1 \leqslant p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \underset{0<t<T}{ess\sup}\|u(t)\|_X \text{ for } p = \infty.$$

Denote $u(t)(x) = u(x,t)$, $\dot{u}(t) = u'(t) = u_t(t) = \frac{\partial u}{\partial t}(t)$, $u_x(t) = \frac{\partial u}{\partial x}(t)$, $u_{xx}(t) = \frac{\partial^2 u}{\partial x^2}(t)$.

With $f \in C^k(\bar{\Omega} \times [0,T^*] \times \mathbb{R}^2)$, $f = f(x,t,y_1,y_2)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \dfrac{\partial f}{\partial t}$, $D_{2+i} f = \dfrac{\partial f}{\partial y_i}$, $i = 1,2$ and $D^\alpha f = D_1^{\alpha_1} \cdots D_4^{\alpha_4} f$, $\alpha = (\alpha_1, \cdots, \alpha_4) \in \mathbb{Z}_+^4$, $|\alpha| = \alpha_1 + \cdots + \alpha_4 \leqslant k$, $D^0 f = D^{(0,\cdots,0)} f = f$.

For a fixed constant $T^* > 0$, we make the following assumptions:

(H_1) $\tilde{u}_0 \in H^2 \cap H_0^1$;
(H_2) $g \in H^1(0,T^*)$;
(H_3) $\alpha \in C^1([0,T^*])$, $\alpha(t) \geqslant \alpha_* > 0$ for all $t \in [0,T^*]$;
(H_4) $\mu \in C^2(\bar{\Omega} \times [0,T^*] \times \mathbb{R}_+)$, $\mu(x,t,z) \geqslant \mu_* > 0$, $\forall (x,t,v) \in \bar{\Omega} \times [0,T^*] \times \mathbb{R}_+$;
(H_5) $f \in C^1(\bar{\Omega} \times [0,T^*] \times \mathbb{R}^2)$, $f(x,t,0,v) = 0$, $\forall (x,t,v) \in \{1,R\} \times [0,T^*] \times \mathbb{R}$.

DEFINITION 2.7. A weak solution of (1.1)–(1.3) is a function $u \in C([0,T];H^2 \cap H_0^1)$ such that $u' \in L^\infty(0,T;H^2 \cap H_0^1)$, and u satisfies the following variational problem

$$\begin{cases} \langle u'(t), w\rangle + \alpha(t) a(u'(t), w) + \langle \mu[u](t) u_x(t), w_x\rangle \\ \qquad = \int_0^t g(t-s) a(u(s), w)\, ds + \langle f[u](t), w\rangle, \quad \forall w \in H_0^1, \text{ a.e., } t \in (0,T), \\ u(x,0) = \tilde{u}_0(x), \end{cases} \tag{2.12}$$

where

$$\begin{aligned} \mu[u](x,t) &= \mu(x,t,\|u(t)\|_0^2) \geqslant \mu_* > 0, \\ f[u](x,t) &= f(x,t,u(x,t),u_x(x,t)). \end{aligned} \tag{2.13}$$

For each $T \in (0,T^*]$, let W_T

$$W_T = \left\{ v \in C([0,T];H^2 \cap H_0^1) : v' \in L^\infty(0,T;H^2 \cap H_0^1) \right\}, \tag{2.14}$$

be the Banach space with the associated norm

$$\|v\|_{W_T} = \max\left\{ \|v\|_{C([0,T];H^2 \cap H_0^1)},\ \|v'\|_{L^\infty(0,T;H^2 \cap H_0^1)} \right\}. \tag{2.15}$$

Also, we consider the Banach space $W_1(T)$ (see Lions [15])

$$W_1(T) = \left\{ v \in C([0,T];H_0^1) : v' \in L^2\left(0,T;H_0^1\right) \right\}, \tag{2.16}$$

with the norm defined by

$$\|v\|_{W_1(T)} = \|v\|_{C([0,T];H_0^1)} + \|v'\|_{L^2\left(0,T;H_0^1\right)}. \tag{2.17}$$

For $M > 0$, we put

$$\bar{B}(M,T) = \left\{ v \in W_T : \|v\|_{W_T} \leqslant M \right\}, \tag{2.18}$$

and

$$\begin{aligned} K_M(f) &= \|f\|_{C^1(\Omega_M)} = \sum_{\gamma \in \mathbb{Z}_+^4,\ |\gamma| \leqslant 1} \|D^\gamma f\|_{C(\Omega_M)}, \\ \hat{K}_M(\mu) &= \|\mu\|_{C^2(\hat{\Omega}_M)} = \sum_{\gamma \in \mathbb{Z}_+^3,\ |\gamma| \leqslant 2} \|D^\gamma \mu\|_{C(\hat{\Omega}_M)}, \end{aligned} \tag{2.19}$$

$$\|\alpha\|_{C^1([0,T^*])} = \|\alpha\|_{C([0,T^*])} + \|\alpha'\|_{C([0,T^*])},$$

where

$$\|f\|_{C(\Omega_M)} = \sup\{|f(x,t,y_1,y_2)| : (x,t,y_1,y_2) \in \Omega_M\},$$
$$\|\mu\|_{C(\hat{\Omega}_M)} = \sup\{|\mu(x,t,z)| : (x,t,z) \in \hat{\Omega}_M\},$$
$$\|\alpha\|_{C([0,T^*])} = \sup_{0\leqslant t\leqslant T^*} |\alpha(t)|,$$

with

$$\Omega_M = [1,R] \times [0,T^*] \times \left[-\sqrt{R-1}M, \sqrt{R-1}M\right] \times [-\gamma_0 M, \gamma_0 M],$$
$$\hat{\Omega}_M = [1,R] \times [0,T^*] \times [0,M^2].$$

We use the successive approximation method with the first approximation $u_0 \equiv 0$. Suppose that

$$u_{m-1} \in \bar{B}(M,T), \tag{2.20}$$

we find $u_m \in \bar{B}(M,T),\ m \geqslant 1$ to be the solution of the following problem

$$\begin{cases} \langle u_m'(t), w\rangle + \alpha(t)a(u_m'(t), w) + a_m(t; u_m(t), w) \\ \qquad = \int_0^t g(t-s)a(u_m(s), w)\,ds + \langle F_m(t), w\rangle, \\ \qquad\qquad \text{for all} \quad w \in H_0^1, \quad \text{a.e.,}\ t \in (0,T), \\ u_m(0) = \tilde{u}_0, \end{cases} \tag{2.21}$$

where

$$a_m(t; u, w) = \langle \mu_m(t) u_x, w_x\rangle = \int_1^R x\mu_m(x,t)u_x(x)w_x(x)dx,\ \forall u, w \in H_0^1, \tag{2.22}$$
$$\mu_m(x,t) = \mu[u_{m-1}](x,t) = \mu\left(x,t,\|u_{m-1}(t)\|_0^2\right),$$
$$F_m(x,t) = f[u_{m-1}](x,t) = f(x,t,u_{m-1}(x,t),\nabla u_{m-1}(x,t)).$$

Then we have the following theorem.

THEOREM 2.8. *Assume that $\tilde{u}_0$, g, α, μ, f satisfy (H_1)–(H_5), respectively, then there exist the constants $M > 0$ and $T > 0$ such that the problem* (2.21)–(2.22) *admits $u_m \in \bar{B}(M,T)$.*

Proof. (i) *Faedo-Galerkin approximations.* Consider the basis $\{w_j\}$ for L^2 as in Lemma 2.5, we search for a finite-dimensional approximate solution of the problem (2.21)–(2.22) having the following form

$$u_m^{(k)}(t) = \sum_{j=1}^k c_{mj}^{(k)}(t) w_j, \tag{2.23}$$

where $c_{mj}^{(k)}$, $j = 1, \cdots, k$ are determined by the following linear ordinary differential equations

$$\begin{cases} \left\langle \ddot{u}_m^{(k)}(t), w_j \right\rangle + \alpha(t) a(\dot{u}_m^{(k)}(t), w_j) + a_m(t; u_m^{(k)}(t), w_j) \\ \quad = \int_0^t g(t-s) a(u_m^{(k)}(s), w_j) ds + \left\langle F_m(t), w_j \right\rangle, \quad 1 \leqslant j \leqslant k, \\ u_m^{(k)}(0) = \tilde{u}_{0k}, \end{cases} \tag{2.24}$$

in which

$$\tilde{u}_{0k} = \sum_{j=1}^k \alpha_j^{(k)} w_j \to \tilde{v}_0 \text{ strongly in } H^2 \cap H_0^1. \tag{2.25}$$

Using the contraction mapping principle, it is not difficult to prove the existence of the approximate solution $u_m^{(k)}(t)$ of (2.24) in $[0, T]$.

(ii) *Priori estimates*. Next, the following priori estimates show the boundness of the approximate solution $u_m^{(k)}(t)$.

Put

$$\begin{aligned} S_m^{(k)}(t) &= \left\| \sqrt{\mu_m(t)} u_{mx}^{(k)}(t) \right\|_0^2 + \left\| \sqrt{\mu_m(t)} A u_m^{(k)}(t) \right\|_0^2 \\ &+ \left\| \dot{u}_{mx}^{(k)}(t) \right\|_0^2 + \alpha(t) \left\| A \dot{u}_m^{(k)}(t) \right\|_0^2 \\ &+ 2 \int_0^t \left[\left\| \ddot{u}_m^{(k)}(s) \right\|_0^2 + \left\| \ddot{u}_{mx}^{(k)}(s) \right\|_0^2 + \alpha(s) \left(\left\| \dot{u}_m^{(k)}(s) \right\|_a^2 + \left\| A \dot{u}_m^{(k)}(s) \right\|_0^2 \right) \right] ds. \end{aligned} \tag{2.26}$$

From (2.24) and (2.26), we obtain

$$\begin{aligned} \lambda_* \bar{S}_m^{(k)}(t) \leqslant S_m^{(k)}(t) &= \left\| \sqrt{\mu_m(0)} \tilde{u}_{0kx} \right\|_0^2 + \left\| \sqrt{\mu_m(0)} A \tilde{u}_{0k} \right\|_0^2 \\ &+ \int_0^t ds \int_1^R x \mu_m'(x,s) \left(\left| u_{mx}^{(k)}(x,s) \right|^2 + \left| A u_m^{(k)}(x,s) \right|^2 \right) dx \\ &- 2g(0) \int_0^t \left(\left\| u_{mx}^{(k)}(s) \right\|_0^2 + \left\| A u_m^{(k)}(s) \right\|_0^2 \right) ds \\ &+ 2 \int_0^t g(t-s) \left[a(u_m^{(k)}(s), u_m^{(k)}(t)) + \langle A u_m^{(k)}(s), A u_m^{(k)}(t) + A \dot{u}_m^{(k)}(t) \rangle \right] ds \\ &- 2 \int_0^t dr \int_0^r g'(r-s) \left[a(u_m^{(k)}(s), u_m^{(k)}(r)) + \langle A u_m^{(k)}(s), A u_m^{(k)}(r) \rangle \right] ds \\ &+ 2 \int_0^t \left\langle F_m(s), \dot{u}_m^{(k)}(s) + A \dot{u}_m^{(k)}(s) \right\rangle ds + 2 \int_0^t \left\langle x \mu_{mx}(s) u_{mx}^{(k)}(s), A \dot{u}_m^{(k)}(s) \right\rangle ds \\ &+ \left\langle \frac{\partial}{\partial x} \left[x \mu_m(t) u_{mx}^{(k)}(t) \right], A \dot{u}_m^{(k)}(t) \right\rangle + \left\langle F_m(t), A \dot{u}_m^{(k)}(t) \right\rangle \\ &= \left\| \sqrt{\mu_m(0)} \tilde{u}_{0kx} \right\|_0^2 + \left\| \sqrt{\mu_m(0)} A \tilde{u}_{0k} \right\|_0^2 + \sum_{j=1}^8 I_j, \end{aligned} \tag{2.27}$$

where $\lambda_* = \min\{1, \mu_*, \alpha_*\}$ and

$$\bar{S}_m^{(k)}(t) = \left\| u_m^{(k)}(t) \right\|_{H^2 \cap H_0^1}^2 + \left\| \dot{u}_m^{(k)}(t) \right\|_{H^2 \cap H_0^1}^2 + \int_0^t \left\| \dot{u}_m^{(k)}(s) \right\|_{H^2 \cap H_0^1}^2 ds. \tag{2.28}$$

We shall estimate the right-hand side terms of (2.32) as follows.

— *Estimate of* I_1+I_2. First, we have

$$\begin{aligned}\left|\mu_m'(x,t)\right| &\leqslant \hat{K}_M(\mu)\left(1+2\|u_{m-1}(t)\|_0\left\|u_{m-1}'(t)\right\|_0\right) \\ &\leqslant \hat{K}_M(\mu)\left(1+2M^2\right)\equiv \mu_M^*.\end{aligned} \tag{2.29}$$

Then, I_1+I_2 is estimated as follows

$$\begin{aligned}I_1+I_2 &= \int_0^t ds\int_1^R x\mu_m'(x,s)\left(\left|u_{mx}^{(k)}(x,s)\right|^2+\left|Au_m^{(k)}(x,s)\right|^2\right)dx \\ &\quad -2g(0)\int_0^t\left(\left\|u_{mx}^{(k)}(s)\right\|_0^2+\left\|Au_m^{(k)}(s)\right\|_0^2\right)ds \\ &\leqslant (\mu_M^*+2|g(0)|)\int_0^t\left(\left\|u_{mx}^{(k)}(s)\right\|_0^2+\left\|Au_m^{(k)}(s)\right\|_0^2\right)ds \\ &\leqslant (\mu_M^*+2|g(0)|)\int_0^t \bar{S}_m^{(k)}(s)\,ds.\end{aligned} \tag{2.30}$$

— *Estimate of* $I_3,\ I_4,\ I_5$. Using Cauchy-Schwarz inequality and the inequality $2ab\leqslant \beta a^2+\frac{1}{\beta}b^2$, for all $a,\ b\in\mathbb{R}$, with $\beta=\frac{\lambda_*}{6}$, and the estimation

$$\|v\|_0\leqslant \frac{\sqrt{2R}(R-1)}{2}\|v_x\|_0,\ \forall v\in H_0^1,$$

then $I_3,\ I_4,\ I_5$ are estimated as follows

$$\begin{aligned}I_3 &= 2\int_0^t g(t-s)\left[a(u_m^{(k)}(s),u_m^{(k)}(t))+\langle Au_m^{(k)}(s),Au_m^{(k)}(t)+A\dot{u}_m^{(k)}(t)\rangle\right]ds \\ &\leqslant 2\int_0^t |g(t-s)|\left[\left\|u_{mx}^{(k)}(s)\right\|_0\left\|u_{mx}^{(k)}(t)\right\|_0+\left\|Au_m^{(k)}(s)\right\|_0\left\|Au_m^{(k)}(t)+A\dot{u}_m^{(k)}(t)\right\|_0\right]ds \\ &\leqslant 2\int_0^t |g(t-s)|\left[\left\|u_{mx}^{(k)}(s)\right\|_0\left\|u_{mx}^{(k)}(t)\right\|_0\right. \\ &\qquad\left.+\left\|Au_m^{(k)}(s)\right\|_0\left(\left\|Au_m^{(k)}(t)\right\|_0+\left\|A\dot{u}_m^{(k)}(t)\right\|_0\right)\right]ds \\ &\leqslant 2\int_0^t |g(t-s)|\left[\sqrt{\bar{S}_m^{(k)}(s)}\sqrt{\bar{S}_m^{(k)}(t)}+\sqrt{\bar{S}_m^{(k)}(s)}\sqrt{2}\sqrt{\bar{S}_m^{(k)}(t)}\right]ds \\ &= 2\left(1+\sqrt{2}\right)\int_0^t |g(t-s)|\sqrt{\bar{S}_m^{(k)}(s)}\sqrt{\bar{S}_m^{(k)}(t)}ds \\ &\leqslant \beta\bar{S}_m^{(k)}(t)+\frac{1}{\beta}\left(1+\sqrt{2}\right)^2\|g\|_{L^2(0,T^*)}^2\int_0^t \bar{S}_m^{(k)}(s)\,ds;\end{aligned} \tag{2.31}$$

$$\begin{aligned}I_4 &= -2\int_0^t dr\int_0^r g'(r-s)\left[a(u_m^{(k)}(s),u_m^{(k)}(r))+\langle Au_m^{(k)}(s),Au_m^{(k)}(r)\rangle\right]ds \\ &\leqslant 2\int_0^t dr\int_0^r \left|g'(r-s)\right|\left[\left\|u_{mx}^{(k)}(s)\right\|_0\left\|u_{mx}^{(k)}(r)\right\|_0+\left\|Au_m^{(k)}(s)\right\|_0\left\|Au_m^{(k)}(r)\right\|_0\right]ds\end{aligned}$$

$$
\begin{aligned}
&\leqslant 2\int_0^t dr\int_0^r \left|g'(r-s)\right|\sqrt{\bar{S}_m^{(k)}(s)}\sqrt{\bar{S}_m^{(k)}(r)}ds\\
&\leqslant 2\sqrt{T^*}\left\|g'\right\|_{L^2(0,T^*)}\int_0^t \bar{S}_m^{(k)}(s)\,ds;
\end{aligned}
$$

$$
\begin{aligned}
I_5 &= 2\int_0^t \left\langle F_m(s),\dot{u}_m^{(k)}(s)+A\dot{u}_m^{(k)}(s)\right\rangle ds\\
&\leqslant 2\int_0^t \|F_m(s)\|_0\left(\left\|\dot{u}_m^{(k)}(s)\right\|_0+\left\|A\dot{u}_m^{(k)}(s)\right\|_0\right)ds\\
&\leqslant 2K_M(f)\int_0^t\left(\frac{\sqrt{2R}(R-1)}{2}\left\|\dot{u}_{mx}^{(k)}(s)\right\|_0+\left\|A\dot{u}_m^{(k)}(s)\right\|_0\right)ds\\
&\leqslant 2K_M(f)\left(1+\frac{\sqrt{2R}(R-1)}{2}\right)\int_0^t\sqrt{\bar{S}_m^{(k)}(s)}ds\\
&\leqslant TK_M^2(f)\tilde{R}^2\left(1+\frac{\sqrt{2R}(R-1)}{2}\right)^2+\int_0^t\bar{S}_m^{(k)}(s)\,ds\\
&= TK_M^2(f)\tilde{R}^2+\int_0^t\bar{S}_m^{(k)}(s)\,ds,
\end{aligned}
$$

where $\tilde{R}=1+\dfrac{\sqrt{2R}(R-1)}{2}$.

—- *Estimate of* I_6. In order to estimate I_6, we need to the following estimate

$$|\mu_{mx}(x,t)|=|D_1\mu[u_{m-1}](x,t)|\leqslant \hat{K}_M(\mu).$$

Then

$$
\begin{aligned}
I_6 &= 2\int_0^t\left\langle x\mu_{mx}(s)u_{mx}^{(k)}(s),A\dot{u}_m^{(k)}(s)\right\rangle ds\\
&\leqslant 2R\hat{K}_M(\mu)\int_0^t\left\|u_{mx}^{(k)}(s)\right\|_0\left\|A\dot{u}_m^{(k)}(s)\right\|_0 ds\\
&\leqslant 2R\hat{K}_M(\mu)\int_0^t\bar{S}_m^{(k)}(s)\,ds.
\end{aligned}
\tag{2.32}
$$

— *Estimate of* I_7.

$$
\begin{aligned}
I_7 &= \left\langle\frac{\partial}{\partial x}\left[x\mu_m(t)u_{mx}^{(k)}(t)\right],A\dot{u}_m^{(k)}(t)\right\rangle\\
&\leqslant\left\|\frac{\partial}{\partial x}\left[x\mu_m(t)u_{mx}^{(k)}(t)\right]\right\|_0\left\|A\dot{u}_m^{(k)}(t)\right\|_0\\
&\leqslant\beta\left\|A\dot{u}_m^{(k)}(t)\right\|_0^2+\frac{1}{\beta}\left\|\frac{\partial}{\partial x}\left[x\mu_m(t)u_{mx}^{(k)}(t)\right]\right\|_0^2\\
&\leqslant\beta\bar{S}_m^{(k)}(t)+\frac{1}{\beta}\left\|\frac{\partial}{\partial x}\left[x\mu_m(t)u_{mx}^{(k)}(t)\right]\right\|_0^2.
\end{aligned}
\tag{2.33}
$$

We will estimate the term $\left\|\frac{\partial}{\partial x}\left[x\mu_m(t)u_{mx}^{(k)}(t)\right]\right\|_0^2$ as follows.

$$\begin{aligned}\frac{\partial}{\partial x}\left[x\mu_m(t)u_{mx}^{(k)}(t)\right] &= \mu_m(t)\frac{\partial}{\partial x}\left[xu_{mx}^{(k)}(t)\right]+\mu_{mx}(t)\left[xu_{mx}^{(k)}(t)\right]\\ &= -x\mu_m(t)Au_m^{(k)}(t)+\mu_{mx}(t)\left[xu_{mx}^{(k)}(t)\right],\end{aligned}$$

hence

$$\begin{aligned}\frac{\partial^2}{\partial x\partial t}\left[x\mu_m(t)u_{mx}^{(k)}(t)\right] = &-x\mu_m(t)A\dot{u}_m^{(k)}(t)-x\mu_m'(t)Au_m^{(k)}(t)\\ &+\mu_{mx}'(t)\left[xu_{mx}^{(k)}(t)\right]+\mu_{mx}(t)\left[x\dot{u}_{mx}^{(k)}(t)\right].\end{aligned} \tag{2.34}$$

On the other hand, using the following equalities

$$\begin{aligned}\mu_m(x,t) &= \mu[u_{m-1}](x,t)=\mu\left(x,t,\|u_{m-1}(t)\|_0^2\right)\\ \mu_{mx}(x,t) &= D_1\mu[u_{m-1}](x,t)=D_1\mu\left(x,t,\|u_{m-1}(t)\|_0^2\right),\\ \mu_m'(x,t) &= D_2\mu[u_{m-1}](x,t)+2D_3\mu[u_{m-1}](x,t)\langle u_{m-1}(t),u_{m-1}'(t)\rangle;\\ \mu_{mx}'(x,t) &= D_1D_2\mu[u_{m-1}](x,t)+2D_1D_3\mu[u_{m-1}](x,t)\langle u_{m-1}(t),u_{m-1}'(t)\rangle,\end{aligned}$$

we obtain

$$\begin{aligned}|\mu_m(x,t)| &\leqslant \hat{K}_M(\mu),\ \ |\mu_{mx}(x,t)|\leqslant \hat{K}_M(\mu),\\ \left|\mu_m'(x,t)\right| &\leqslant \hat{K}_M(\mu)\left(1+2M^2\right)\equiv \mu_M^*,\\ \left|\mu_{mx}'(x,t)\right| &\leqslant \hat{K}_M(\mu)\left(1+2M^2\right)\equiv \mu_M^*.\end{aligned}$$

Therefore

$$\begin{aligned}&\left\|\frac{\partial^2}{\partial x\partial t}\left[x\mu_m(t)u_{mx}^{(k)}(t)\right]\right\|_0\\ &= \left\|x\mu_m(t)A\dot{u}_m^{(k)}(t)\right\|_0+\left\|x\mu_m'(t)Au_m^{(k)}(t)\right\|_0\\ &\quad+\left\|\mu_{mx}'(t)\left[xu_{mx}^{(k)}(t)\right]\right\|_0+\left\|\mu_{mx}(t)\left[x\dot{u}_{mx}^{(k)}(t)\right]\right\|_0\\ &\leqslant R\hat{K}_M(\mu)\left\|A\dot{u}_m^{(k)}(t)\right\|_0+R\mu_M^*\left\|Au_m^{(k)}(t)\right\|_0+R\mu_M^*\left\|u_{mx}^{(k)}(t)\right\|_0+R\hat{K}_M(\mu)\left\|\dot{u}_{mx}^{(k)}(t)\right\|_0\\ &\leqslant R\mu_M^*\left(\left\|A\dot{u}_m^{(k)}(t)\right\|_0+\left\|Au_m^{(k)}(t)\right\|_0+\left\|u_{mx}^{(k)}(t)\right\|_0+\left\|\dot{u}_{mx}^{(k)}(t)\right\|_0\right)\\ &\leqslant 2R\mu_M^*\left(\left\|A\dot{u}_m^{(k)}(t)\right\|_0^2+\left\|Au_m^{(k)}(t)\right\|_0^2+\left\|u_{mx}^{(k)}(t)\right\|_0^2+\left\|\dot{u}_{mx}^{(k)}(t)\right\|_0^2\right)^{1/2}\\ &\leqslant 2R\mu_M^*\sqrt{\bar{S}_m^{(k)}(t)}.\end{aligned} \tag{2.35}$$

It follows that

$$\begin{aligned}
&\left\|\frac{\partial}{\partial x}\left[x\mu_m(t)u_{mx}^{(k)}(t)\right]\right\|_0^2 \qquad (2.36)\\
&\leqslant \left\|\frac{\partial}{\partial x}\left[x\mu_m(0)u_{mx}^{(k)}(0)\right]+\int_0^t \frac{\partial^2}{\partial x\partial s}\left[x\mu_m(s)u_{mx}^{(k)}(s)\right]ds\right\|_0^2\\
&\leqslant 2\left\|\frac{\partial}{\partial x}\left[x\mu_m(0)\tilde{u}_{0kx}\right]\right\|_0^2+2\left(\int_0^t\left\|\frac{\partial^2}{\partial x\partial s}\left[x\mu_m(s)u_{mx}^{(k)}(s)\right]\right\|_0 ds\right)^2\\
&\leqslant 2\left\|\frac{\partial}{\partial x}\left[x\mu_m(0)\tilde{u}_{0kx}\right]\right\|_0^2+2T^*\int_0^t\left\|\frac{\partial^2}{\partial x\partial s}\left[x\mu_m(s)u_{mx}^{(k)}(s)\right]\right\|_0^2 ds\\
&\leqslant 2\left\|\frac{\partial}{\partial x}\left[x\mu_m(0)\tilde{u}_{0kx}\right]\right\|_0^2+8T^*R^2\left(\mu_M^*\right)^2\int_0^t \bar{S}_m^{(k)}(s)ds.
\end{aligned}$$

Finally, by using (2.33) and (2.36), the term I_7 is estimated as follows.

$$\begin{aligned}
I_7 &= \left\langle \frac{\partial}{\partial x}\left[x\mu_m(t)u_{mx}^{(k)}(t)\right], A\dot{u}_m^{(k)}(t)\right\rangle \qquad (2.37)\\
&\leqslant \left\|\frac{\partial}{\partial x}\left[x\mu_m(t)u_{mx}^{(k)}(t)\right]\right\|_0 \left\|A\dot{u}_m^{(k)}(t)\right\|_0\\
&\leqslant \beta\left\|A\dot{u}_m^{(k)}(t)\right\|_0^2+\frac{1}{\beta}\left\|\frac{\partial}{\partial x}\left[x\mu_m(t)u_{mx}^{(k)}(t)\right]\right\|_0^2\\
&\leqslant \beta\bar{S}_m^{(k)}(t)+\frac{2}{\beta}\left\|\frac{\partial}{\partial x}\left[x\mu_m(0)\tilde{u}_{0kx}\right]\right\|_0^2+\frac{8}{\beta}T^*R^2\left(\mu_M^*\right)^2\int_0^t \bar{S}_m^{(k)}(s)ds.
\end{aligned}$$

— *Estimate of* I_8. Using Cauchy-Schwarz inequality, we get that

$$I_8 = \left\langle F_m(t), A\dot{u}_m^{(k)}(t)\right\rangle \leqslant \frac{1}{4\beta}\left\|F_m(t)\right\|_0^2+\beta\bar{S}_m^{(k)}(t). \qquad (2.38)$$

On the other hand, we have that

$$F_m(t) = F_m(0)+\int_0^t F_m'(s)ds, \qquad (2.39)$$

and

$$F_m'(x,t) = D_2 f[u_{m-1}](x,t)+D_3 f[u_{m-1}](x,t)u_{m-1}'(x,t)+D_4 f[u_{m-1}](x,t)\nabla u_{m-1}'(x,t).$$

It leads to

$$\left|F_m'(x,t)\right| \leqslant K_M(f)\left[1+\left|u_{m-1}'(x,t)\right|+\left|\nabla u_{m-1}'(x,t)\right|\right],$$

and

$$\left\|F_m'(t)\right\|_0 \leqslant K_M(f)\left[\sqrt{\frac{R^2-1}{2}}+\left\|u_{m-1}'(t)\right\|_0+\left\|\nabla u_{m-1}'(t)\right\|_0\right] \leqslant K_M(f)\left[\sqrt{\frac{R^2-1}{2}}+2M\right] \equiv F_M^*. \tag{2.40}$$

Then, from (2.39)–(2.40), we have

$$\|F_m(t)\|_0 \leqslant \|F_m(0)\|_0+\int_0^t\left\|F_m'(s)\right\|_0 ds \leqslant \|f(\cdot,0,\tilde{u}_0,\tilde{u}_{0x}\|_0+TF_M^*. \tag{2.41}$$

Since (2.38), it follows from (2.41) that

$$\begin{aligned} I_8 = \left\langle F_m(t), A\dot{u}_m^{(k)}(t)\right\rangle &\leqslant \frac{1}{4\beta}\|F_m(t)\|_0^2+\beta\bar{S}_m^{(k)}(t) \\ &\leqslant \frac{1}{4\beta}\left(\|f(\cdot,0,\tilde{u}_0,\tilde{u}_{0x}\|_0+TF_M^*\right)^2+\beta\bar{S}_m^{(k)}(t) \\ &\leqslant \frac{1}{2\beta}\left[\|f(\cdot,0,\tilde{u}_0,\tilde{u}_{0x}\|_0^2+T^2(F_M^*)^2\right]+\beta\bar{S}_m^{(k)}(t). \end{aligned} \tag{2.42}$$

Choosing $\beta=\frac{\lambda_*}{6}$, and combining (2.30)–(2.32), (2.37), and (2.42), it implies from (2.27) that

$$\bar{S}_m^{(k)}(t) \leqslant \bar{S}_{0mk}+Td_1(M)+d_2(M)\int_0^t\bar{S}_m^{(k)}(s)\,ds, \tag{2.43}$$

where

$$\left\{\begin{aligned} \bar{S}_{0mk} &= \frac{6}{\lambda_*^2}\|f(\cdot,0,\tilde{u}_0,\tilde{u}_{0x})\|_0^2+\frac{2}{\lambda_*}\left(\left\|\sqrt{\mu_m(0)}\tilde{u}_{0kx}\right\|_0^2+\left\|\sqrt{\mu_m(0)}A\tilde{u}_{0k}\right\|_0^2\right) \\ &\quad +\frac{24}{\lambda_*^2}\left\|\frac{\partial}{\partial x}[x\mu_m(0)\tilde{u}_{0kx}]\right\|_0^2, \\ d_1(M) &= \frac{2}{\lambda_*}\left(K_M^2(f)\tilde{R}^2+\frac{3}{\lambda_*}T^*(F_M^*)^2\right), \\ d_2(M) &= \frac{2}{\lambda_*}\left(1+\mu_M^*+2|g(0)|+2\sqrt{T^*}\|g\|_{H^1(0,T^*)}+2R\hat{K}_M(\mu)\right) \\ &\quad +\frac{12}{\lambda_*^2}\left(1+\sqrt{2}\right)^2\|g\|_{H^1(0,T^*)}^2+\frac{96}{\lambda_*^2}T^*R^2(\mu_M^*)^2. \end{aligned}\right. \tag{2.44}$$

Thanking the convergence shown in (2.25), there exists a constant $M>0$ independent of k and m such that

$$\bar{S}_{0mk} \leqslant \frac{1}{2}M^2, \quad \text{for all } \ m,\, k\in\mathbb{N}. \tag{2.45}$$

By choosing M as above, we can choose $T \in (0, T^*]$ such that

$$\left(\frac{1}{2}M^2 + Td_1(M)\right)\exp\left(Td_2(M)\right) \leqslant M^2, \tag{2.46}$$

and

$$k_T = 2\sqrt{T\eta_1(M)}\exp\left(T\eta_2(M)\right) < 1, \tag{2.47}$$

where

$$\eta_1(M) = \frac{12}{\lambda_*^2}\left[M^4\hat{K}_M^2(\mu) + 4K_M^2(f)\right], \tag{2.48}$$
$$\eta_2(M) = \frac{1}{\lambda_*}\left[\mu_M^* + 2|g(0)| + \frac{6}{\lambda_*}\|g\|_{H^1(0,T^*)}^2 + 2\sqrt{T^*}\|g\|_{H^1(0,T^*)}\right].$$

Applying Gronwall's lemma, we deduce from (2.43), (2.45), and (2.46) that

$$\bar{S}_m^{(k)}(t) \leqslant M^2\exp\left(-Td_2(M)\right)\exp\left(td_2(M)\right) \leqslant M^2, \tag{2.49}$$

for all $t \in [0, T]$, for all m and k. Therefore, we have

$$u_m^{(k)} \in \bar{B}(M, T), \quad \text{for all } m \text{ and } k \in \mathbb{N}. \tag{2.50}$$

Due to (2.50), there exists a subsequence of $\{u_m^{(k)}\}$, still denoted by $\{u_m^{(k)}\}$ such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty(0,T;H^2\cap H_0^1) \text{ weakly*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in} \quad L^\infty(0,T;H^2\cap H_0^1) \text{ weakly*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in} \quad L^2(0,T;H^2\cap H_0^1) \text{ weakly}, \\ u_m \in \bar{B}(M,T). \end{cases} \tag{2.51}$$

Taking to the limitations in (2.24) and (2.25), we have that u_m satisfies (2.21) and (2.22) in $L^2(0,T)$. Theorem 2.8 is proved. □

By Theorem 2.8, the local existence and uniqueness of solutions for the problem (1.1)–(1.3) are proved and shown in Theorem 2.9 below.

THEOREM 2.9. *Suppose that* (H_1)–(H_5) *are satisfied. Then, the recurrent sequence* $\{u_m\}$ *defined by* (2.20)–(2.22) *strongly converges to* u *in* $W_1(T)$, *and* $u \in \bar{B}(M,T)$ *is the unique weak solution of the problem* (1.1)–(1.3).

Moreover, the following estimate is valid

$$\|u_m - u\|_{W_1(T)} \leqslant C_T k_T^m, \quad \text{for all } m \in \mathbb{N}, \tag{2.52}$$

where $k_T \in [0,1)$ *is defined as in (2.47) and* C_T *is a constant depending only on* T, $\tilde{u}_0$, g, α, μ, f *and* k_T.

Proof. First, we prove the local existence of (1.1)–(1.3). It is necessary to prove that $\{u_m\}$ (in Theorem 2.8) is a Cauchy sequence in $W_1(T)$. Let $v_m = u_{m+1} - u_m$.

Then v_m satisfies the variational proplem

$$\begin{cases} \langle v_m'(t),w\rangle+\alpha(t)a\left(v_m'(t),w\right)+a_{m+1}\left(t;v_m(t),w\right)+\langle[\mu_{m+1}(t)-\mu_m(t)]\,u_{mx}(t),w_x\rangle \\ \quad =\int_0^t g(t-s)a\left(v_m(s),w\right)ds+\langle F_{m+1}(t)-F_m(t),w\rangle, \\ \qquad\qquad\qquad\qquad\qquad \text{for all } \ w\in H_0^1, \ \text{a.e.,} \ t\in(0,T), \\ v_m(0)=0, \end{cases} \tag{2.53}$$

where

$$\mu_m(x,t)=\mu[u_{m-1}](x,t)=\mu\left(x,t,\|u_{m-1}(t)\|_0^2\right), \tag{2.54}$$
$$F_m(x,t)=f[u_{m-1}](x,t)=f\left(x,t,u_{m-1}(x,t),\nabla u_{m-1}(x,t)\right).$$

Taking $w=v_m'(t)$ in $(2.53)_1$ and then integrating in t, we get

$$\begin{aligned} \lambda_*\bar{Z}_m(t) &\leqslant \int_0^t ds\int_1^R x\mu_{m+1}'(x,s)v_{mx}^2(x,s)dx-2g(0)\int_0^t \|v_{mx}(s)\|_0^2\,ds \\ &\quad +2\int_0^t g(t-s)\langle v_{mx}(s),v_{mx}(t)\rangle ds-2\int_0^t dr\int_0^r g'(r-s)\langle v_{mx}(s),v_{mx}(r)\rangle ds \\ &\quad -\int_0^t \langle[\mu_{m+1}(s)-\mu_m(s)]\,u_{mx}(s),v_{mx}'(s)\rangle ds \\ &\quad +2\int_0^t \langle F_{m+1}(s)-F_m(s),v_m'(s)\rangle ds \\ &=\bar{Z}_1+\cdots+\bar{Z}_6, \end{aligned} \tag{2.55}$$

where $\lambda_*=\min\{1,\mu_*,\alpha_*\}$ and

$$\bar{Z}_m(t)=\|v_{mx}(t)\|_0^2+\int_0^t\left(\left\|v_m'(s)\right\|_0^2+\left\|v_{mx}'(s)\right\|_0^2\right)ds. \tag{2.56}$$

Next, we have to estimate the integrals on the right-hand side of (2.55).

By using the hypotheses (H_2), (H_4), and the inequalitiy $2ab\leqslant\delta a^2+\frac{1}{\delta}b^2$, $\forall a,b\in\mathbb{R}$, with $\delta=\frac{\lambda_*}{6}$, the terms $\bar{Z}_1,\ \cdots,\ \bar{Z}_4$ are estimated as follows

$$\begin{aligned} \bar{Z}_1 &=\int_0^t ds\int_1^R x\mu_{m+1}'(x,s)v_{mx}^2(x,s)dx\leqslant\mu_M^*\int_0^t \bar{Z}_m(s)ds; \\ \bar{Z}_2 &=-2g(0)\int_0^t \|v_{mx}(s)\|_0^2\,ds\leqslant 2\,|g(0)|\int_0^t \bar{Z}_m(s)ds; \\ \bar{Z}_3 &=2\int_0^t g(t-s)\langle v_{mx}(s),v_{mx}(t)\rangle ds \\ &\leqslant 2\int_0^t |g(t-s)|\,\|v_{mx}(s)\|_0\,\|v_{mx}(t)\|_0\,ds \\ &\leqslant 2\int_0^t |g(t-s)|\sqrt{\bar{Z}_m(s)}\sqrt{\bar{Z}_m(t)}ds \end{aligned} \tag{2.57}$$

$$\leqslant \delta \bar{Z}_m(t) + \frac{1}{\delta}\int_0^t g^2(s)ds \int_0^t \bar{Z}_m(s)ds$$
$$\leqslant \delta \bar{Z}_m(t) + \frac{1}{\delta}\|g\|^2_{L^2(0,T^*)}\int_0^t \bar{Z}_m(s)ds;$$
$$\bar{Z}_4 = -2\int_0^t dr \int_0^r g'(r-s)\langle v_{mx}(s), v_{mx}(r)\rangle ds$$
$$\leqslant 2\int_0^t dr \int_0^r \left|g'(r-s)\right| \sqrt{\bar{Z}_m(s)}\sqrt{\bar{Z}_m(r)}ds$$
$$\leqslant 2\sqrt{T^*}\left\|g'\right\|_{L^2(0,T^*)}\int_0^t \bar{Z}_m(s)ds.$$

In order to estimate $\bar{Z}_5$, we use the following estimation

$$|\mu_{m+1}(x,t) - \mu_m(x,t)|$$
$$= \left|\mu\left(x,t,\|u_m(t)\|_0^2\right) - \mu\left(x,t,\|u_{m-1}(t)\|_0^2\right)\right|$$
$$\leqslant \hat{K}_M(\mu)\left|\|u_m(t)\|_0^2 - \|u_{m-1}(t)\|_0^2\right| \leqslant 2M\hat{K}_M(\mu)\|v_{m-1}(t)\|_0$$
$$\leqslant 2M\hat{K}_M(\mu)\|v_{m-1}\|_{W_1(T)},$$

then we deduce that

$$\bar{Z}_5 = -\int_0^t \langle[\mu_{m+1}(s) - \mu_m(s)]\, u_{mx}(s), v'_{mx}(s)\rangle ds \tag{2.58}$$
$$\leqslant \int_0^t \|\mu_{m+1}(s) - \mu_m(s)\|_{L^\infty}\|u_{mx}(s)\|_0 \left\|v'_{mx}(s)\right\|_0 ds$$
$$\leqslant 2M^2\hat{K}_M(\mu)\|v_{m-1}\|_{W_1(T)}\int_0^t \left\|v'_{mx}(s)\right\|_0 ds$$
$$\leqslant \frac{1}{\delta}TM^4\hat{K}_M^2(\mu)\|v_{m-1}\|^2_{W_1(T)} + \delta\int_0^t \left\|v'_{mx}(s)\right\|_0^2 ds$$
$$\leqslant \frac{1}{\delta}TM^4\hat{K}_M^2(\mu)\|v_{m-1}\|^2_{W_1(T)} + \delta\bar{Z}_m(t).$$

Applying the mean value theorem to the fucntion f, we have

$$\|F_{m+1}(t) - F_m(t)\|_0 \leqslant K_M(f)\left[\|v_{m-1}(t)\|_0 + \|\nabla v_{m-1}(t)\|_0\right] = 2K_M(f)\|v_{m-1}\|_{W_1(T)}.$$

Hence, we can estimate $\bar{Z}_6$ as follows

$$\bar{Z}_6 = 2\int_0^t \langle F_{m+1}(s) - F_m(s), v'_m(s)\rangle ds \tag{2.59}$$
$$\leqslant 4K_M(f)\|v_{m-1}\|_{W_1(T)}\int_0^t \left\|v'_m(s)\right\|_0 ds$$
$$\leqslant \frac{1}{\delta}4TK_M^2(f)\|v_{m-1}\|^2_{W_1(T)} + \delta\int_0^t \left\|v'_m(s)\right\|_0^2 ds$$
$$\leqslant \frac{1}{\delta}4TK_M^2(f)\|v_{m-1}\|^2_{W_1(T)} + \delta\bar{Z}_m(t).$$

Choosing $\delta = \frac{\lambda_*}{6}$, using (2.57), (2.58) and (2.59), we deduce from (2.55) that

$$\bar{Z}_m(t) \leqslant T\eta_1(M) \|v_{m-1}\|^2_{W_1(T)} + 2\eta_2(M) \int_0^t \bar{Z}_m(s)\, ds, \tag{2.60}$$

where $\eta_1(M),\ \eta_2(M)$ are defined by (2.48).

Using Gronwall's lemma, we have

$$\bar{Z}_m(t) \leqslant T\eta_1(M) \exp(2T\eta_2(M)) \|v_{m-1}\|^2_{W_1(T)}. \tag{2.61}$$

It leads to

$$\|v_m\|_{W_1(T)} \leqslant k_T \|v_{m-1}\|_{W_1(T)}, \quad \forall m \in \mathbb{N}, \tag{2.62}$$

where $k_T < 1$ is defined as in (2.47), this implies that

$$\left\|u_{m+p} - u_m\right\|_{W_1(T)} \leqslant \frac{M}{1-k_T} k_T^m, \ \forall m,\ p \in \mathbb{N}. \tag{2.63}$$

The above inequality ensures that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \to u \quad \text{strongly in} \quad W_1(T). \tag{2.64}$$

Note that $u_m \in \bar{B}(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in} \quad L^\infty(0,T;H^2\cap H_0^1) \quad \text{weakly*}, \\ u'_{m_j} \to u' & \text{in} \quad L^\infty(0,T;H^2\cap H_0^1) \quad \text{weakly*}, \\ u \in \bar{B}(M,T). \end{cases} \tag{2.65}$$

We note that

$$\|F_m(t) - f[u](t)\|_0 \leqslant 2K_M(f) \|u_{m-1} - u\|_{W_1(T)},$$

it follows from (2.64) that

$$F_m \to f[u] \quad \text{strongly in} \quad L^\infty(0,T;L^2). \tag{2.66}$$

Similarly

$$|\mu_m(t) - \mu[u](t)| \leqslant 2M\hat{K}_M(\mu) \|u_{m-1} - u\|_{W_1(T)},$$

it implies that

$$\mu_m \to \mu[u] \quad \text{strongly in} \quad L^\infty(0,T). \tag{2.67}$$

Letting $m = m_j \to \infty$ in (2.21), (2.22) and using (2.64), $(2.65)_3$, (2.66) and (2.67), we get that there exists $u \in \bar{B}(M,T)$ satisfying (2.12)–(2.13). The proof of the solution existence is completed.

Finally, we need to prove the uniqueness of solutions.

Let $u_1,\ u_2 \in \bar{B}(M,T)$ be two weak solutions of the problem (2.11)–(2.12). Then $u = u_1 - u_2$ satisfies the variational problem

$$\begin{cases} \langle u'(t), w\rangle + \alpha(t) a\left(u'(t), w\right) + \langle \mu[u_1](t) u_x(t), w_x\rangle + \langle \bar{\mu}(t) u_{2x}(t), w_x\rangle \\ \qquad = \int_0^t g(t-s) a\left(u(s), w\right) ds + \langle \bar{f}(t), w\rangle, \\ \qquad\qquad \text{for all } \ w \in H_0^1, \ \text{a.e., } \ t \in (0,T), \\ u(0) = 0, \end{cases} \tag{2.68}$$

where

$$\bar{\mu}(x,t) = \mu[u_1](x,t) - \mu[u_2](x,t), \tag{2.69}$$
$$\bar{f}(x,t) = f[u_1](x,t) - f[u_2](x,t),$$
$$\mu_i(x,t) = \mu[u_i](t) = \mu\left(x, t, \|u_i(t)\|_0^2\right), \quad f[u_i](x,t) = f\left(x, t, u_i(x,t), u_{ix}(x,t)\right).$$

Taking $w = u'(t)$ and integrating in time from 0 to t, we get

$$\begin{aligned} \lambda_* \bar{Z}(t) \leqslant & \int_0^t ds \int_1^R x \mu_1'(x,s) u_x^2(x,s) dx - 2g(0) \int_0^t \|u_x(s)\|_0^2 ds \\ & + 2\int_0^t g(t-s) \langle u_x(s), u_x(t)\rangle ds \\ & - 2\int_0^t dr \int_0^r g'(r-s) \langle u_x(s), u_x(r)\rangle ds \\ & - 2\int_0^t \langle \bar{\mu}(s) u_{2x}(s), u_x'(s)\rangle ds + 2\int_0^t \langle \bar{f}(s), u'(s)\rangle ds, \end{aligned} \tag{2.70}$$

where $\lambda_* = \min\{1, \mu_*, \alpha_*\}$ and

$$\bar{Z}(t) = \|u_x(t)\|_0^2 + \int_0^t \left(\left\|u'(s)\right\|_0^2 + \left\|u_x'(s)\right\|_0^2\right) ds. \tag{2.71}$$

Similarly, the integrals on the right-hand side of (2.70) are computed as follows

$$\bar{Z}(t) \leqslant \tilde{\rho}_M \textstyle\int_0^t \bar{Z}(s) ds, \tag{2.72}$$

where

$$\begin{aligned} \tilde{\rho}_M = & \frac{2}{\lambda_*}\left[\mu_M^* + 2|g(0)| + 2\sqrt{T^*}\left\|g'\right\|_{L^2(0,T^*)}\right] + \frac{12}{\lambda_*^2}\|g\|_{L^2(0,T^*)}^2 \\ & + 2M^4 \hat{K}_M^2(\mu) R(R-1)^2 + \frac{12}{\lambda_*^2} K_M^2(f)\left(1 + \frac{\sqrt{2R}(R-1)}{2}\right)^2. \end{aligned}$$

Therefore, using Gronwall's lemma, it follows that $\bar{Z}(t) \equiv 0$, i.e., $u_1 \equiv u_2$. The uniqueness of solutions is confirmed. Consequently, Theorem 2.9 is proved completely. □

3. Continuous dependence

In this section, we consider the continuous dependence of solutions for the problem (1.1)–(1.3) on the functions $f,\ \mu,\ \alpha,\ g$. Let $\tilde{u}_0 \in H^2 \cap H_0^1$ and $f,\ \mu,\ \alpha,\ g$ satisfy the assumptions $(H_2)-(H_5)$. By Theorem 2.9, the problem (1.1)–(1.3) admits a unique weak solution $u = u(f,\mu,\alpha,g)$ depending on $f,\ \mu,\ \alpha,\ g$.

For each fixed $\alpha_* > 0$ and $\mu_* > 0$, we put

$$\begin{aligned}\mathscr{F}(\alpha_*,\mu_*) = \{(f,\mu,\alpha,g) \in C^1(\bar{\Omega}\times[0,T^*]\times\mathbb{R}^2)\times C^2(\bar{\Omega}\times[0,T^*]\times\mathbb{R}_+)\\ \times C^1([0,T^*])\times H^1(0,T^*):\\ f(x,t,0,v) = 0,\ \forall (x,t,v)\in\{1,R\}\times[0,T^*]\times\mathbb{R};\\ \mu(x,t,z)\geqslant \mu_* > 0,\ \forall (x,t,v)\in\bar{\Omega}\times[0,T^*]\times\mathbb{R}_+;\\ \alpha(t)\geqslant\alpha_* > 0,\ \forall t\in[0,T^*]\}.\end{aligned} \tag{3.1}$$

Then we have the following theorem.

THEOREM 3.1. *Let $T^* > 0$ and $\tilde{u}_0 \in H^2\cap H_0^1$. Then, there exists a positive constant T such that the solution of the problem* (1.1)–(1.3) *is stable with the datum $f,\ \mu,\ \alpha,\ g$, i.e. if $(f,\mu,\alpha,g),\ (f_j,\mu_j,\alpha_j,g_j)\in\mathscr{F}(\alpha_*,\mu_*)$ such that*

$$\begin{aligned}E_j \equiv \sup_{M>0}\left\|f_j-f\right\|_{C^1(\Omega_M)} + \sup_{M>0}\left\|\mu_j-\mu\right\|_{C^2(\hat{\Omega}_M)}\\ +\left\|\alpha_j-\alpha\right\|_{C^1([0,T^*])}+\left\|g_j-g\right\|_{H^1(0,T^*)} \to 0,\ \text{as}\ j\to\infty,\end{aligned} \tag{3.2}$$

then

$$u_j \to u \quad \text{strongly in } W_1(T), \quad \text{as } j\to\infty, \tag{3.3}$$

where $u_j = u(f_j,\mu_j,\alpha_j,g_j)$. Moreover, we have the estimation

$$\left\|u_j-u\right\|_{W_1(T)} \leqslant C_T E_j, \quad \forall j\in\mathbb{N}, \tag{3.4}$$

where C_T is a positive constant only dependent on $T,\ f,\ \mu,\ \alpha,\ g,$ and $\tilde{u}_0$.

Proof. First, we note that if $(f,\mu,\alpha,g),\ (f_j,\mu_j,\alpha_j,g_j)\in\mathscr{F}(\alpha_*,\mu_*)$, and the following additional condition is fulfilled

$$\begin{aligned}\sup_{M>0}\left\|f_j-f\right\|_{C^1(\Omega_M)} &= \sup_{M>0}\sum_{\beta\in\mathbb{Z}_+^4,\ |\beta|\leqslant 1}\left\|D^\beta f_j - D^\beta f\right\|_{C(\Omega_M)} \to 0,\ \text{as}\ j\to\infty,\\ \sup_{M>0}\left\|\mu_j-\mu\right\|_{C^2(\hat{\Omega}_M)} &= \sup_{M>0}\sum_{\gamma\in\mathbb{Z}_+^3,\ |\gamma|\leqslant 2}\left\|D^\gamma\mu_j - D^\gamma\mu\right\|_{C(\hat{\Omega}_M)} \to 0,\ \text{as}\ j\to\infty,\\ \left\|\alpha_j-\alpha\right\|_{C^1([0,T^*])} &+ \left\|g_j-g\right\|_{H^1(0,T^*)} \to 0,\ \text{as}\ j\to\infty,\end{aligned} \tag{3.5}$$

then there exists $j_0 \in \mathbb{N}$ (independent of M) such that

$$\begin{cases} \left\|D^\beta f_j\right\|_{C(\Omega_M)} \leqslant 1+\left\|D^\beta f\right\|_{C(\Omega_M)}, \forall \beta \in \mathbb{Z}_+^4, |\beta| \leqslant 1, \forall M>0, \forall j \geqslant j_0, \\ \left\|D^\gamma \mu_j\right\|_{C(\hat{\Omega}_M)} \leqslant 1+\left\|D^\gamma \mu\right\|_{C(\hat{\Omega}_M)}, \forall \gamma \in \mathbb{Z}_+^3, |\gamma| \leqslant 2, \forall M>0, \forall j \geqslant j_0, \\ \left\|\alpha_j\right\|_{C^1([0,T^*])} \leqslant 1+\|\alpha\|_{C^1([0,T^*])}, \forall j \geqslant j_0, \\ \left\|g_j\right\|_{H^1(0,T^*)} \leqslant 1+\|g\|_{H^1(0,T^*)}, \forall j \geqslant j_0. \end{cases} \tag{3.6}$$

By setting the constants $K_M(f)$ and $\hat{K}_M(\mu)$ as in (2.19), and due to (H_2)–(H_5), we deduce from the above estimates that

$$\begin{cases} K_M(f_j) \leqslant D_M(f) \equiv 1+K_M(f)+\sum_{i=1}^{j_0-1} K_M(f_i), \forall M>0, \forall j \in \mathbb{N}, \\ \hat{K}_M(\mu_j) \leqslant \hat{D}_M(\mu) \equiv 1+\hat{K}_M(\mu)+\sum_{i=1}^{j_0-1} \hat{K}_M(\mu_j), \forall M>0, \forall j \in \mathbb{N}, \\ \left\|\alpha_j\right\|_{C^1([0,T^*])} \leqslant D_1(\alpha) \equiv 1+\|\alpha\|_{C^1([0,T^*])}+\sum_{i=1}^{j_0-1}\left\|\alpha_j\right\|_{C^1([0,T^*])}, \forall j \in \mathbb{N}, \\ \left\|g_j\right\|_{H^1(0,T^*)} \leqslant D_2(g) \equiv 1+\|g\|_{H^1(0,T^*)}+\sum_{i=1}^{j_0-1}\left\|g_j\right\|_{H^1(0,T^*)}, \forall j \in \mathbb{N}. \end{cases} \tag{3.7}$$

Therefore, the Faedo-Galerkin approximation sequence $\{u_m^{(k)}\}$ corresponding to $(f,\mu,\alpha,g)=(f_j,\mu_j,\alpha_j,g_j),\ j \in \mathbb{N}$ also satisfies the priori estimates as in Theorem 2.8 and

$$u_m^{(k)} \in \bar{B}(M,T), \quad \text{for all} \quad m \quad \text{and} \quad k \in \mathbb{N}, \tag{3.8}$$

where M and T are independent of j. Indeed, in the process, we can choose two positive constants M and T as in (2.45)–(2.47) with replacing $K_M(f),\ \hat{K}_M(\mu),\ \|\alpha\|_{C^1([0,T^*])}$, $\|g\|_{H^1(0,T^*)}$ by $D_M(f),\ \hat{D}_M(\mu),\ D_1(\alpha),\ D_2(g)$, respectively.

Hence, the limitation u_j of $\{u_m^{(k)}\}$ as $k \to +\infty$ and later $m \to +\infty$ is the unique weak solution of (1.1)–(1.3) corresponding to $(f,\mu,\alpha,g)=(f_j,\mu_j,\alpha_j,g_j),\ j \in \mathbb{N}$, and satisfying

$$u_j \in \bar{B}(M,T), \quad \text{for all} \quad j \in \mathbb{N}. \tag{3.9}$$

Moreover, by using the same arguments as in the proof of Theorem 2.9, we can prove that the limitation u of $\{u_j\}$ as $j \to +\infty$, is the unique weak solution of (1.1)–(1.3) and $u \in \bar{B}(M,T)$.

Put

$$\begin{aligned} \bar{f}_j(x,t) &= f_j[u_j](x,t)-f[u](x,t), \\ \bar{g}_j(t) &= g_j(t)-g(t),\ \bar{\alpha}_j(t)=\alpha_j(t)-\alpha(t), \\ \bar{\mu}_j(x,t) &= \mu_j[u_j](x,t)-\mu[u](x,t), \\ \tilde{\mu}(x,t) &= \mu[u](x,t), \end{aligned} \tag{3.10}$$

then $\bar{u}_j = u_j - u$ satisfies the variational problem

$$\begin{cases} \langle \bar{u}_j'(t), w\rangle + \alpha_j(t)a(\bar{u}_j'(t), w) + \langle \tilde{\mu}(t)\bar{u}_{jx}(t), w_x\rangle \\ \qquad = -\bar{\alpha}_j(t)a(u'(t), w) - \langle \bar{\mu}_j(t)u_{jx}(t), w_x\rangle \\ \qquad\quad + \int_0^t g_j(t-s)a(\bar{u}_j(s), w)\,ds + \int_0^t \bar{g}_j(t-s)a(u(s), w)\,ds \\ \qquad\quad + \langle \bar{f}_j(t), w\rangle,\ \forall w \in H_0^1,\ \ a.e.,\ \ t \in (0,T), \\ \bar{u}_j(x,0) = 0. \end{cases} \tag{3.11}$$

Taking $v = \bar{u}_j'(t)$ in (3.11)$_1$ and then integrating in t, we get

$$\begin{aligned} \bar{\mu}_*\bar{S}_j(t) \leqslant{}& \int_0^t ds \int_1^R x\tilde{\mu}'(x,s)\bar{u}_{jx}^2(x,s)dx - 2g_j(0)\int_0^t \|\bar{u}_{jx}(s)\|_0^2 ds \\ &- 2\bar{g}_j(0)\int_0^t \langle u_x(s), \bar{u}_{jx}(s)\rangle ds + 2\int_0^t g_j(t-s)\langle \bar{u}_{jx}(s), \bar{u}_{jx}(t)\rangle ds \\ &+ 2\int_0^t \bar{g}_j(t-s)\langle u_x(s), \bar{u}_{jx}(t)\rangle ds \\ &- 2\int_0^t dr \int_0^r g_j'(r-s)\langle \bar{u}_{jx}(s), \bar{u}_{jx}(r)\rangle ds \\ &- 2\int_0^t dr \int_0^r \bar{g}_j'(r-s)\langle u_x(s), \bar{u}_{jx}(r)\rangle ds - 2\int_0^t \bar{\alpha}_j(s)\langle u_x'(s), \bar{u}_{jx}'(s)\rangle ds \\ &- 2\int_0^t \langle \bar{\mu}_j(s)u_{jx}(s), \bar{u}_{jx}'(s)\rangle ds + 2\int_0^t \langle \bar{f}_j(s), \bar{u}_j'(s)\rangle ds \ = \Sigma_{j=1}^{10} I_j, \end{aligned} \tag{3.12}$$

where $\bar{\mu}_* = \min\{1, \alpha_*, \mu_*\}$ and

$$\bar{S}_j(t) = \|\bar{u}_{jx}(t)\|_0^2 + \int_0^t \left(\|\bar{u}_j'(s)\|_0^2 + \|\bar{u}_{jx}'(s)\|_0^2\right) ds. \tag{3.13}$$

We will estimate the terms I_j on the right-hand side of (3.12). Because the imbedding $H^1(0,T^*) \hookrightarrow C^0([0,T^*])$ is continuous, hence $\|g\|_{C^0([0,T^*])} \leqslant D_{T^*}\|g\|_{H^1(0,T^*)}$ for all $g \in H^1(0,T^*)$.

By the following inequality

$$|\tilde{\mu}'(x,t)| \leqslant \hat{K}_M(\mu)\left(1 + 2M^2\right),$$

the terms $I_1 - I_8$ on the right-hand side of (3.12) are estimated as follows

$$\begin{aligned} I_1 &= \int_0^t ds \int_1^R x\tilde{\mu}'(x,s)\bar{u}_{jx}^2(x,s)dx \leqslant \hat{K}_M(\mu)\left(1+2M^2\right)\int_0^t \|\bar{u}_{jx}(s)\|_0^2 ds \\ &\leqslant \hat{K}_M(\mu)\left(1+2M^2\right)\int_0^t \bar{S}_j(s)ds; \\ I_2 &= -2g_j(0)\int_0^t \|\bar{u}_{jx}(s)\|_0^2 ds \leqslant 2D_{T^*}\|g_j\|_{H^1(0,T^*)}\int_0^t \bar{S}_j(s)ds \\ &\leqslant 2D_{T^*}D_1(g)\int_0^t \bar{S}_j(s)ds; \end{aligned} \tag{3.14}$$

$$
\begin{aligned}
I_3 &= -2\bar{g}_j(0)\int_0^t \langle u_x(s), \bar{u}_{jx}(s)\rangle ds \\
&\leqslant 2D_{T^*}\left\|\bar{g}_j\right\|_{H^1(0,T^*)}\int_0^t \|u_x(s)\|_0 \sqrt{\bar{S}_j(s)}ds \\
&\leqslant D_{T^*}^2\left\|\bar{g}_j\right\|_{H^1(0,T^*)}^2\left[\int_0^t \|u_x(s)\|_0\, ds\right]^2 + \int_0^t \bar{S}_j(s)ds \\
&\leqslant (T^*M)^2 D_{T^*}^2\left\|\bar{g}_j\right\|_{H^1(0,T^*)}^2 + \int_0^t \bar{S}_j(s)ds; \\
I_4 &= 2\int_0^t g_j(t-s)\langle \bar{u}_{jx}(s), \bar{u}_{jx}(t)\rangle ds \\
&\leqslant 2\int_0^t \left|g_j(t-s)\right|\left\|\bar{u}_{jx}(s)\right\|_0\left\|\bar{u}_{jx}(t)\right\|_0 ds \\
&\leqslant \delta\left\|\bar{u}_{jx}(t)\right\|_0^2 + \frac{1}{\delta}\int_0^t g_j^2(t-s)ds\int_0^t \left\|\bar{u}_{jx}(s)\right\|_0^2 ds \\
&\leqslant \delta\bar{S}_j(t) + \frac{1}{\delta}T^*D_{T^*}^2\left\|g_j\right\|_{H^1(0,T^*)}^2\int_0^t \bar{S}_j(s)ds \\
&\leqslant \delta\bar{S}_j(t) + \frac{1}{\delta}T^*D_{T^*}^2 D_1^2(g)\int_0^t \bar{S}_j(s)ds,\ \forall\delta>0; \\
I_5 &= 2\int_0^t \bar{g}_j(t-s)\langle u_x(s), \bar{u}_{jx}(t)\rangle ds \leqslant 2\int_0^t \bar{g}_j(t-s)\langle u_x(s), \bar{u}_{jx}(t)\rangle ds \\
&\leqslant 2\int_0^t \left|\bar{g}_j(t-s)\right|\|u_x(s)\|_0\left\|\bar{u}_{jx}(t)\right\|_0 ds \\
&\leqslant \delta\left\|\bar{u}_{jx}(t)\right\|_0^2 + \frac{1}{\delta}\int_0^t \bar{g}_j^2(t-s)ds\int_0^t \|u_x(s)\|_0^2 ds \\
&\leqslant \delta\bar{S}_j(t) + \frac{1}{\delta}D_{T^*}^2\left\|\bar{g}_j\right\|_{H^1(0,T^*)}^2\int_0^t \|u_x(s)\|_0^2 ds \\
&\leqslant \delta\bar{S}_j(t) + \frac{1}{\delta}T^*M^2D_{T^*}^2\left\|\bar{g}_j\right\|_{H^1(0,T^*)}^2,\ \forall\delta>0; \\
I_6 &= -2\int_0^t dr\int_0^r g_j'(r-s)\langle \bar{u}_{jx}(s), \bar{u}_{jx}(r)\rangle ds \\
&\leqslant 2\int_0^t dr\int_0^r \left|g_j'(r-s)\right|\left\|\bar{u}_{jx}(s)\right\|_0\left\|\bar{u}_{jx}(r)\right\|_0 ds \\
&\leqslant 2\int_0^t dr\int_0^r \left|g_j'(r-s)\right|\sqrt{\bar{S}_j(s)}\sqrt{\bar{S}_j(r)}ds \\
&\leqslant 2\sqrt{T^*}\left\|g_j'\right\|_{L^2(0,T^*)}\int_0^t \bar{S}_j(s)ds \\
&\leqslant 2\sqrt{T^*}\left\|g_j\right\|_{H^1(0,T^*)}\int_0^t \bar{S}_j(s)ds \leqslant 2\sqrt{T^*}D_1(g)\int_0^t \bar{S}_j(s)ds; \\
I_7 &= -2\int_0^t dr\int_0^r \bar{g}_j'(r-s)\langle u_x(s), \bar{u}_{jx}(r)\rangle ds \\
&\leqslant 2\int_0^t dr\int_0^r \left|\bar{g}_j'(r-s)\right|\|u_x(s)\|_0\left\|\bar{u}_{jx}(r)\right\|_0 ds
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant 2\int_0^t \left\|\bar{u}_{jx}(r)\right\|_0 dr \int_0^r \left|\bar{g}_j'(r-s)\right| \|u_x(s)\|_0\, ds \\
&\leqslant 2\left[\int_0^t \left\|\bar{u}_{jx}(r)\right\|_0^2 dr\right]^{1/2} \left[\int_0^t dr \left(\int_0^r \left|\bar{g}_j'(r-s)\right| \|u_x(s)\|_0\, ds\right)^2\right]^{1/2} \\
&\leqslant 2\left[\int_0^t \left\|\bar{u}_{jx}(r)\right\|_0^2 dr\right]^{1/2} \left[\int_0^t dr \left(\int_0^r \left|\bar{g}_j'(s)\right|^2 ds \int_0^r \|u_x(s)\|_0^2\, ds\right)\right]^{1/2} \\
&\leqslant 2\left[\int_0^t \left\|\bar{u}_{jx}(r)\right\|_0^2 dr\right]^{1/2} \sqrt{T^*}\left\|\bar{g}_j'\right\|_{L^2(0,T^*)} \left[\int_0^T \|u_x(s)\|_0^2\, ds\right]^{1/2} \\
&\leqslant 2\sqrt{T^*}M\left[\int_0^t \bar{S}_j(r)dr\right]^{1/2} \sqrt{T^*}\left\|\bar{g}_j'\right\|_{L^2(0,T^*)} \\
&\leqslant (T^*M)^2\left\|\bar{g}_j\right\|_{H^1(0,T^*)}^2 + \int_0^t \bar{S}_j(r)dr;
\end{aligned}
$$

$$
\begin{aligned}
I_8 &= -2\int_0^t \bar{\alpha}_j(s)\langle u_x'(s), \bar{u}_{jx}'(s)\rangle ds \\
&\leqslant 2\left\|\bar{\alpha}_j\right\|_{C^1([0,T^*])} \int_0^t \left\|u_x'(s)\right\|_0 \left\|\bar{u}_{jx}'(s)\right\|_0 ds \\
&\leqslant \delta\int_0^t \left\|\bar{u}_{jx}'(s)\right\|_0^2 ds + \frac{1}{\delta}\left\|\bar{\alpha}_j\right\|_{C^1([0,T^*])}^2 \int_0^t \left\|u_x'(s)\right\|_0^2 ds \\
&\leqslant \delta\bar{S}_j(t) + \frac{1}{\delta}T^*M^2\left\|\bar{\alpha}_j\right\|_{C^1([0,T^*])}^2,\ \forall\delta > 0.
\end{aligned}
$$

We shall estimate I_9 as follows

$$
\begin{aligned}
I_9 &= -2\int_0^t \langle\bar{\mu}_j(s)u_{jx}(s), \bar{u}_{jx}'(s)\rangle ds \leqslant 2\int_0^t \left\|\bar{\mu}_j(s)u_{jx}(s)\right\|_0 \left\|\bar{u}_{jx}'(s)\right\|_0 ds \qquad (3.15)\\
&\leqslant \delta\int_0^t \left\|\bar{u}_{jx}'(s)\right\|_0^2 ds + \frac{1}{\delta}\int_0^t \left\|\bar{\mu}_j(s)u_{jx}(s)\right\|_0^2 ds \\
&\leqslant \delta\bar{S}_j(t) + \frac{1}{\delta}\int_0^t \left\|\bar{\mu}_j(s)u_{jx}(s)\right\|_0^2 ds.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\left|\bar{\mu}_j(x,t)\right| &= \left|\mu_j[u_j](x,t) - \mu[u](x,t)\right| \\
&\leqslant \left|\mu_j[u_j](x,t) - \mu[u_j](x,t)\right| + \left|\mu[u_j](x,t) - \mu[u](x,t)\right| \\
&\leqslant \left\|\mu_j - \mu\right\|_{C(\hat{\Omega}_M)} + 2M\hat{K}_M(\mu)\left\|\bar{u}_j(t)\right\|_0 \\
&\leqslant \sup_{M>0}\left\|\mu_j - \mu\right\|_{C^2(\hat{\Omega}_M)} + 2M\hat{K}_M(\mu)\frac{\sqrt{2R}(R-1)}{2}\left\|\bar{u}_{jx}(t)\right\|_0 \\
&\leqslant \sup_{M>0}\left\|\mu_j - \mu\right\|_{C^2(\hat{\Omega}_M)} + M\hat{K}_M(\mu)\sqrt{2R}(R-1)\sqrt{\bar{S}_j(t)}.
\end{aligned}
$$

This leads to

$$\begin{aligned}
&\int_0^t \left\|\bar{\mu}_j(s)u_{jx}(s)\right\|_0^2 ds \\
&\leqslant \int_0^t \left(\sup_{M>0}\left\|\mu_j-\mu\right\|_{C^2(\hat{\Omega}_M)} + M\hat{K}_M(\mu)\sqrt{2R}(R-1)\sqrt{\bar{S}_j(s)}\right)^2 \left\|u_{jx}(s)\right\|_0^2 ds \\
&\leqslant 2M^2\int_0^t \left[\left(\sup_{M>0}\left\|\mu_j-\mu\right\|_{C^2(\hat{\Omega}_M)}\right)^2 + 2M^2\hat{K}_M^2(\mu)R(R-1)^2\bar{S}_j(s)\right] ds \\
&\leqslant 2M^2T^*\left(\sup_{M>0}\left\|\mu_j-\mu\right\|_{C^2(\hat{\Omega}_M)}\right)^2 + 4M^4\hat{K}_M^2(\mu)R(R-1)^2\int_0^t \bar{S}_j(s)ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
I_9 &= -2\int_0^t \langle\bar{\mu}_j(s)u_{jx}(s), \bar{u}'_{jx}(s)\rangle ds \leqslant \delta\bar{S}_j(t) + \frac{1}{\delta}\int_0^t \left\|\bar{\mu}_j(s)u_{jx}(s)\right\|_0^2 ds \qquad (3.16)\\
&\leqslant \delta\bar{S}_j(t) + \frac{1}{\delta}\left[2M^2T^*\left(\sup_{M>0}\left\|\mu_j-\mu\right\|_{C^2(\hat{\Omega}_M)}\right)^2 + 4M^4\hat{K}_M^2(\mu)R(R-1)^2\int_0^t \bar{S}_j(s)ds\right] \\
&= \delta\bar{S}_j(t) + \frac{2}{\delta}M^2T^*\left(\sup_{M>0}\left\|\mu_j-\mu\right\|_{C^2(\hat{\Omega}_M)}\right)^2 + \frac{4}{\delta}M^4\hat{K}_M^2(\mu)R(R-1)^2\int_0^t \bar{S}_j(s)ds.
\end{aligned}$$

Note that

$$\begin{aligned}
I_{10} &= 2\int_0^t \langle\bar{f}_j(s), \bar{u}'_j(s)\rangle ds \leqslant 2\int_0^t \left\|\bar{f}_j(s)\right\|_0 \left\|\bar{u}'_j(s)\right\|_0 ds \\
&\leqslant \delta\bar{S}_j(t) + \frac{1}{\delta}\int_0^t \left\|\bar{f}_j(s)\right\|_0^2 ds.
\end{aligned}$$

On the other hand, the term $\left\|\bar{f}_j(s)\right\|_0$ can be estimated as follows

$$\begin{aligned}
\left|\bar{f}_j(x,t)\right| &= \left|f_j[u_j](x,t) - f[u](x,t)\right| \\
&\leqslant \left|f_j[u_j](x,t) - f[u_j](x,t)\right| + \left|f[u_j](x,t) - f[u](x,t)\right| \\
&\leqslant \left\|f_j - f\right\|_{C(\Omega_M)} + K_M(f)\left(1 + \frac{\sqrt{2R}(R-1)}{2}\right)\left\|\bar{u}_{jx}(t)\right\|_0 \\
&\leqslant \left\|f_j - f\right\|_{C(\Omega_M)} + \tilde{R}K_M(f)\sqrt{\bar{S}_j(t)}.
\end{aligned}$$

Hence

$$\begin{aligned}
I_{10} &= 2\int_0^t \langle\bar{f}_j(s), \bar{u}'_j(s)\rangle ds \leqslant \delta\bar{S}_j(t) + \frac{1}{\delta}\int_0^t \left\|\bar{f}_j(s)\right\|_0^2 ds \qquad (3.17)\\
&\leqslant \delta\bar{S}_j(t) + \frac{1}{\delta}\int_0^t \left[\left\|f_j - f\right\|_{C(\Omega_M)} + \tilde{R}K_M(f)\sqrt{\bar{S}_j(s)}\right]^2 ds
\end{aligned}$$

$$\leqslant \delta \bar{S}_j(t) + \frac{2}{\delta}\int_0^t \left[\|f_j - f\|^2_{C(\Omega_M)} + \tilde{R}^2 K_M^2(f)\bar{S}_j(s)\right] ds$$
$$\leqslant \delta \bar{S}_j(t) + \frac{2}{\delta}T^* \|f_j - f\|^2_{C(\Omega_M)} + \frac{2}{\delta}\tilde{R}^2 K_M^2(f)\int_0^t \bar{S}_j(s)ds, \ \forall \delta > 0.$$

Finally, by choosing $\delta = \frac{\bar{\mu}_*}{10}$, it follows from (3.12), (3.14), (3.16), and (3.17) that

$$\bar{S}_j(t) \leqslant D_1(M)E_j^2 + 2D_2(M)\int_0^t \bar{S}_j(s)ds, \tag{3.18}$$

where

$$D_1(M) = \frac{2T^*}{\bar{\mu}_*}\left[\left(1 + D_{T^*}^2\right)T^* M^2 + \frac{10}{\bar{\mu}_*}\left(2 + 3M^2 + M^2 D_{T^*}^2\right)\right], \tag{3.19}$$
$$D_2(M) = \frac{1}{\bar{\mu}_*}\left[2 + \left(1 + 2M^2\right)\hat{K}_M(\mu) + 2\left(D_{T^*} + \sqrt{T^*}\right)D_1(g)\right]$$
$$+ \frac{10}{\bar{\mu}_*^2}\left[T^* D_{T^*}^2 D_1^2(g) + 4R(R-1)^2 M^4 \hat{K}_M^2(\mu) + 2\tilde{R}^2 K_M^2(f)\right].$$

Using Gronwall's lemma, we have

$$\bar{S}_j(t) \leqslant D_1(M)E_j^2 \exp(2TD_2(M)). \tag{3.20}$$

This derive that

$$\|u_j - u\|_{W_1(T)} = \|\bar{u}_j\|_{W_1(T)} \leqslant 2\sqrt{D_1(M)}\exp(TD_2(M))E_j \tag{3.21}$$
$$\equiv C_T E_j, \ \forall j \in \mathbb{N},$$

where

$$C_T = 2\sqrt{D_1(M)}\exp(TD_2(M)).$$

Theorem 3.1 is proved. □

4. Conclusion

The paper has examined an initial boundary value problem for the viscoelastic pseudo-parabolic equation of Carrier type. Utilizing the linear approximation, the Faedo-Galerkin method, and arguments of compactness, we have derived the existence and uniqueness of solutions for the problem. In addition, under several appropriate assumptions, we proved that the solution continuously depends on the relaxation functions g, α, and the nonlinear components μ in the problem. Also, it appears that there exist several difficulties in establishing suitable conditions for finding results such as existence, uniqueness, long-time behavior, and blow-up to the boundary value problem (1.1)–(1.2) associated with the nonlocal initial condition in time as follows

$$u(x,0) = \tilde{u}_0(x) + \sum_{i=1}^N \beta_i u(x,T_i) + \int_0^T h(x,t)\Phi(u(x,t))\,dt,$$

where β_i, T_i are constants, with $0 < T_1 < T_2 < \cdots < T_N < T$, and h, Φ are given functions. Therefore, these obstacles are still open problems.

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