Differential
Equations
& Applications
Volume 17, Number 3 (2025), 133–162
doi:10.7153/dea-2025-17-09

EIGENVALUE PROBLEMS FOR A COUPLED SYSTEM OF SINGULAR KATUGAMPOLA FRACTIONAL DIFFERENTIAL EQUATIONS WITH FOUR–POINT BOUNDARY CONDITIONS

WENGUI YANG

(*Communicated by S. K. Ntouyas*)

Abstract. This paper investigates the nonlinear eigenvalue problems (NEPs) for a class of singular Katugampola fractional differential systems with four-point coupled boundary conditions. Firstly, we establish the Green's functions of the aforementioned NEPs and their fundamental analytic properties. Secondly, utilizing some classical fixed-point theorems, some explicit eigenvalue interval-dependent criteria are derived for the existence of at least one positive solution to the addressed NEPs. As applications, some examples are presented to illustrate the feasibility and effectiveness of our main results.

1. Introduction

We consider the nonlinear eigenvalue problem (NEP) for singular Katugampola fractional differential system with four-point coupled boundary conditions as follows

$$\begin{cases} \mathscr{D}_{a+}^{\alpha,\rho}\phi(t)+\lambda\mathscr{F}(t,\phi(t),\psi(t))=0, \quad t\in(a,b), \ \ \lambda>0, \\ \mathscr{D}_{a+}^{\beta,\rho}\psi(t)+\lambda\mathscr{G}(t,\phi(t),\psi(t))=0, \quad t\in(a,b), \ \ \lambda>0, \\ \gamma^{i}\phi(a)=\gamma^{j}\psi(a)=0, \ \ \phi(b)=p\psi(\xi), \ \psi(b)=q\phi(\eta), \ \xi,\eta\in(a,b), \end{cases} \tag{1}$$

where $0\leqslant i\leqslant m-2$, $0\leqslant j\leqslant n-2$, $\mathscr{D}_{a+}^{\alpha,\rho}$ and $\mathscr{D}_{a+}^{\beta,\rho}$ stand for the Katugampola fractional derivatives of orders $\alpha\in(m-1,m]$ and $\beta\in(n-1,n]$ for some positive integers $m,n\geqslant 3$, respectively; $\gamma=t^{1-\rho}d/dt$, $\rho>0$; λ,p,q are three parameters satisfying $0<pq(\eta^{\rho}-a^{\rho})^{\alpha-1}(\xi^{\rho}-a^{\rho})^{\beta-1}<(b^{\rho}-a^{\rho})^{\alpha+\beta-2}$; the nonlinear functions f and g are continuous and may be singular at $t=a$ and $t=b$, respectively.

Fractional differential equations (FDEs) break through the locality limitations of traditional integer order models by introducing non-integer order derivatives such as Riemann-Liouville derivatives and Caputo derivatives, and can accurately characterize

Mathematics subject classification (2020): 34A08, 34B16, 34B18.

Keywords and phrases: Katugampola fractional differential equations, eigenvalue problems, four-point boundary conditions, Green's functions, Krasnoselskii's fixed-point theorem.

This research is supported by the Scientific and Technological Breakthrough Project of Henan Province under Grant No. 252102210086 and the High-level Talent Fund Project of Sanmenxia Polytechnic under Grant No. SZYGCCRC-2021-009.

Paper DEA-17-09

complex system dynamics with memory, heritability, and non local effects. Since Leibniz and L'Hôpital first explored the mathematical significance of fractional derivatives in 1695, the theory of fractional calculus has gradually improved. In the waning years of the 20th century, their theoretical framework has rapidly evolved through interdisciplinary integration in mathematics, physics, and engineering: in neuroscience, fractional models precisely capture the long-range memory characteristics of neuronal electrical signals; in environmental science, they simulate anomalous diffusion behaviors of pollutants in heterogeneous media. This transformation from a "mathematical tool" to a "cross-disciplinary universal model" marks a paradigm shift in differential equation research. In the past decades, there have been various types of fractional calculus such as Hadamard, conformable, proportional, generalized Riemann-Liouville, generalized Hadamard, generalized conformable, generalized proportional-type fractional integrals and fractional derivatives, etc. In 2011, Katugampola [11] introduced a kind of new generalized Riemann-Liouville fractional integral and fractional derivative, which can be called as Katugampola fractional calculus. Later, Katugampola FDEs have attracted a lot of attention from scholars at home and abroad.

Boundary value problems (BVPs) involving nonlinear FDEs play a pivotal role across a diverse spectrum of scientific and engineering domains, spanning disciplines such as physics, continuum mechanics, chemical kinetics, and multiple branches of engineering. For the recent development of fractional BVPs, the reader may consult the works [3, 7, 17, 19, 22, 23, 24, 29] and the references incited therein. Using Green's function properties and the generalized Avery-Henderson fixed-point theorem, Zhang et al. [28] proved the existence of three positive solutions for a class of p-Laplacian FDEs. By utilizing the properties of the Green's function and applying the Guo-Krasnoselskii fixed-point theorem, the existence of positive solutions were obtained for a singular system of nonlinear fractional q-difference equations featuring coupled integral boundary conditions and two parameters [26]. By using some classical fixed-point theorems, Ahmad and Ntouyas systematically analyzed the theoretical properties of coupled fractional differential systems under multiple nonlocal boundary conditions in their seminal work [1]. Using the cone-based fixed point theory and approximation techniques, Jiang et al. [10] established the existence of a positive solution for a singular nonlinear semipositone fractional differential system with coupled boundary conditions. By combining the Leggett-Williams fixed-point theorem and fixed point index theory, Rao [18] studied the existence of multiple positive solutions for p-Laplacian FDEs. Using Krasnoselskii's and Leggett-Williams' fixed-point theorems, Djourdem and Benaicha [5] established single and triple positive solutions for nonlinear FDEs with multi-term integral and multi-point boundary conditions. By using the comparison principle combined with the method of upper and lower methods and fixed-point theorems, the author [25, 27] considered the monotone iterative solutions and positive solutions for a system of nonlinear Hadamard FDEs, respectively. Besides, employing the monotone iterative technique and Banach contraction principle, Li et al. [13] derived some explicit schemes for approximating extreme/unique positive solutions of fractional differential systems with integral boundary conditions on infinite intervals. Combining the upper and lower solutions method with Schauder fixed-point theorem, Azzaoui et al. [4] considered the positive solutions of nonlinear Riemann-Liouville fractional BVPs in a

Sobolev spaces.

On the other hand, more and more researchers have shown growing interest in the systems of FDEs. For example, uisng the Leray-Schauder alternative and Krasnoselskii's fixed-point theorem, Subramanian et al. [21] studied existence and Hyers-Ulam stability of solutions for a coupled system of generalized Liouville-Caputo FDEs with Katugampola integral boundary conditions. By means of the properties of the Green's function, fractional counterpart of the Lyapunov inequality was obtained for higher order Katugampola FDEs [12]. Based on the classical fixed-point theorems, existence and uniqueness of solutions for nonlinear Katugampola fractional BVPs were investigated [2]. By using a Lyapunov-type inequality, Łupińska and Schmeidel [16] derived criteria for the existence and non-existence of solutions to FDEs subject to fractional boundary conditions involving the Katugampola derivative. Using coincidence degree theory, Srivastava et al. [20] obtained the existence of solutions for Katugampola FDEs with Riemann-Stieltjes integral boundary conditions at resonance. By using a combined approach of the Guo-Krasnoselskii and Leggett-Williams fixed-point theorems, Luca and Tudorache [14] considered positive solutions for a system of nonlinear Hadamard FDEs involving nonlocal coupled Riemann-Stieltjes integral boundary conditions on an infinite interval. By means of Schauder fixed-point theorem, Henderson et al. [8] investigated the existence/nonexistence of positive solutions for a system of ρ-Laplacian Riemann-Liouville FDEs with coupled Riemann-Stieltjes integral nonlocal boundary conditions and positive parameters.

Nevertheless, to our knowledge, limited literature exists on the existence of positive solutions for nonlinear BVPs involving coupled systems of Katugampola FDEs. Motivated by prior advancements in nonlinear fractional BVPs, we will investigate the existence of positive solutions for NEP (1) under specified boundary conditions. In Section 2, we will leverage foundational lemmas and preliminary results to construct the associated Green's function and analyze its critical properties. Section 3 will establish and rigorously prove the main existence theorems using Krasnoselskii's fixed-point theorem. To validate the theoretical findings, Section 4 will provide illustrative examples demonstrating the applicability of our results.

2. Preliminaries

To facilitate subsequent analysis of NEP (1), we begin by reviewing the fundamental principles from Katugampola fractional calculus theory. For more comprehensive overviews, the reader refer to see [11, 12, 15].

DEFINITION 1. ([11, 15]) Let $\alpha > 0$, $\rho > 0$, $-\infty < a < b \leqslant \infty$, $p \geqslant 1$, and $\phi \in L^p(a,b)$. The left-sided and right-sided Katugampola fractional integrals of order α are respectively defined as follows

$$\mathscr{I}_{a+}^{\alpha,\rho}\phi(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^\rho - s^\rho)^{\alpha-1} \phi(s) \frac{ds}{s^{1-\rho}},$$

$$\mathscr{I}_{b-}^{\alpha,\rho}\phi(t)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_t^b (s^\rho-t^\rho)^{\alpha-1}\phi(s)\frac{ds}{s^{1-\rho}} \text{ for } t\in(a,b).$$

DEFINITION 2. ([12, 15]) Let $\alpha>0$, $\rho>0$, $n=[\alpha]+1$, $0<a<b\leqslant\infty$, $p\geqslant 1$, and $\phi\in L^p(a,b)$. The left-sided and right-sided Katugampola fractional derivatives of order α are respectively defined as follows

$$\mathscr{D}_{a+}^{\alpha,\rho}\phi(t)=\gamma^n\mathscr{I}_{a+}^{n-\alpha,\rho}\phi(t)=\frac{\gamma^n}{\Gamma(n-\alpha)}\int_a^t\Big(\frac{t^\rho-s^\rho}{\rho}\Big)^{n-\alpha-1}\phi(s)\frac{ds}{s^{1-\rho}},\quad \gamma=t^{1-\rho}\frac{d}{dt},$$

$$\mathscr{D}_{b-}^{\alpha,\rho}\phi(t)=(-\gamma)^n\mathscr{I}_{b-}^{n-\alpha,\rho}\phi(t)=\frac{(-\gamma)^n}{\Gamma(n-\alpha)}\int_t^b\Big(\frac{s^\rho-t^\rho}{\rho}\Big)^{n-\alpha-1}\phi(s)\frac{ds}{s^{1-\rho}} \text{ for } t\in(a,b).$$

REMARK 1. Let $\rho\to 1$ and $\rho\to 0$, the Katugampola fractional integrals and derivatives reduce to the Riemann-Liouville and Hadamard fractional integrals and derivatives, respectively.

Let $\mathbb{AC}[a,b]$ stand for the collection of all absolutely continuous functions on the interval $[a,b]$. Furthermore, we define $\mathbb{AC}_\gamma^n[a,b]=\{\phi:[a,b]\to\mathbb{R}$ and $\gamma^{n-1}\phi\in\mathbb{AC}[a,b],\gamma=t^{1-p}d/dt\}$, $\mathbb{AC}_\gamma^1[a,b]=\mathbb{AC}[a,b]$.

LEMMA 1. ([9]) *Let $\alpha>0$, $n=[\alpha]+1$, $\phi\in L(a,b)$, $\mathscr{I}_{a+}^{\alpha,\rho}\phi\in\mathbb{AC}_\gamma^n[a,b]$ and $\mathscr{I}_{b-}^{\alpha,\rho}\phi\in\mathbb{AC}_\gamma^n[a,b]$. Then*

$$\mathscr{I}_{a+}^{\alpha,\rho}\mathscr{D}_{a+}^{\alpha,\rho}\phi(t)=\phi(t)+\sum_{j=1}^n\hat{c}_j\Big(\frac{t^\rho-a^\rho}{\rho}\Big)^{\alpha-j},$$

$$\mathscr{I}_{b-}^{\alpha,\rho}\mathscr{D}_{b-}^{\alpha,\rho}\phi(t)=\phi(t)+\sum_{j=1}^n\check{c}_j\Big(\frac{b^\rho-t^\rho}{\rho}\Big)^{\alpha-j},$$

where $\hat{c}_j=-\mathscr{D}_{a+}^{\alpha-j,\rho}\phi(a)/\Gamma(\alpha-j+1)$ and $\check{c}_j=(-1)^{j+1}\mathscr{D}_{b-}^{\alpha-j,\rho}\phi(b)/\Gamma(\alpha-j+1)$.

LEMMA 2. ([9, 15]) *Let $\alpha>0$, $a>0$, $\rho>0$, and $\kappa>\alpha-1$. Then*

$$\mathscr{D}_{a+}^{\alpha,\rho}\Big(\frac{t^\rho-a^\rho}{\rho}\Big)^\kappa=\frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\alpha+1)}\Big(\frac{t^\rho-a^\rho}{\rho}\Big)^{\kappa-\alpha}\quad\text{and}\quad\mathscr{D}_{b-}^{\alpha,\rho}\Big(\frac{t^\rho-a^\rho}{\rho}\Big)^{\alpha-i}=0,$$

where $i=1,2,\ldots,[\alpha]+1$.

LEMMA 3. *Let $0<a<b<\infty$, $\bar{h}>2$ and $\varphi:[a,b]\to\mathbb{R}$ be a continuous function. Then the unique solution of the BVP $\mathscr{D}_{a+}^{\bar{h},\rho}\phi(t)+\varphi(t)=0$, $\gamma^i\phi(a)=\phi(b)=0$, $0\leqslant i\leqslant[\bar{h}]-1$, is $\phi(t)=\int_a^b G_{\bar{h}}(t,s)\varphi(s)\frac{ds}{s^{1-\rho}}$, where*

$$G_{\bar{h}}(t,s)=\frac{\rho^{1-\bar{h}}}{\Gamma(\bar{h})}\begin{cases}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\bar{h}-1}(b^\rho-s^\rho)^{\bar{h}-1}-(t^\rho-s^\rho)^{\bar{h}-1}, & s\leqslant t,\\ \Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\bar{h}-1}(b^\rho-s^\rho)^{\bar{h}-1}, & t\leqslant s,\end{cases}\tag{2}$$

$$=\frac{(b^\rho-a^\rho)^{\overline{h}-1}}{\rho^{\overline{h}-1}\Gamma(\overline{h})}\begin{cases}\left(\dfrac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\left(\dfrac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}-\left(\dfrac{t^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}, & s\leqslant t,\\ \left(\dfrac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\left(\dfrac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}, & t\leqslant s.\end{cases}$$

Proof. Following Lemmas 1 and 2 and the proof of [15, Theorem 3], we can easily obtain Lemma 3. □

LEMMA 4. *Let $0<a<b<\infty$, the Green's function $G_{\overline{h}}(t,s)$ defined by (2) has the following properties:*

(I) *$G_{\overline{h}}(t,s)$ is continuous function on $(t,s)\in[a,b]^2$ and $G_{\overline{h}}(t,s)>0$ for $t,s\in(a,b)$;*

(II) *$\rho_{\overline{h}}(t)\sigma_{\overline{h}}(s)\leqslant\frac{\rho^{\overline{h}-1}\Gamma(\overline{h})}{(b^\rho-a^\rho)^{\overline{h}-1}}G_{\overline{h}}(t,s)\leqslant(\overline{h}-1)\sigma_{\overline{h}}(s)$ and $\rho_{\overline{h}}(t)\sigma_{\overline{h}}(s)\leqslant\frac{\rho^{\overline{h}-1}\Gamma(\overline{h})}{(b^\rho-a^\rho)^{\overline{h}-1}}G_{\overline{h}}(t,s)\leqslant(\overline{h}-1)\rho_{\overline{h}}(t)$ for $t,s\in[a,b]$, where $\rho_{\overline{h}}(t)=(\frac{t^\rho-a^\rho}{b^\rho-a^\rho})^{\overline{h}-1}(\frac{b^\rho-t^\rho}{b^\rho-a^\rho})$ and $\sigma_{\overline{h}}(t)=(\frac{b^\rho-t^\rho}{b^\rho-a^\rho})^{\overline{h}-1}(\frac{t^\rho-a^\rho}{b^\rho-a^\rho})$ for $\overline{h}>2$ and $t\in[a,b]$.*

Proof. Based on the properties of power functions, we can easily derive property (I).

For $a\leqslant s\leqslant t\leqslant b$ and $\overline{h}>2$, then $y=x^{\overline{h}-2}$ is monotonically increasing on $(0,\infty)$. Furthermore, we acquire

$$\begin{aligned}\frac{\rho^{\overline{h}-1}\Gamma(\overline{h})}{(b^\rho-a^\rho)^{\overline{h}-1}}G_{\overline{h}}(t,s)=&\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}-\left(\frac{t^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\\ =&(\overline{h}-1)\int_{\frac{t^\rho-s^\rho}{b^\rho-a^\rho}}^{(\frac{t^\rho-a^\rho}{b^\rho-a^\rho})(\frac{b^\rho-s^\rho}{b^\rho-a^\rho})}x^{\overline{h}-2}dx\\ \leqslant&(\overline{h}-1)\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-2}\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\frac{b^\rho-s^\rho}{b^\rho-a^\rho}-\frac{t^\rho-s^\rho}{b^\rho-a^\rho}\right)\\ =&(\overline{h}-1)\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-2}\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-2}\left(\frac{s^\rho-a^\rho}{b^\rho-a^\rho}\right)\left(\frac{b^\rho-t^\rho}{b^\rho-a^\rho}\right)\\ \leqslant&(\overline{h}-1)\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-2}\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)\left(\frac{s^\rho-a^\rho}{b^\rho-a^\rho}\right)=(\overline{h}-1)\sigma_{\overline{h}}(s),\end{aligned}$$

and

$$\begin{aligned}\frac{\rho^{\overline{h}-1}\Gamma(\overline{h})}{(b^\rho-a^\rho)^{\overline{h}-1}}G_{\overline{h}}(t,s)=&\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}-\left(\frac{t^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\\ =&\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}-\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)\left(\frac{s^\rho-a^\rho}{b^\rho-a^\rho}\right)\right)^{\overline{h}-1}\\ &-\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}-\frac{s^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-2}\left(\frac{t^\rho-s^\rho}{b^\rho-a^\rho}\right)\\ \geqslant&\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}-\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)\left(\frac{s^\rho-a^\rho}{b^\rho-a^\rho}\right)\right)^{\overline{h}-1}\\ &-\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}-\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)\left(\frac{s^\rho-a^\rho}{b^\rho-a^\rho}\right)\right)^{\overline{h}-2}\left(\frac{t^\rho-s^\rho}{b^\rho-a^\rho}\right)\end{aligned}$$

$$
\begin{aligned}
&=\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}-\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)\left(\frac{s^\rho-a^\rho}{b^\rho-a^\rho}\right)\right)^{\overline{h}-2}\\
&\quad\cdot\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}-\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)\left(\frac{s^\rho-a^\rho}{b^\rho-a^\rho}\right)-\frac{t^\rho-s^\rho}{b^\rho-a^\rho}\right)\\
&=\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-2}\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-2}\left(\frac{s^\rho-a^\rho}{b^\rho-a^\rho}\right)\left(\frac{b^\rho-t^\rho}{b^\rho-a^\rho}\right)\\
&\geqslant\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\left(\frac{b^\rho-t^\rho}{b^\rho-a^\rho}\right)\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\left(\frac{s^\rho-a^\rho}{b^\rho-a^\rho}\right)\\
&=\rho_{\overline{h}}(t)\sigma_{\overline{h}}(s).
\end{aligned}
$$

For $a\leqslant t\leqslant s\leqslant b$, then we obtain

$$
\begin{aligned}
\frac{\rho^{\overline{h}-1}\Gamma(\overline{h})}{(b^\rho-a^\rho)^{\overline{h}-1}}G_{\overline{h}}(t,s)&=\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\\
&=\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-2}\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)\\
&\leqslant(\overline{h}-1)\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)\\
&\leqslant(\overline{h}-1)\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\left(\frac{s^\rho-a^\rho}{b^\rho-a^\rho}\right)=(\overline{h}-1)\sigma_{\overline{h}}(s),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\rho^{\overline{h}-1}\Gamma(\overline{h})}{(b^\rho-a^\rho)^{\overline{h}-1}}G_{\overline{h}}(t,s)&=\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\\
&\geqslant\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\left(\frac{b^\rho-t^\rho}{b^\rho-a^\rho}\right)\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\left(\frac{s^\rho-a^\rho}{b^\rho-a^\rho}\right)\\
&=\rho_{\overline{h}}(t)\sigma_{\overline{h}}(s).
\end{aligned}
$$

On the other hand, for $a\leqslant s\leqslant t\leqslant b$, then we gain

$$
\begin{aligned}
\frac{\rho^{\overline{h}-1}\Gamma(\overline{h})}{(b^\rho-a^\rho)^{\overline{h}-1}}G_{\overline{h}}(t,s)&\leqslant(\overline{h}-1)\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-2}\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-2}\left(\frac{s^\rho-a^\rho}{b^\rho-a^\rho}\right)\left(\frac{b^\rho-t^\rho}{b^\rho-a^\rho}\right)\\
&\leqslant(\overline{h}-1)\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-2}\left(\frac{b^\rho-t^\rho}{b^\rho-a^\rho}\right)\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)=(\overline{h}-1)\rho_{\overline{h}}(t).
\end{aligned}
$$

For $a\leqslant t\leqslant s\leqslant b$, then we claim

$$
\begin{aligned}
\frac{\rho^{\overline{h}-1}\Gamma(\overline{h})}{(b^\rho-a^\rho)^{\overline{h}-1}}G_{\overline{h}}(t,s)&=\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\\
&=\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-2}\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)\\
&\leqslant(\overline{h}-1)\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)\\
&\leqslant(\overline{h}-1)\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\overline{h}-1}\left(\frac{b^\rho-t^\rho}{b^\rho-a^\rho}\right)=(\overline{h}-1)\rho_{\overline{h}}(t).
\end{aligned}
$$

This completes the proof of the lemma. □

Now, we will investigate the corresponding Green's function for NEP (1) and its fundamental properties.

LEMMA 5. *Let $\varphi, \varkappa : [a,b] \to \mathbb{R}$ be two continuous functions. Then the BVP*

$$\begin{cases} \mathscr{D}_{a+}^{\alpha,\rho}\phi(t) + \varphi(t) = 0, \;\; \mathscr{D}_{a+}^{\beta,\rho}\psi(t) + \varkappa(t) = 0, \quad t \in (a,b), \\ \gamma^j\phi(a) = \gamma^j\psi(a) = 0, \;\; \phi(b) = p\psi(\xi), \; \psi(b) = q\phi(\eta), \; \xi,\eta \in (a,b), \end{cases} \tag{3}$$

where $0 \leqslant i \leqslant m-2$, $0 \leqslant j \leqslant n-2$, $\alpha \in (m-1,m]$, $\beta \in (n-1,n]$, $m,n \geqslant 3$, has the following integral form of the solution

$$\begin{cases} \phi(t) = \displaystyle\int_a^b K_1(t,s)\varphi(s)\frac{ds}{s^{1-\rho}} + \int_a^b H_1(t,s)\varkappa(s)\frac{ds}{s^{1-\rho}}, \\ \psi(t) = \displaystyle\int_a^b K_2(t,s)\varkappa(s)\frac{ds}{s^{1-\rho}} + \int_a^b H_2(t,s)\varphi(s)\frac{ds}{s^{1-\rho}}, \end{cases} \tag{4}$$

where

$$\begin{aligned} K_1(t,s) =& G_\alpha(t,s) + \frac{pq(t^\rho - a^\rho)^{\alpha-1}}{C(\alpha,\beta)(\xi^\rho - a^\rho)^{1-\beta}} G_\alpha(\eta,s), \\ H_1(t,s) =& \frac{p(t^\rho - a^\rho)^{\alpha-1}}{C(\alpha,\beta)(b^\rho - a^\rho)^{1-\beta}} G_\beta(\xi,s), \\ K_2(t,s) =& G_\beta(t,s) + \frac{pq(t^\rho - a^\rho)^{\beta-1}}{C(\alpha,\beta)(\eta^\rho - a^\rho)^{1-\alpha}} G_\beta(\xi,s), \\ H_2(t,s) =& \frac{q(t^\rho - a^\rho)^{\beta-1}}{C(\alpha,\beta)(b^\rho - a^\rho)^{1-\alpha}} G_\alpha(\eta,s), \\ C(\alpha,\beta) =& (b^\rho - a^\rho)^{\alpha+\beta-2} - pq(\eta^\rho - a^\rho)^{\alpha-1}(\xi^\rho - a^\rho)^{\beta-1}. \end{aligned} \tag{5}$$

Proof. It follows from Lemma 1 that the solution of Katugampola fractional differential system (3) can be expressed as the subsequent equivalent integral equations

$$\begin{cases} \phi(t) = c_{11}\Big(\dfrac{t^\rho - a^\rho}{\rho}\Big)^{\alpha-1} + \cdots + c_{1m}\Big(\dfrac{t^\rho - a^\rho}{\rho}\Big)^{\alpha-m} - \dfrac{\rho^{1-\alpha}}{\Gamma(\alpha)}\displaystyle\int_a^t (t^\rho - s^\rho)^{\alpha-1}\varphi(s)\frac{ds}{s^{1-\rho}}, \\ \psi(t) = c_{21}\Big(\dfrac{t^\rho - a^\rho}{\rho}\Big)^{\beta-1} + \cdots + c_{2n}\Big(\dfrac{t^\rho - a^\rho}{\rho}\Big)^{\beta-n} - \dfrac{\rho^{1-\beta}}{\Gamma(\beta)}\displaystyle\int_a^t (t^\rho - s^\rho)^{\beta-1}\varkappa(s)\frac{ds}{s^{1-\rho}}. \end{cases} \tag{6}$$

Using Lemma 2 and $\gamma^j\phi(a) = \gamma^j\psi(a) = 0$, $0 \leqslant i \leqslant m-2$, $0 \leqslant j \leqslant n-2$, we observe $c_{1m} = c_{1(m-1)} = \cdots = c_{12} = 0$ and $c_{2n} = c_{2(n-1)} = \cdots = c_{22} = 0$. Thus, system (6)

degenerates into the following forms

$$\begin{cases} \phi(t) = c_{11}\Big(\dfrac{t^\rho - a^\rho}{\rho}\Big)^{\alpha-1} - \dfrac{\rho^{1-\alpha}}{\Gamma(\alpha)}\displaystyle\int_a^t (t^\rho - s^\rho)^{\alpha-1}\varphi(s)\frac{ds}{s^{1-\rho}}, \\ \psi(t) = c_{21}\Big(\dfrac{t^\rho - a^\rho}{\rho}\Big)^{\beta-1} - \dfrac{\rho^{1-\beta}}{\Gamma(\beta)}\displaystyle\int_a^t (t^\rho - s^\rho)^{\beta-1}\varkappa(s)\frac{ds}{s^{1-\rho}}. \end{cases} \tag{7}$$

Based on the boundary conditions $\phi(b) = p\psi(\xi)$ and $\psi(b) = q\phi(\eta)$, from (7), we derive

$$\begin{cases} c_{11}\Big(\dfrac{b^\rho - a^\rho}{\rho}\Big)^{\alpha-1} - p\Big(\dfrac{\xi^\rho - a^\rho}{\rho}\Big)^{\beta-1} c_{21} = \dfrac{\rho^{1-\alpha}}{\Gamma(\alpha)}\displaystyle\int_a^b (b^\rho - s^\rho)^{\alpha-1}\varphi(s)\frac{ds}{s^{1-\rho}} \\ \qquad\qquad - \dfrac{p\rho^{1-\beta}}{\Gamma(\beta)}\displaystyle\int_a^\xi (\xi^\rho - s^\rho)^{\beta-1}\varkappa(s)\frac{ds}{s^{1-\rho}}, \\ c_{21}\Big(\dfrac{b^\rho - a^\rho}{\rho}\Big)^{\beta-1} - q\Big(\dfrac{\eta^\rho - a^\rho}{\rho}\Big)^{\alpha-1} c_{11} = \dfrac{\rho^{1-\beta}}{\Gamma(\beta)}\displaystyle\int_a^b (b^\rho - s^\rho)^{\beta-1}\varkappa(s)\frac{ds}{s^{1-\rho}} \\ \qquad\qquad - \dfrac{q\rho^{1-\alpha}}{\Gamma(\alpha)}\displaystyle\int_a^\eta (\eta^\rho - s^\rho)^{\alpha-1}\varphi(s)\frac{ds}{s^{1-\rho}}. \end{cases} \tag{8}$$

Solving for both c_{11} and c_{21} in system of equations (8), we achieve

$$\begin{aligned} c_{11} =& \frac{\rho^{\alpha-1}(b^\rho - a^\rho)^{\beta-1}}{C(\alpha,\beta)}\Bigg(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b (b^\rho - s^\rho)^{\alpha-1}\varphi(s)\frac{ds}{s^{1-\rho}} \\ & - \frac{p\rho^{1-\beta}}{\Gamma(\beta)}\int_a^\xi (\xi^\rho - s^\rho)^{\beta-1}\varkappa(s)\frac{ds}{s^{1-\rho}}\Bigg) + \frac{p\rho^{\alpha-1}(\xi^\rho - a^\rho)^{\beta-1}}{C(\alpha,\beta)} \\ & \cdot\Bigg(\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_a^b (b^\rho - s^\rho)^{\beta-1}\varkappa(s)\frac{ds}{s^{1-\rho}} - \frac{q\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^\eta (\eta^\rho - s^\rho)^{\alpha-1}\varphi(s)\frac{ds}{s^{1-\rho}}\Bigg), \\ c_{21} =& \frac{\rho^{\beta-1}(b^\rho - a^\rho)^{\alpha-1}}{C(\alpha,\beta)}\Bigg(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b (b^\rho - s^\rho)^{\alpha-1}\varphi(s)\frac{ds}{s^{1-\rho}} \\ & - \frac{p\rho^{1-\beta}}{\Gamma(\beta)}\int_a^\xi (\xi^\rho - s^\rho)^{\beta-1}\varkappa(s)\frac{ds}{s^{1-\rho}}\Bigg) + \frac{q\rho^{\beta-1}(\eta^\rho - a^\rho)^{\alpha-1}}{C(\alpha,\beta)} \\ & \cdot\Bigg(\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_a^b (b^\rho - s^\rho)^{\beta-1}\varkappa(s)\frac{ds}{s^{1-\rho}} - \frac{q\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^\eta (\eta^\rho - s^\rho)^{\alpha-1}\varphi(s)\frac{ds}{s^{1-\rho}}\Bigg). \end{aligned} \tag{9}$$

Substituting (9) into (7) yields the following result

$$\begin{aligned}
\phi(t) =& -\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t (t^\rho - s^\rho)^{\alpha-1}\varphi(s)\frac{ds}{s^{1-\rho}} + \frac{(b^\rho - a^\rho)^{\beta-1}(t^\rho - a^\rho)^{\alpha-1}}{C(\alpha,\beta)} \\
&\cdot\left(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b (b^\rho - s^\rho)^{\alpha-1}\varphi(s)\frac{ds}{s^{1-\rho}} - \frac{p\rho^{1-\beta}}{\Gamma(\beta)}\int_a^\xi (\xi^\rho - s^\rho)^{\beta-1}\varkappa(s)\frac{ds}{s^{1-\rho}}\right) \\
&+ \frac{p(\xi^\rho - a^\rho)^{\beta-1}(t^\rho - a^\rho)^{\alpha-1}}{C(\alpha,\beta)} \\
&\cdot\left(\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_a^b (b^\rho - s^\rho)^{\beta-1}\varkappa(s)\frac{ds}{s^{1-\rho}} - \frac{q\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^\eta (\eta^\rho - s^\rho)^{\alpha-1}\varphi(s)\frac{ds}{s^{1-\rho}}\right) \\
=& -\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t (t^\rho - s^\rho)^{\alpha-1}\varphi(s)\frac{ds}{s^{1-\rho}} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\alpha-1} \\
&\cdot\int_a^b (b^\rho - s^\rho)^{\alpha-1}\varphi(s)\frac{ds}{s^{1-\rho}} - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\alpha-1}\int_a^b (b^\rho - s^\rho)^{\alpha-1}\varphi(s)\frac{ds}{s^{1-\rho}} \\
&+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{(b^\rho - a^\rho)^{\beta-1}(t^\rho - a^\rho)^{\alpha-1}}{C(\alpha,\beta)}\int_a^b (b^\rho - s^\rho)^{\alpha-1}\varphi(s)\frac{ds}{s^{1-\rho}} \\
&- \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{pq(\xi^\rho - a^\rho)^{\beta-1}(t^\rho - a^\rho)^{\alpha-1}}{C(\alpha,\beta)}\int_a^\eta (\eta^\rho - s^\rho)^{\alpha-1}\varphi(s)\frac{ds}{s^{1-\rho}} \\
&+ \frac{\rho^{1-\beta}}{\Gamma(\beta)}\frac{p(b^\rho - a^\rho)^{\beta-1}(t^\rho - a^\rho)^{\alpha-1}}{C(\alpha,\beta)} \\
&\cdot\left(\left(\frac{\xi^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\beta-1}\int_a^b (b^\rho - s^\rho)^{\beta-1}\varkappa(s)\frac{ds}{s^{1-\rho}} - \int_a^\xi (\xi^\rho - s^\rho)^{\beta-1}\varkappa(s)\frac{ds}{s^{1-\rho}}\right) \\
=& \int_a^b G_\alpha(t,s)\varphi(s)\frac{ds}{s^{1-\rho}} + \frac{pq(\xi^\rho - a^\rho)^{\beta-1}(t^\rho - a^\rho)^{\alpha-1}}{C(\alpha,\beta)}\int_a^b G_\alpha(\eta,s)\varphi(s)\frac{ds}{s^{1-\rho}} \\
&+ \frac{p(b^\rho - a^\rho)^{\beta-1}(t^\rho - a^\rho)^{\alpha-1}}{C(\alpha,\beta)}\int_a^b G_\beta(\xi,s)\varkappa(s)\frac{ds}{s^{1-\rho}} \\
=& \int_a^b \left(G_\alpha(t,s) + \frac{pq(t^\rho - a^\rho)^{\alpha-1}}{C(\alpha,\beta)(\xi^\rho - a^\rho)^{1-\beta}}G_\alpha(\eta,s)\right)\varphi(s)\frac{ds}{s^{1-\rho}} \\
&+ \int_a^b \frac{p(t^\rho - a^\rho)^{\alpha-1}}{C(\alpha,\beta)(b^\rho - a^\rho)^{1-\beta}}G_\beta(\xi,s)\varkappa(s)\frac{ds}{s^{1-\rho}}.
\end{aligned}$$

Similarly, we acquire

$$\psi(t)=\int_a^b\left(G_\beta(t,s)+\frac{pq(t^\rho-a^\rho)^{\beta-1}}{C(\alpha,\beta)(\eta^\rho-a^\rho)^{1-\alpha}}G_\beta(\xi,s)\right)\varkappa(s)\frac{ds}{s^{1-\rho}}$$
$$+\int_a^b\frac{q(t^\rho-a^\rho)^{\beta-1}}{C(\alpha,\beta)(b^\rho-a^\rho)^{1-\alpha}}G_\alpha(\eta,s)\varphi(s)\frac{ds}{s^{1-\rho}}.$$

Hence, we observe

$$\begin{cases}\phi(t)=\displaystyle\int_a^b K_1(t,s)\varphi(s)\frac{ds}{s^{1-\rho}}+\int_a^b H_1(t,s)\varkappa(s)\frac{ds}{s^{1-\rho}},\\ \psi(t)=\displaystyle\int_a^b K_2(t,s)\varkappa(s)\frac{ds}{s^{1-\rho}}+\int_a^b H_2(t,s)\varphi(s)\frac{ds}{s^{1-\rho}},\end{cases}$$

which implies (4). □

It follows from Lemma 5 that NEP (1) can be rewritten as an integral form

$$\begin{cases}\phi(t)=\lambda\left(\displaystyle\int_a^b K_1(t,s)\mathscr{F}(s,\phi(s),\psi(s))\frac{ds}{s^{1-\rho}}+\int_a^b H_1(t,s)\mathscr{G}(s,\phi(s),\psi(s))\frac{ds}{s^{1-\rho}}\right),\\ \psi(t)=\lambda\left(\displaystyle\int_a^b K_2(t,s)\mathscr{G}(s,\phi(s),\psi(s))\frac{ds}{s^{1-\rho}}+\int_a^b H_2(t,s)\mathscr{F}(s,\phi(s),\psi(s))\frac{ds}{s^{1-\rho}}\right).\end{cases}\tag{10}$$

LEMMA 6. *Let $0<a<b<\infty$, the given functions $K_1(t,s)$ and $H_1(t,s)$ in (5) satisfy the following inequalities*

$$\frac{pq\rho_\alpha(\eta)(b^\rho-a^\rho)^{2(\alpha-1)}(\xi^\rho-a^\rho)^{\beta-1}}{\rho^{\alpha-1}\Gamma(\alpha)C(\alpha,\beta)}\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\alpha-1}\sigma_\alpha(s)\leqslant K_1(t,s)$$
$$\leqslant\frac{(b^\rho-a^\rho)^{2\alpha+\beta-3}+pq(b^\rho-a^\rho)^{\alpha-1}(\xi^\rho-a^\rho)^{\beta-1}}{\rho^{\alpha-1}\Gamma(\alpha-1)C(\alpha,\beta)}\cdot[(b^\rho-a^\rho)^{\alpha-1}-(\eta^\rho-a^\rho)^{\alpha-1}]\sigma_\alpha(s),\tag{11}$$

$$\frac{p\rho_\beta(\xi)(b^\rho-a^\rho)^{\alpha+2\beta-3}}{\rho^{\beta-1}\Gamma(\beta)C(\alpha,\beta)}\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\alpha-1}\sigma_\beta(s)\leqslant H_1(t,s)$$
$$\leqslant\frac{p(b^\rho-a^\rho)^{\alpha+2\beta-3}}{\rho^{\beta-1}\Gamma(\beta-1)C(\alpha,\beta)}\sigma_\beta(s),\tag{12}$$

$$K_1(t,s) \leqslant \frac{(b^\rho-a^\rho)^{2\alpha+\beta-3}+pq(b^\rho-a^\rho)^{\alpha-1}(\xi^\rho-a^\rho)^{\beta-1}\cdot[(b^\rho-a^\rho)^{\alpha-1}-(\eta^\rho-a^\rho)^{\alpha-1}]}{\rho^{\alpha-1}\Gamma(\alpha-1)C(\alpha,\beta)}\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\alpha-1}, \tag{13}$$

$$H_1(t,s) \leqslant \frac{p(b^\rho-a^\rho)^{\alpha+2\beta-3}}{\rho^{\beta-1}\Gamma(\beta-1)C(\alpha,\beta)}\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\beta-1} \quad \text{for } t,s\in[a,b]. \tag{14}$$

Proof. Firstly, we will prove that (11) is correct. From Lemmas 4 and 5, we gain

$$\begin{aligned}
K_1(t,s) =& G_\alpha(t,s)+\frac{pq(t^\rho-a^\rho)^{\alpha-1}}{C(\alpha,\beta)(\xi^\rho-a^\rho)^{1-\beta}}G_\alpha(\eta,s)\\
\leqslant& \frac{(b^\rho-a^\rho)^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)}(\alpha-1)\sigma_\alpha(s)\\
&+\frac{pq(t^\rho-a^\rho)^{\alpha-1}}{C(\alpha,\beta)(\xi^\rho-a^\rho)^{1-\beta}}\cdot\frac{(b^\rho-a^\rho)^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)}(\alpha-1)\sigma_\alpha(s)\\
=& \frac{(b^\rho-a^\rho)^{2\alpha+\beta-3}+pq(b^\rho-a^\rho)^{\alpha-1}(\xi^\rho-a^\rho)^{\beta-1}\cdot[(b^\rho-a^\rho)^{\alpha-1}-(\eta^\rho-a^\rho)^{\alpha-1}]}{\rho^{\alpha-1}\Gamma(\alpha-1)C(\alpha,\beta)}\sigma_\alpha(s), \quad t,s\in[a,b].
\end{aligned}$$

On the other hand, using Lemmas 4 and 5, we also obtain

$$\begin{aligned}
K_1(t,s) =& G_\alpha(t,s)+\frac{pq(t^\rho-a^\rho)^{\alpha-1}}{C(\alpha,\beta)(\xi^\rho-a^\rho)^{1-\beta}}G_\alpha(\eta,s)\\
\geqslant& \frac{pq(t^\rho-a^\rho)^{\alpha-1}}{C(\alpha,\beta)(\xi^\rho-a^\rho)^{1-\beta}}G_\alpha(\eta,s)\\
\geqslant& \frac{pq(t^\rho-a^\rho)^{\alpha-1}}{C(\alpha,\beta)(\xi^\rho-a^\rho)^{1-\beta}}\frac{(b^\rho-a^\rho)^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)}\rho_\alpha(\eta)\sigma_\alpha(s)\\
=& \frac{pq\rho_\alpha(\eta)(b^\rho-a^\rho)^{2(\alpha-1)}(\xi^\rho-a^\rho)^{\beta-1}}{\rho^{\alpha-1}\Gamma(\alpha)C(\alpha,\beta)}\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\alpha-1}\sigma_\alpha(s), \quad t,s\in[a,b].
\end{aligned}$$

Secondly, we will demonstrate that (12) holds for $t,s\in[a,b]$. It follows from Lemmas 4 and 5 that

$$\begin{aligned}
H_1(t,s) =& \frac{p(t^\rho-a^\rho)^{\alpha-1}}{C(\alpha,\beta)(b^\rho-a^\rho)^{1-\beta}}G_\beta(\xi,s) \leqslant \frac{p(b^\rho-a^\rho)^{\alpha-1}}{C(\alpha,\beta)(b^\rho-a^\rho)^{1-\beta}}G_\beta(\xi,s)\\
\leqslant& \frac{p(b^\rho-a^\rho)^{\alpha-1}}{C(\alpha,\beta)(b^\rho-a^\rho)^{1-\beta}}\cdot\frac{(b^\rho-a^\rho)^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)}(\beta-1)\sigma_\beta(s)\\
=& \frac{p(b^\rho-a^\rho)^{\alpha+2\beta-3}}{\rho^{\beta-1}\Gamma(\beta-1)C(\alpha,\beta)}\sigma_\beta(s),\ t,s\in[a,b].
\end{aligned}$$

On the other hand, by means of Lemmas 4 and 5, we also acquire

$$\begin{aligned} H_1(t,s) &\geqslant \frac{p(t^\rho-a^\rho)^{\alpha-1}}{C(\alpha,\beta)(b^\rho-a^\rho)^{1-\beta}}\cdot\frac{(b^\rho-a^\rho)^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)}\rho_\beta(\xi)\sigma_\beta(s) \\ &= \frac{p\rho_\beta(\xi)(b^\rho-a^\rho)^{\alpha+2\beta-3}}{\rho^{\beta-1}\Gamma(\beta)C(\alpha,\beta)}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha-1}\sigma_\beta(s),\ t,s\in[a,b]. \end{aligned}$$

Thirdly, we will show (13) and (14) are efficacious for any $t,s\in[a,b]$. Noticing $\rho_\alpha(t)\leqslant(\frac{t^\rho-a^\rho}{b^\rho-a^\rho})^{\alpha-1}$, $\rho_\beta(t)\leqslant(\frac{t^\rho-a^\rho}{b^\rho-a^\rho})^{\beta-1}$, $\rho_\alpha(\eta)\leqslant 1$, $\rho_\beta(\xi)\leqslant 1$, and utilizing Lemmas 4 and 5, we can claim

$$\begin{aligned} K_1(t,s) &\leqslant \frac{(b^\rho-a^\rho)^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)}(\alpha-1)\rho_\alpha(t) \\ &\quad+\frac{pq(b^\rho-a^\rho)^{\alpha-1}}{C(\alpha,\beta)(\xi^\rho-a^\rho)^{1-\beta}}\cdot\frac{(b^\rho-a^\rho)^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)}(\alpha-1)\rho_\alpha(t) \\ &\leqslant \frac{\begin{array}{r}(b^\rho-a^\rho)^{2\alpha+\beta-3}+pq(b^\rho-a^\rho)^{\alpha-1}(\xi^\rho-a^\rho)^{\beta-1}\\ \cdot[(b^\rho-a^\rho)^{\alpha-1}-(\eta^\rho-a^\rho)^{\alpha-1}]\end{array}}{\rho^{\alpha-1}\Gamma(\alpha-1)C(\alpha,\beta)}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha-1},\ t,s\in[a,b], \end{aligned}$$

and

$$\begin{aligned} H_1(t,s) &\leqslant \frac{p(t^\rho-a^\rho)^{\alpha-1}}{C(\alpha,\beta)(b^\rho-a^\rho)^{1-\beta}}\cdot\frac{(b^\rho-a^\rho)^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)}(\beta-1)\rho_\beta(t) \\ &\leqslant \frac{p(b^\rho-a^\rho)^{\alpha+2\beta-3}}{\rho^{\beta-1}\Gamma(\beta-1)C(\alpha,\beta)}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\beta-1},\quad t,s\in[a,b]. \end{aligned}$$

The lemma has thus been established. □

By an analogous approach, we obtain

LEMMA 7. *Let $0<a<b<\infty$, the given functions $K_2(t,s)$ and $H_2(t,s)$ in (5) satisfy the following inequalities*

$$\begin{aligned} &\frac{pq\rho_\beta(\xi)(b^\rho-a^\rho)^{2(\beta-1)}(\eta^\rho-a^\rho)^{\alpha-1}}{\rho^{\beta-1}\Gamma(\beta)C(\alpha,\beta)}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\beta-1}\sigma_\beta(s)\leqslant K_2(t,s) \\ &\qquad\leqslant \frac{\begin{array}{r}(b^\rho-a^\rho)^{\alpha+2\beta-3}+pq(b^\rho-a^\rho)^{\beta-1}(\eta^\rho-a^\rho)^{\alpha-1}\\ \cdot[(b^\rho-a^\rho)^{\beta-1}-(\xi^\rho-a^\rho)^{\beta-1}]\end{array}}{\rho^{\beta-1}\Gamma(\beta-1)C(\alpha,\beta)}\sigma_\beta(s), \end{aligned}$$

$$\frac{q\rho_\alpha(\eta)(b^\rho-a^\rho)^{2\alpha+\beta-3}}{\rho^{\alpha-1}\Gamma(\alpha)C(\alpha,\beta)}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\beta-1}\sigma_\alpha(s)\leqslant H_2(t,s)\leqslant\frac{q(b^\rho-a^\rho)^{2\alpha+\beta-3}}{\rho^{\alpha-1}\Gamma(\alpha-1)C(\alpha,\beta)}\sigma_\alpha(s),$$

$$K_2(t,s)\leqslant\frac{\begin{array}{r}(b^\rho-a^\rho)^{\alpha+2\beta-3}+pq(b^\rho-a^\rho)^{\beta-1}(\eta^\rho-a^\rho)^{\alpha-1}\\ \cdot[(b^\rho-a^\rho)^{\beta-1}-(\xi^\rho-a^\rho)^{\beta-1}]\end{array}}{\rho^{\beta-1}\Gamma(\beta-1)C(\alpha,\beta)}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\beta-1},$$

$$H_2(t,s) \leqslant \frac{q(b^\rho - a^\rho)^{2\alpha+\beta-3}}{\rho^{\alpha-1}\Gamma(\alpha-1)C(\alpha,\beta)} \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} \quad \textit{for } t,s \in [a,b].$$

REMARK 2. Let $0 < a < b < \infty$. Combining Lemmas 6 and 7, for any $t,s \in [a,b]$, then we have the following relationships

$$\begin{aligned}
&\ell_1 \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} \sigma_\alpha(s) \leqslant K_1(t,s) \leqslant \ell_2 \sigma_\alpha(s), \qquad K_1(t,s) \leqslant \ell_2 \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1},\\
&\ell_1 \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} \sigma_\beta(s) \leqslant H_1(t,s) \leqslant \ell_2 \sigma_\beta(s), \qquad H_1(t,s) \leqslant \ell_2 \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1},\\
&\ell_1 \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\beta-1} \sigma_\beta(s) \leqslant K_2(t,s) \leqslant \ell_2 \sigma_\beta(s), \qquad K_2(t,s) \leqslant \ell_2 \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\beta-1},\\
&\ell_1 \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\beta-1} \sigma_\alpha(s) \leqslant H_2(t,s) \leqslant \ell_2 \sigma_\alpha(s), \qquad H_2(t,s) \leqslant \ell_2 \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\beta-1},
\end{aligned}$$

where

$$\begin{aligned}
\ell_1 = \min\Bigg\{ &\frac{\min\{q\rho_\alpha(\eta)(b^\rho - a^\rho)^{2\alpha+\beta-3}, pq\rho_\alpha(\eta)(b^\rho - a^\rho)^{2(\alpha-1)}(\xi^\rho - a^\rho)^{\beta-1}\}}{\rho^{\alpha-1}\Gamma(\alpha)C(\alpha,\beta)},\\
&\frac{\min\{p\rho_\beta(\xi)(b^\rho - a^\rho)^{\alpha+2\beta-3}, pq\rho_\beta(\xi)(b^\rho - a^\rho)^{2(\beta-1)}(\eta^\rho - a^\rho)^{\alpha-1}\}}{\rho^{\beta-1}\Gamma(\beta)C(\alpha,\beta)}\Bigg\},\\
\ell_2 = \max\Bigg\{ &\frac{\max\Big\{q(b^\rho - a^\rho)^{2\alpha+\beta-3}, \begin{array}{l}(b^\rho - a^\rho)^{2\alpha+\beta-3} + pq(b^\rho - a^\rho)^{\alpha-1}(\xi^\rho\\ -a^\rho)^{\beta-1}[(b^\rho - a^\rho)^{\alpha-1} - (\eta^\rho - a^\rho)^{\alpha-1}]\end{array}\Big\}}{\rho^{\alpha-1}\Gamma(\alpha-1)C(\alpha,\beta)},\\
&\frac{\max\Big\{p(b^\rho - a^\rho)^{\alpha+2\beta-3}, \begin{array}{l}(b^\rho - a^\rho)^{\alpha+2\beta-3} + pq(b^\rho - a^\rho)^{\beta-1}(\eta^\rho\\ -a^\rho)^{\alpha-1}[(b^\rho - a^\rho)^{\beta-1} - (\xi^\rho - a^\rho)^{\beta-1}]\end{array}\Big\}}{\rho^{\beta-1}\Gamma(\beta-1)C(\alpha,\beta)}\Bigg\}.
\end{aligned}$$

The subsequent analysis builds upon the foundational hypothesis that

(H) Let $0 < a < b < \infty$, $\mathscr{F},\mathscr{G} : (a,b) \times [0,+\infty)^2 \to (-\infty,+\infty)$ are continuous functions satisfying for $t \in (a,b)$, $\phi,\psi \geqslant 0$,

$$\begin{aligned}
-\mathscr{Q}_1(t) \leqslant \mathscr{F}(t,\phi,\psi) &\leqslant \mathscr{P}_1(t)\mathscr{F}_\star(t,\phi,\psi),\\
-\mathscr{Q}_2(t) \leqslant \mathscr{G}(t,\phi,\psi) &\leqslant \mathscr{P}_2(t)\mathscr{G}_\star(t,\phi,\psi),
\end{aligned}$$

where $\mathscr{F}_*,\mathscr{G}_* \in C([a,b] \times [0,+\infty)^2, [0,+\infty))$ and $\mathscr{P}_k, \mathscr{Q}_k \in C((a,b),[0,+\infty))$ fulfill the following conditions $0 < \int_a^b \mathscr{P}_k(s)\frac{ds}{s^{1-\rho}} < +\infty$ and $0 < \int_a^b \mathscr{Q}_k(s)\frac{ds}{s^{1-\rho}} < +\infty$ for $k = 1,2$.

LEMMA 8. *Under the validity of condition* (H)*, then the BVP*

$$\begin{cases}
-\mathscr{D}^{\alpha,\rho}_{a+}\varpi_1(t) = \lambda\mathscr{Q}_1(t), \quad -\mathscr{D}^{\beta,\rho}_{a+}\varpi_2(t) = \lambda\mathscr{Q}_2(t), \ t \in (a,b), \ \lambda > 0,\\
\gamma^j\varpi_1(a) = \gamma^j\varpi_2(a) = 0, \ \varpi_1(b) = p\varpi_2(\xi), \ \varpi_2(b) = q\varpi_1(\eta), \ \xi,\eta \in (a,b),
\end{cases}$$

where $0 \leqslant i \leqslant m-2$, $0 \leqslant j \leqslant n-2$, $\alpha \in (m-1,m]$, $\beta \in (n-1,n]$, $m,n \geqslant 3$, *has an unique solution*

$$\begin{cases} \varpi_1(t) = \lambda \left(\int_a^b K_1(t,s)\mathscr{Q}_1(s)\frac{ds}{s^{1-\rho}} + \int_a^b H_1(t,s)\mathscr{Q}_2(s)\frac{ds}{s^{1-\rho}} \right), \\ \varpi_2(t) = \lambda \left(\int_a^b K_2(t,s)\mathscr{Q}_2(s)\frac{ds}{s^{1-\rho}} + \int_a^b H_2(t,s)\mathscr{Q}_1(s)\frac{ds}{s^{1-\rho}} \right), \end{cases} \tag{15}$$

which satisfies

$$\begin{cases} \varpi_1(t) \leqslant \lambda \ell_2 \left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho} \right)^{\alpha-1} \int_a^b (\mathscr{Q}_1(s) + \mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}, \quad t \in [a,b], \\ \varpi_2(t) \leqslant \lambda \ell_2 \left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho} \right)^{\beta-1} \int_a^b (\mathscr{Q}_1(s) + \mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}, \quad t \in [a,b]. \end{cases} \tag{16}$$

Proof. By combining Lemma 5 with Remark 2 under condition (H), we conclude that equations (15) and (16) hold. □

Let $\mathbb{E} = [a,b]^2$, then $\mathbb{E}$ is a Banach space with the norm $\|(\phi,\psi)\|_1 = \|\phi\| + \|\psi\|$, $\|\phi\| = \max_{t\in[a,b]} |\phi(t)|$, $\|\psi\| = \max_{t\in[a,b]} |\psi(t)|$ for any $(\phi,\psi) \in \mathbb{E}$. Let $\mathbb{P} = \{(\phi,\psi) \in \mathbb{E} : \phi(t) \geqslant \ell(\frac{t^\rho - a^\rho}{b^\rho - a^\rho})^{\alpha-1}\|\phi\|,\ \psi(t) \geqslant \ell(\frac{t^\rho - a^\rho}{b^\rho - a^\rho})^{\beta-1}\|\psi\| \text{ for } t \in [a,b]\}$, where $0 < \ell = \ell_1/\ell_2 < 1$. Then $\mathbb{P}$ stands for a cone of $\mathbb{E}$.

In the subsequent analysis, we investigate the singular BVP of the form

$$\begin{cases} \mathscr{D}_{a+}^{\alpha,\rho}\Phi(t) + \lambda(\mathscr{F}(t,[\Phi(t)-\varpi_1(t)]^*,[\Psi(t)-\varpi_2(t)]^*) + \mathscr{Q}_1(t)) = 0,\ t \in (a,b), \\ \mathscr{D}_{a+}^{\beta,\rho}\Psi(t) + \lambda(\mathscr{G}(t,[\Phi(t)-\varpi_1(t)]^*,[\Psi(t)-\varpi_2(t)]^*) + \mathscr{Q}_2(t)) = 0,\ \lambda > 0, \\ \gamma^j\Phi(a) = \gamma^j\Psi(a) = 0,\ \ \Phi(b) = p\Psi(\xi),\ \Psi(b) = q\Phi(\eta),\ \xi,\eta \in (a,b), \end{cases} \tag{17}$$

where $0 \leqslant i \leqslant m-2$, $0 \leqslant j \leqslant n-2$, an adjusted function $[\beth(t)]^*$ is defined by $[\beth(t)]^* = \beth(t)$, if $\beth(t) \geqslant 0$, and $[\beth(t)]^* = 0$, if $\beth(t) < 0$ for any $\beth \in \mathbb{C}[a,b]$.

LEMMA 9. *Let* $(\Phi,\Psi) \in \mathbb{C}[a,b]^2$ *be a positive solution of system* (17) *with* $\Phi(t) > \varpi_1(t)$ *and* $\Psi(t) > \varpi_2(t)$ *for any* $t \in (a,b)$, *then* $(\Phi - \varpi_1, \Psi - \varpi_2)$ *is a positive solution of NEP* (1).

Proof. In fact, if $(\Phi,\Psi) \in \mathbb{C}[a,b]^2$ be a positive solution of system (17) satisfying $\Phi(t) > \varpi_1(t)$ and $\Psi(t) > \varpi_2(t)$ for any $t \in (a,b)$, then, from (17) and the definition of the adjusted function $[\cdot]^*$, we derive

$$\begin{cases} \mathscr{D}_{a+}^{\alpha,\rho}\Phi(t) + \lambda(\mathscr{F}(t,\Phi(t)-\varpi_1(t),\Psi(t)-\varpi_2(t)) + \mathscr{Q}_1(t)) = 0,\ t \in (a,b), \\ \mathscr{D}_{a+}^{\beta,\rho}\Psi(t) + \lambda(\mathscr{G}(t,\Phi(t)-\varpi_1(t),\Psi(t)-\varpi_2(t)) + \mathscr{Q}_2(t)) = 0,\ \lambda > 0, \\ \gamma^j\Phi(a) = \gamma^j\Psi(a) = 0,\ \ \Phi(b) = p\Psi(\xi),\ \Psi(b) = q\Phi(\eta),\ \xi,\eta \in (a,b), \end{cases} \tag{18}$$

where $0 \leqslant i \leqslant m-2$, $0 \leqslant j \leqslant n-2$.

Let $\mu = \Phi - \varpi_1$ and $\nu = \Psi - \varpi_2$, then $\mathscr{D}_{a+}^{\alpha,\rho}\mu(t) = \mathscr{D}_{a+}^{\alpha,\rho}\Phi(t) - \mathscr{D}_{a+}^{\alpha,\rho}\varpi_1(t)$ and $\mathscr{D}_{a+}^{\beta,\rho}\nu(t) = \mathscr{D}_{a+}^{\beta,\rho}\Psi(t) - \mathscr{D}_{a+}^{\beta,\rho}\varpi_2(t)$ for $t \in (a,b)$, which indicate that

$$\begin{cases} -\mathscr{D}_{a+}^{\alpha,\rho}\mu(t) = -\mathscr{D}_{a+}^{\alpha,\rho}\Phi(t) + \mathscr{D}_{a+}^{\alpha,\rho}\varpi_1(t) = -\mathscr{D}_{a+}^{\alpha,\rho}\Phi(t) - \lambda\mathscr{Q}_1(t), & t \in (a,b), \\ -\mathscr{D}_{a+}^{\beta,\rho}\nu(t) = -\mathscr{D}_{a+}^{\beta,\rho}\Psi(t) + \mathscr{D}_{a+}^{\beta,\rho}\varpi_2(t) = -\mathscr{D}_{a+}^{\beta,\rho}\Psi(t) - \lambda\mathscr{Q}_2(t), & t \in (a,b). \end{cases}$$

Therefore, system (18) can be formulated as

$$\begin{cases} \mathscr{D}_{a+}^{\alpha,\rho}\mu(t) + \lambda\mathscr{F}(t,\mu(t),\nu(t)) = 0, \ \ \mathscr{D}_{a+}^{\beta,\rho}\nu(t) + \lambda\mathscr{G}(t,\mu(t),\nu(t)) = 0, \ t \in (a,b), \\ \gamma^j\mu(a) = \gamma^j\nu(a) = 0, \ \ \mu(b) = p\nu(\xi), \ \nu(b) = q\mu(\eta), \ \xi,\eta \in (a,b), \end{cases}$$

where $\lambda > 0$, $0 \leqslant i \leqslant m-2$, $0 \leqslant j \leqslant n-2$. That is, $(\Phi - \varpi_1, \Psi - \varpi_2)$ is a positive solution of NEP (1). The proof of Lemma 9 is completed. □

It follows from Lemma 5 that system (18) can be modeled as an integral system

$$\begin{cases} \Phi(t) = \lambda\Big(\displaystyle\int_a^b K_1(t,s)(\mathscr{F}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*) + \mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}} \\ \qquad\quad + \displaystyle\int_a^b H_1(t,s)(\mathscr{G}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\omega_2(s)]^*) + \mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\Big), \\ \Psi(t) = \lambda\Big(\displaystyle\int_a^b K_2(t,s)(\mathscr{G}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*) + \mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}} \\ \qquad\quad + \displaystyle\int_a^b H_2(t,s)(\mathscr{F}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*) + \mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}\Big), \end{cases} \tag{19}$$

for $t \in [a,b]$.

Let (Φ,Ψ) be a solution to the system (18), then this implies that (Φ,Ψ) satisfies the criteria for a solution to the corresponding system of integral equations (19). Introduce an operator $\mathbf{T} : \mathbb{P} \to \mathbb{P}$ by $\mathbf{T}(\Phi,\Psi) = (\mathbf{T}_1(\Phi,\Psi), \mathbf{T}_2(\Phi,\Psi))$, where operators $\mathbf{T}_k : \mathbb{P} \to \mathbb{C}[a,b]$ $(k = 1,2)$ are defined by

$$\begin{aligned} \mathbf{T}_1(\Phi,\Psi)(t) = \lambda\Big(&\int_a^b K_1(t,s)(\mathscr{F}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*) + \mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}} \\ &+ \int_a^b H_1(t,s)(\mathscr{G}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\omega_2(s)]^*) + \mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\Big), \end{aligned}$$

$$\mathbf{T}_2(\Phi,\Psi)(t) =\lambda\bigg(\int_a^b K_2(t,s)(\mathscr{G}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}$$
$$+\int_a^b H_2(t,s)(\mathscr{F}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}\bigg), \tag{20}$$

for $t\in[a,b]$. Evidently, any fixed point $(\Phi,\Psi)\in\mathbb{P}$ of the operator $\mathbf{T}$ directly corresponds to a solution of system (18).

LEMMA 10. *Assume that condition* (H) *remains valid, then* $\mathbf{T}:\mathbb{P}\to\mathbb{P}$ *is a completely continuous operator.*

Proof. For any fixed $(\Phi,\Psi)\in\mathbb{P}$, there exists a positive constant $\mathscr{L}$ such that $\|(\Phi,\Psi)\|_1\leqslant\mathscr{L}$. And we obtain

$$[\Phi(s)-\varpi_1(s)]^*\leqslant\Phi(s)\leqslant\|\Phi\|\leqslant\|(\Phi,\Psi)\|_1\leqslant\mathscr{L},$$
$$[\Psi(s)-\varpi_2(s)]^*\leqslant\Psi(s)\leqslant\|\Psi\|\leqslant\|(\Phi,\Psi)\|_1\leqslant\mathscr{L},\ \ s\in[a,b].$$

Based on Remark 2 and (20), then we derive for any $t\in[a,b]$

$$\mathbf{T}_1(\Phi,\Psi)(t) =\lambda\bigg(\int_a^b K_1(t,s)(\mathscr{F}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}$$
$$+\int_a^b H_1(t,s)(\mathscr{G}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\omega_2(s)]^*)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\bigg)$$
$$\leqslant\lambda\ell_2\bigg(\int_a^b\sigma_\alpha(s)(\mathscr{P}_1(s)\mathscr{F}_\star(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}$$
$$+\int_a^b\sigma_\beta(s)(\mathscr{P}_2(s)\mathscr{G}_\star(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\omega_2(s)]^*)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\bigg)$$
$$\leqslant\lambda\mathfrak{L}\ell_2\bigg(\int_a^b\sigma_\alpha(s)(\mathscr{P}_1(s)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}+\int_a^b\sigma_\beta(s)(\mathscr{P}_2(s)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\bigg)$$
$$\leqslant\lambda\mathfrak{L}\ell_2\bigg(\int_a^b(\mathscr{P}_1(s)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}+\int_a^b(\mathscr{P}_2(s)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\bigg)<+\infty,$$

where $\mathfrak{L}=\max\{\max_{t\in[a,b],\phi,\psi\in[0,\mathscr{L}]}\mathscr{F}_*(t,\phi,\psi),\max_{t\in[a,b],\phi,\psi\in[0,\mathscr{L}]}\mathscr{G}_*(t,\phi,\psi)\}+1$. Similarly, we get

$$\mathbf{T}_2(\Phi,\Psi)(t)\leqslant\lambda\mathfrak{L}\ell_2\bigg(\int_a^b(\mathscr{P}_1(s)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}+\int_a^b(\mathscr{P}_2(s)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\bigg)<+\infty,$$

which implies $\mathbf{T}:\mathbb{P}\to\mathbb{E}$ is well defined.

Next, we show that $\mathbf{T}:\mathbb{P}\to\mathbb{P}$. For any fixed $(\Phi,\Psi)\in\mathbb{P}$, $t\in[a,b]$, by Remark 2 and (20), we have

$$\begin{aligned}
\mathbf{T}_1(\Phi,\Psi)(t)\leqslant&\lambda\ell_2\bigg(\int_a^b\sigma_\alpha(s)(\mathscr{F}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}\\
&+\int_a^b\sigma_\beta(s)(\mathscr{G}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\omega_2(s)]^*)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\bigg),\\
\mathbf{T}_2(\Phi,\Psi)(t)\leqslant&\lambda\ell_2\bigg(\int_a^b\sigma_\beta(s)(\mathscr{G}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\\
&+\int_a^b\sigma_\alpha(s)(\mathscr{F}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\omega_2(s)]^*)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}\bigg),
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\mathbf{T}_1(\Phi,\Psi)\|\leqslant&\lambda\ell_2\bigg(\int_a^b\sigma_\alpha(s)(\mathscr{F}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}\\
&+\int_a^b\sigma_\beta(s)(\mathscr{G}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\omega_2(s)]^*)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\bigg),\\
\|\mathbf{T}_2(\Phi,\Psi)\|\leqslant&\lambda\ell_2\bigg(\int_a^b\sigma_\beta(s)(\mathscr{G}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\\
&+\int_a^b\sigma_\alpha(s)(\mathscr{F}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\omega_2(s)]^*)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}\bigg).
\end{aligned}\tag{21}$$

On the other hand, it follows from Remark 2 and (20) that we also acquire

$$\begin{aligned}
&\mathbf{T}_1(\Phi,\Psi)(t)\\
&\geqslant\lambda\ell_1\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha-1}\bigg(\int_a^b\sigma_\alpha(s)(\mathscr{F}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}\\
&\quad+\int_a^b\sigma_\beta(s)(\mathscr{G}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\omega_2(s)]^*)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\bigg),
\end{aligned}$$

$$\begin{aligned}&\mathbf{T}_2(\Phi,\Psi)(t)\\ &\geqslant\lambda\ell_1\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\beta-1}\bigg(\int_a^b\sigma_\beta(s)(\mathscr{G}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\\ &\quad+\int_a^b\sigma_\alpha(s)(\mathscr{F}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\omega_2(s)]^*)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}\bigg).\end{aligned}\tag{22}$$

It follows from (21) and (22) that we achieve

$$\begin{aligned}\mathbf{T}_1(\Phi,\Psi)(t)&\geqslant\ell\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha-1}\|\mathbf{T}_1(\Phi,\Psi)\|,\\ \mathbf{T}_2(\Phi,\Psi)(t)&\geqslant\ell\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\beta-1}\|\mathbf{T}_2(\Phi,\Psi)\|,\ \ t\in[a,b],\end{aligned}$$

which reveals that $\mathbf{T}(\mathbb{P})\subset\mathbb{P}$. It is straightforward to verify from the Ascoli-Arzela theorem that $\mathbf{T}:\mathbb{P}\to\mathbb{P}$ is completely continuous. We have thus established the proof of Lemma 10. □

LEMMA 11. ([6]) *Let $\mathbb{P}\subset\mathbb{X}$ be a cone for a Banach space $\mathbb{X}$ and Ω_1,Ω_2 denote two bounded open subsets of $\mathbb{X}$ with $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$. Moreover, assume that $\mathbf{S}:\mathbb{P}\to\mathbb{P}$ be a completely continuous operator such that, either*

(a) $\|\mathbf{S}x\|\leqslant\|x\|$, $x\in\mathbb{P}\cap\partial\Omega_1$, $\|\mathbf{S}x\|\geqslant\|x\|$, $x\in\mathbb{P}\cap\partial\Omega_2$, *or*

(b) $\|\mathbf{S}x\|\geqslant\|x\|$, $x\in\mathbb{P}\cap\partial\Omega_1$, $\|\mathbf{S}x\|\leqslant\|x\|$, $x\in\mathbb{P}\cap\partial\Omega_2$.

Then $\mathbf{S}$ has a fixed point $x\in\mathbb{P}\cap(\overline{\Omega}_2\backslash\Omega_1)$.

3. Main results

In order to facilitate, for any $R>0$, $\mathscr{F}_\star^R=\max\{\mathscr{F}_\star(t,\phi,\psi):t\in[a,b],\phi,\psi\geqslant 0,\phi+\psi\leqslant R\}$, $\mathscr{G}_\star^R=\max\{\mathscr{G}_\star(t,\phi,\psi):t\in[a,b],\phi,\psi\geqslant 0,\phi+\psi\leqslant R\}$, and fixed $[\varepsilon_1,\varepsilon_2]\subset(a,b)$, we introduce the following notations:

$$\vartheta_\alpha=\ell_2\int_a^b\sigma_\alpha(s)(\mathscr{P}_1(s)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}},\qquad \vartheta_\beta=\ell_2\int_a^b\sigma_\beta(s)(\mathscr{P}_2(s)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}},$$

$$\widetilde{\mathscr{F}}(\infty)=\liminf_{\phi+\psi\uparrow+\infty,\phi,\psi\geqslant0}\min_{t\in[\varepsilon_1,\varepsilon_2]}\mathscr{F}(t,\phi,\psi),\qquad \widetilde{\mathscr{G}}(\infty)=\liminf_{\phi+\psi\uparrow+\infty,\phi,\psi\geqslant0}\min_{t\in[\varepsilon_1,\varepsilon_2]}\mathscr{G}(t,\phi,\psi),$$

$$\underline{\mathscr{F}}(\infty)=\liminf_{\phi+\psi\uparrow+\infty,\phi,\psi\geqslant0}\min_{t\in[\varepsilon_1,\varepsilon_2]}\frac{\mathscr{F}(t,\phi,\psi)}{\phi+\psi},\qquad \underline{\mathscr{G}}(\infty)=\liminf_{\phi+\psi\uparrow+\infty,\phi,\psi\geqslant0}\min_{t\in[\varepsilon_1,\varepsilon_2]}\frac{\mathscr{G}(t,\phi,\psi)}{\phi+\psi},$$

$$\overline{\mathscr{F}_\star}(\infty)=\limsup_{\phi+\psi\uparrow+\infty,\phi,\psi\geqslant0}\max_{t\in[a,b]}\frac{\mathscr{F}_\star(t,\phi,\psi)}{\phi+\psi},\qquad \overline{\mathscr{G}_\star}(\infty)=\limsup_{\phi+\psi\uparrow+\infty,\phi,\psi\geqslant0}\max_{t\in[a,b]}\frac{\mathscr{G}_\star(t,\phi,\psi)}{\phi+\psi}.$$

THEOREM 1. *Under the validity of condition* (H)*, the following condition* (C_1) *holds:* $\underline{\mathscr{F}}(\infty)=+\infty$ *or* $\underline{\mathscr{G}}(\infty)=+\infty$*. Then there exists* $\hat{\lambda}>0$ *such that for any* $\lambda\in(0,\hat{\lambda})$*, NEP* (1) *has at least one positive solution* $(\hat{\phi},\hat{\psi})$*. Furthermore,* $(\hat{\phi},\hat{\psi})$ *satisfies* $\hat{\phi}(t)\geqslant\hat{w}(\frac{t^\rho-a^\rho}{b^\rho-a^\rho})^{\alpha-1}$ *and* $\hat{\psi}(t)\geqslant\hat{w}(\frac{t^\rho-a^\rho}{b^\rho-a^\rho})^{\beta-1}$ *for any* $t\in[a,b]$ *and some positive constant* $\hat{w}$*.*

Proof. We firstly choose a constant R_1 satisfying $R_1>\max\{\frac{2\ell_2}{\ell},\frac{2\ell_2}{p\ell^2}(\frac{b^\rho-a^\rho}{\xi^\rho-a^\rho})^{\beta-1},\frac{2\ell_2}{q\ell^2}(\frac{b^\rho-a^\rho}{\eta^\rho-a^\rho})^{\alpha-1}\}\cdot\int_a^b(\mathscr{P}_2(s)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}$. Let $\hat{\lambda}=\min\big\{1,\frac{R_1}{2[(\mathscr{F}_\star^{R_1}+1)\vartheta_\alpha+(\mathscr{G}_\star^{R_1}+1)\vartheta_\beta]}\big\}$ and $0<\lambda<\hat{\lambda}$. Set $\Omega_1=\{(\Phi,\Psi)\in\mathbb{E}:\|(\Phi,\Psi)\|_1<R_1\}$. For any $(\Phi,\Psi)\in\mathbb{P}\cap\partial\Omega_1$, $s\in[a,b]$, we observe $[\Phi(s)-\varpi_1(s)]^*\leqslant\Phi(s)\leqslant\|\Phi\|$ and $[\Psi(s)-\varpi_2(s)]^*\leqslant\Psi(s)\leqslant\|\Psi\|$. Employing Remark 2 and (20), we derive for any $(\Phi,\Psi)\in\mathbb{P}\cap\partial\Omega_1$

$$
\begin{aligned}
&\mathbf{T}_1(\Phi,\Psi)(t)\\
&\leqslant\lambda\ell_2\bigg(\int_a^b\sigma_\alpha(s)(\mathscr{P}_1(s)\mathscr{F}_\star(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}\\
&\quad+\int_a^b\sigma_\beta(s)(\mathscr{P}_2(s)\mathscr{G}_\star(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\omega_2(s)]^*)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\bigg)\\
&\leqslant\lambda\ell_2\bigg(\int_a^b\sigma_\alpha(s)(\mathscr{P}_1(s)\mathscr{F}_\star^{R_1}+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}+\int_a^b\sigma_\beta(s)(\mathscr{P}_2(s)\mathscr{G}_\star^{R_1}+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\bigg)\\
&\leqslant\lambda(\mathscr{F}_\star^{R_1}+1)\ell_2\int_a^b\sigma_\alpha(s)(\mathscr{P}_1(s)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}\\
&\quad+\lambda(\mathscr{G}_\star^{R_1}+1)\ell_2\int_a^b\sigma_\beta(s)(\mathscr{P}_2(s)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\\
&=\lambda[(\mathscr{F}_\star^{R_1}+1)\vartheta_\alpha+(\mathscr{G}_\star^{R_1}+1)\vartheta_\beta]\leqslant\frac{\|(\Phi,\Psi)\|_1}{2}.
\end{aligned}
$$

Similarly, we also have $\|\mathbf{T}_2(\Phi,\Psi)\|\leqslant\|(\Phi,\Psi)\|_1/2$. Synthesizing the above results yields

$$\|\mathbf{T}(\Phi,\Psi)\|_1=\|\mathbf{T}_1(\Phi,\Psi)\|+\|\mathbf{T}_2(\Phi,\Psi)\|\leqslant\|(\Phi,\Psi)\|_1\ \text{ for }\ \forall(\Phi,\Psi)\in\mathbb{P}\cap\partial\Omega_1.\tag{23}$$

On the contrary, pick N_1 as a sufficiently large positive number so that

$$\lambda\frac{\ell\ell_2}{2}\Big(\frac{\varepsilon_1^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{2(\alpha-1)}N_1\int_{\varepsilon_1}^{\varepsilon_2}\sigma_\alpha(s)\frac{ds}{s^{1-\rho}}\geqslant1.\tag{24}$$

According to the condition $\underline{\mathscr{F}}(\infty)=+\infty$, then there exists a positive number L_1 such that

$$\mathscr{F}(t,\phi,\psi)\geqslant N_1(\phi+\psi)\ \text{ for }\ t\in[\varepsilon_1,\varepsilon_2],\ \ \phi,\psi\geqslant0,\ \ \phi+\psi\geqslant L_1.\tag{25}$$

Set $R_2 = \max\{2R_1, \frac{4}{\ell}(\frac{b^\rho - a^\rho}{\varepsilon_1^\rho - a^\rho})^{1-\alpha} L_1\}$ and $\Omega_2 = \{(\Phi, \Psi) \in \mathbb{E} : \|(\Phi, \Psi)\|_1 < R_2\}$. For any $(\Phi, \Psi) \in \mathbb{P} \cap \partial\Omega_2$, we get $\|(\Phi, \Psi)\|_1 = R_2$. Therefore, we have a component Φ or Ψ such that $\|\Phi\| \geqslant R_2/2$ or $\|\Psi\| \geqslant R_2/2$. Without loss of generality, we may assume that $\|\Phi\| \geqslant R_2/2$. Then, for any $(\Phi, \Psi) \in \mathbb{P} \cap \partial\Omega_2$, from (16), we obtain

$$\begin{aligned}
\Phi(t) - \varpi_1(t) \geqslant & \Phi(t) - \lambda \ell_2 \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} \int_a^b (\mathscr{Q}_1(s) + \mathscr{Q}_2(s)) \frac{ds}{s^{1-\rho}} \\
\geqslant & \Phi(t) - \frac{\ell_2 \Phi(t)}{\ell \|\Phi\|} \int_a^b (\mathscr{Q}_1(s) + \mathscr{Q}_2(s)) \frac{ds}{s^{1-\rho}} \\
\geqslant & \Phi(t) - \frac{2\ell_2 \Phi(t)}{\ell R_2} \int_a^b (\mathscr{Q}_1(s) + \mathscr{Q}_2(s)) \frac{ds}{s^{1-\rho}} \\
\geqslant & \frac{1}{2}\Phi(t) \geqslant \frac{\ell}{2}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} \|\Phi\| \geqslant \frac{\ell}{4}\Big(\frac{\varepsilon_1^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} R_2 \geqslant L_1,
\end{aligned}$$

and

$$\begin{aligned}
[\Phi(t) - \varpi_1(t)]^* + [\Psi(t) - \varpi_2(t)]^* \geqslant & [\Phi(t) - \varpi_1(t)]^* \\
= & \Phi(t) - \varpi_1(t) \geqslant \frac{\ell}{4}\Big(\frac{\varepsilon_1^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} R_2 \geqslant L_1. \quad (26)
\end{aligned}$$

Hence, for any $(\Phi, \Psi) \in \mathbb{P} \cap \partial\Omega_2$, $t \in [\varepsilon_1, \varepsilon_2]$, using (25) and (26), we observe

$$\mathscr{F}(t, [\Phi(t) - \varpi_1(t)]^*, [\Psi(t) - \varpi_2(t)]^*) \geqslant N_1([\Phi(t) - \varpi_1(t)]^* + [\Psi(t) - \varpi_2(t)]^*). \quad (27)$$

For any $(\Phi, \Psi) \in \mathbb{P} \cap \partial\Omega_2$ and $t \in [\varepsilon_1, \varepsilon_2]$, then, by (24), (26) and (27), we derive

$$\begin{aligned}
& \mathbf{T}_1(\Phi, \Psi)(t) + \mathbf{T}_2(\Phi, \Psi)(t) \\
& \geqslant \lambda \int_a^b K_1(t,s)(\mathscr{F}(s, [\Phi(s) - \varpi_1(s)]^*, [\Psi(s) - \varpi_2(s)]^*) + \mathscr{Q}_1(s)) \frac{ds}{s^{1-\rho}} \\
& \quad + \lambda \int_a^b H_2(t,s)(\mathscr{F}(s, [\Phi(s) - \varpi_1(s)]^*, [\Psi(s) - \varpi_2(s)]^*) + \mathscr{Q}_1(s)) \frac{ds}{s^{1-\rho}} \\
& \geqslant 2\lambda \ell_1 \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} \int_a^b \sigma_\alpha(s) \mathscr{F}(s, [\Phi(s) - \varpi_1(s)]^*, [\Psi(s) - \varpi_2(s)]^*) \frac{ds}{s^{1-\rho}} \\
& \geqslant 2\lambda \ell_1 \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} \int_{\varepsilon_1}^{\varepsilon_2} \sigma_\alpha(s) \mathscr{F}(s, [\Phi(s) - \varpi_1(s)]^*, [\Psi(s) - \varpi_2(s)]^*) \frac{ds}{s^{1-\rho}}
\end{aligned}$$

$$\geqslant 2\lambda\ell_1\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha-1}\int_{\varepsilon_1}^{\varepsilon_2}\sigma_\alpha(s)N_1([\Phi(t)-\varpi_1(t)]^*+[\Psi(t)-\varpi_2(t)]^*)\frac{ds}{s^{1-\rho}}$$

$$\geqslant \lambda\frac{\ell\ell_2}{2}\Big(\frac{\varepsilon_1^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{2(\alpha-1)}N_1R_2\int_{\varepsilon_1}^{\varepsilon_2}\sigma_\alpha(s)\frac{ds}{s^{1-\rho}}\geqslant R_2=\|(\Phi,\Psi)\|_1.$$

Therefore, we can get

$$\|\mathbf{T}(\Phi,\Psi)\|_1\geqslant\|(\Phi,\Psi)\|_1\ \text{ for }\ \forall(\Phi,\Psi)\in\mathbb{P}\cap\partial\Omega_1. \tag{28}$$

Obviously, when $\underline{\mathscr{G}}(\infty)=+\infty$, the equation (28) holds still.

It follows from (23), (28) and Lemma 11 that $\mathbf{T}$ has a fixed point $(\widehat{\Phi},\widehat{\Psi})\in\mathbb{P}$, and $R_1\leqslant\|(\widehat{\Phi},\widehat{\Psi})\|_1\leqslant R_2$. Since $\|(\widehat{\Phi},\widehat{\Psi})\|_1\geqslant R_1$, so there exists a component $\widehat{\Phi}$ or $\widehat{\Psi}$ such that $\|\widehat{\Phi}\|\geqslant R_1/2$ or $\|\widehat{\Psi}\|\geqslant R_1/2$. Without loss of generality, we may suppose $\|\widehat{\Phi}\|\geqslant R_1/2$, by (16), we obtain

$$\begin{aligned}
\widehat{\Phi}(t)-\varpi_1(t)&\geqslant\widehat{\Phi}(t)-\lambda\ell_2\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha-1}\int_a^b(\mathscr{Q}_1(s)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\\
&\geqslant\widehat{\Phi}(t)-\frac{\ell_2\widehat{\Phi}(t)}{\ell\|\widehat{\Phi}\|}\int_a^b(\mathscr{Q}_1(s)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\\
&\geqslant\Big(1-\frac{2\ell_2}{\ell R_1}\int_a^b(\mathscr{Q}_1(s)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\Big)\widehat{\Phi}(t)\\
&\geqslant\Big(1-\frac{2\ell_2}{\ell R_1}\int_a^b(\mathscr{Q}_1(s)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\Big)\ell\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha-1}\|\widehat{\Phi}\|\\
&\geqslant\Big(1-\frac{2\ell_2}{\ell R_1}\int_a^b(\mathscr{Q}_1(s)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\Big)\frac{1}{2}\ell R_1\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha-1}\\
&=w_1\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha-1},\quad t\in[a,b],
\end{aligned}$$

where $w_1=\frac{1}{2}\ell R_1(1-\frac{2\ell_2}{\ell R_1}\int_a^b(\mathscr{Q}_1(s)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}})>0$. From the boundary conditions of singular BVP (17), we have $\widehat{\Psi}(b)=b\widehat{\Phi}(\eta)>0$, and so we have

$$\|\widehat{\Psi}\|\geqslant\widehat{\Psi}(b)=q\widehat{\Phi}(\eta)\geqslant q\ell\Big(\frac{\eta^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha-1}\|\widehat{\Phi}\|\geqslant\frac{1}{2}q\ell\Big(\frac{\eta^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha-1}R_1,$$

this together with (16), we get

$$\begin{aligned}\widehat{\Psi}(t)-\varpi_2(t)\geqslant&\widehat{\Psi}(t)-\lambda\ell\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\beta-1}\int_a^b(\mathscr{Q}_1(s)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\\ \geqslant&\widehat{\Psi}(t)-\frac{\ell_2\widehat{\Psi}(t)}{\ell\|\widehat{\Psi}\|}\int_a^b(\mathscr{Q}_1(s)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\\ \geqslant&\bigg(1-\frac{2\ell_2}{q\ell^2R_1}\Big(\frac{b^\rho-a^\rho}{\eta^\rho-a^\rho}\Big)^{\alpha-1}\int_a^b(\mathscr{Q}_1(s)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\bigg)\widehat{\Psi}(t)\\ \geqslant&\tilde{w}_2\ell\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\beta-1}\|\widehat{\Psi}\|\\ \geqslant&\tilde{w}_2\frac{1}{2}q\ell^2\Big(\frac{\eta^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha-1}R_1\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\beta-1}\\ =&w_2\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\beta-1},\quad t\in[a,b],\end{aligned}$$

where $w_2=\frac{1}{2}q\ell^2(\frac{\eta^\rho-a^\rho}{b^\rho-a^\rho})^{\alpha-1}R_1\tilde{w}_2>0$ and $\tilde{w}_2=1-\frac{2\ell_2}{q\ell^2R_1}(\frac{b^\rho-a^\rho}{\eta^\rho-a^\rho})^{\alpha-1}\int_a^b(\mathscr{Q}_1(s)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}$.

Let $\hat{\phi}(t)=\widehat{\Phi}(t)-\varpi_1(t)$, $\hat{\psi}(t)=\widehat{\Psi}(t)-\varpi_2(t)$, and $\hat{w}=\min\{w_1,w_2\}$, then we have

$$\hat{\phi}(t)\geqslant\hat{w}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha-1}>0\ \text{ and }\ \hat{\psi}(t)\geqslant\hat{w}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\beta-1}>0\ \text{ for }\ \forall t\in(a,b].$$

It follows Lemma 11 that we know the NEP (1) has at least one positive solution $(\hat{\phi},\hat{\psi})$ satisfying $\hat{\phi}(t)\geqslant\hat{w}(\frac{t^\rho-a^\rho}{b^\rho-a^\rho})^{\alpha-1}$ and $\hat{\psi}(t)\geqslant\hat{w}(\frac{t^\rho-a^\rho}{b^\rho-a^\rho})^{\beta-1}$ for any $t\in[a,b]$. The proof Theorem 1 is completed. □

THEOREM 2. *Under the validity of condition* (H)*, the following condition* (C_2) *holds:* $\overline{\mathscr{F}_\star}(\infty)=\overline{\mathscr{G}_\star}(\infty)=0$ *and* $\widetilde{\mathscr{F}}(\infty)>\Lambda$ *or* $\widetilde{\mathscr{G}}(\infty)>\Lambda$*, where*

$$\Lambda=\frac{\max\{2,\frac{2}{p\ell}(\frac{b^\rho-a^\rho}{\xi^\rho-a^\rho})^{\beta-1},\frac{2}{q\ell}(\frac{b^\rho-a^\rho}{\eta^\rho-a^\rho})^{\alpha-1}\}\cdot\int_a^b(\mathscr{Q}_1(s)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}}{\ell^2\cdot\min\{(\frac{\varepsilon_1^\rho-a^\rho}{b^\rho-a^\rho})^{\alpha-1}\int_{\varepsilon_1}^{\varepsilon_2}\sigma_\alpha(s)\frac{ds}{s^{1-\rho}},(\frac{\varepsilon_1^\rho-a^\rho}{b^\rho-a^\rho})^{\beta-1}\int_{\varepsilon_1}^{\varepsilon_2}\sigma_\beta(s)\frac{ds}{s^{1-\rho}}\}}.$$

Then there exists $\tilde{\lambda}$ *such that NEP* (1) *has at least one positive solution* $(\tilde{\phi},\tilde{\psi})$ *for any* $\lambda\in(\tilde{\lambda},+\infty)$*. Furthermore,* $(\tilde{\phi},\tilde{\psi})$ *satisfies* $\tilde{\phi}(t)\geqslant\tilde{w}(\frac{t^\rho-a^\rho}{b^\rho-a^\rho})^{\alpha-1}$ *and* $\tilde{\psi}(t)\geqslant\tilde{w}(\frac{t^\rho-a^\rho}{b^\rho-a^\rho})^{\beta-1}$ *for* $t\in[a,b]$ *and some positive constant* $\tilde{w}$*.*

Proof. By virtue of the inequality $\widetilde{\mathscr{F}}(\infty) > \Lambda$, there exists positive constant L_1 such that, for any $t \in [\varepsilon_1, \varepsilon_2]$, $\phi, \psi \geqslant 0$, $\phi + \psi \geqslant L_1$,

$$\mathscr{F}(t,\phi,\psi) \geqslant \Lambda = \frac{\max\{2, \frac{2}{p\ell}(\frac{b^\rho - a^\rho}{\xi^\rho - a^\rho})^{\beta-1}, \frac{2}{q\ell}(\frac{b^\rho - a^\rho}{\eta^\rho - a^\rho})^{\alpha-1}\} \cdot \int_a^b (\mathscr{Q}_1(s) + \mathscr{Q}_2(s)) \frac{ds}{s^{1-\rho}}}{\ell^2 \cdot \min\{(\frac{\varepsilon_1^\rho - a^\rho}{b^\rho - a^\rho})^{\alpha-1} \int_{\varepsilon_1}^{\varepsilon_2} \sigma_\alpha(s) \frac{ds}{s^{1-\rho}}, (\frac{\varepsilon_1^\rho - a^\rho}{b^\rho - a^\rho})^{\beta-1} \int_{\varepsilon_1}^{\varepsilon_2} \sigma_\beta(s) \frac{ds}{s^{1-\rho}}\}}. \tag{29}$$

Let $\lambda > \tilde{\lambda}$, where

$$\tilde{\lambda} = \frac{\max\{L_1, L_1\ell^2(\frac{\xi^\rho - a^\rho}{b^\rho - a^\rho})^{\beta-1}, L_1\ell^2(\frac{\eta^\rho - a^\rho}{b^\rho - a^\rho})^{\alpha-1}\}}{\ell_2 \varepsilon_\alpha^\beta \int_a^b (\mathscr{Q}_1(s) + \mathscr{Q}_2(s)) \frac{ds}{s^{1-\rho}}},$$

$$\varepsilon_\alpha^\beta = \min\left\{\left(\frac{\varepsilon_1^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\alpha-1}, \left(\frac{\varepsilon_1^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\beta-1}\right\}.$$

Let $R_1 = \max\{\frac{4\lambda\ell_2}{\ell}, \frac{4\lambda\ell_2}{p\ell^2}(\frac{b^\rho - a^\rho}{\xi^\rho - a^\rho})^{\beta-1}, \frac{4\lambda\ell_2}{q\ell^2}(\frac{b^\rho - a^\rho}{\eta^\rho - a^\rho})^{\alpha-1}\} \int_a^b (\mathscr{Q}_1(s) + \mathscr{Q}_2(s)) \frac{ds}{s^{1-\rho}}$, and $\Omega_1 = \{(\Phi, \Psi) \in \mathbb{E} : \|(\Phi, \Psi)\|_1 < R_1\}$. For any $(\Phi, \Psi) \in \mathbb{P} \cap \partial\Omega_1$, similar to Theorem 1, we may suppose $\|\Phi\| \geqslant R_1/2$, then, for $t \in [a, b]$,

$$\begin{aligned}
\Phi(t) - \varpi_1(t) \geqslant & \ell\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} \|\Phi\| - \lambda\ell_2\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} \int_a^b (\mathscr{Q}_1(s) + \mathscr{Q}_2(s)) \frac{ds}{s^{1-\rho}} \\
\geqslant & \frac{1}{2}\ell\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} R_1 - \lambda\ell_2\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} \int_a^b (\mathscr{Q}_1(s) + \mathscr{Q}_2(s)) \frac{ds}{s^{1-\rho}} \\
\geqslant & \frac{1}{2}\ell\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} \cdot \frac{4\lambda\ell_2}{\ell} \int_a^b (\mathscr{Q}_1(s) + \mathscr{Q}_2(s)) \frac{ds}{s^{1-\rho}} \\
& - \lambda\ell_2\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} \int_a^b (\mathscr{Q}_1(s) + \mathscr{Q}_2(s)) \frac{ds}{s^{1-\rho}} \\
= & \lambda\ell_2\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} \int_a^b (\mathscr{Q}_1(s) + \mathscr{Q}_2(s)) \frac{ds}{s^{1-\rho}} \\
\geqslant & \frac{L_1}{\varepsilon_\alpha^\beta}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1}.
\end{aligned}$$

Thus, for any $(\Phi, \Psi) \in \mathbb{P} \cap \partial\Omega_1$ and $t \in [\varepsilon_1, \varepsilon_2]$, we derive

$$\begin{aligned}
[\Phi(t) - \varpi_1(t)]^* + [\Psi(t) - \varpi_2(t)]^* \geqslant & [\Phi(t) - \varpi_1(t)]^* = \Phi(t) - \varpi_1(t) \\
\geqslant & \frac{L_1}{\varepsilon_\alpha^\beta}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} \geqslant L_1.
\end{aligned} \tag{30}$$

In conjunction with the established results (29) and (30), for any $(\Phi,\Psi)\in\mathbb{P}\cap\partial\Omega_1$ and $t\in[\varepsilon_1,\varepsilon_2]$, we obtain

$$
\begin{aligned}
&\mathbf{T}_1(\Phi,\Psi)(t)+\mathbf{T}_2(\Phi,\Psi)(t)\\
&\geqslant\lambda\int_a^b K_1(t,s)(\mathscr{F}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}\\
&\quad+\lambda\int_a^b H_2(t,s)(\mathscr{F}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}\\
&\geqslant 2\lambda\ell_1\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha-1}\int_a^b\sigma_\alpha(s)\mathscr{F}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*)\frac{ds}{s^{1-\rho}}\\
&\geqslant 2\lambda\ell_1\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha-1}\int_{\varepsilon_1}^{\varepsilon_2}\sigma_\alpha(s)\mathscr{F}(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*)\frac{ds}{s^{1-\rho}}\\
&\geqslant 2\lambda\ell_1\Big(\frac{\varepsilon_1^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha-1}\\
&\quad\cdot\int_{\varepsilon_1}^{\varepsilon_2}\sigma_\alpha(s)\frac{\max\{2,\frac{2}{p\ell}(\frac{b^\rho-a^\rho}{\xi^\rho-a^\rho})^{\beta-1},\frac{2}{q\ell}(\frac{b^\rho-a^\rho}{\eta^\rho-a^\rho})^{\alpha-1}\}\cdot\int_a^b(\mathscr{Q}_1(s)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}}{\ell^2\cdot\min\{(\frac{\varepsilon_1^\rho-a^\rho}{b^\rho-a^\rho})^{\alpha-1}\int_{\varepsilon_1}^{\varepsilon_2}\sigma_\alpha(s)\frac{ds}{s^{1-\rho}},(\frac{\varepsilon_1^\rho-a^\rho}{b^\rho-a^\rho})^{\beta-1}\int_{\varepsilon_1}^{\varepsilon_2}\sigma_\beta(s)\frac{ds}{s^{1-\rho}}\}}\frac{ds}{s^{1-\rho}}\\
&\geqslant\max\left\{\frac{4\lambda\ell_2}{\ell},\frac{4\lambda\ell_2}{p\ell^2}\Big(\frac{b^\rho-a^\rho}{\xi^\rho-a^\rho}\Big)^{\beta-1},\frac{4\lambda\ell_2}{q\ell^2}\Big(\frac{b^\rho-a^\rho}{\eta^\rho-a^\rho}\Big)^{\alpha-1}\right\}\cdot\int_a^b(\mathscr{Q}_1(s)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\\
&=R_1=\|(\Phi,\Psi)\|_1.
\end{aligned}
$$

Accordingly, we have

$$\|\mathbf{T}(\Phi,\Psi)\|_1\geqslant\|(\Phi,\Psi)\|_1\ \text{ for }\ \forall(\Phi,\Psi)\in\mathbb{P}\cap\partial\Omega_1. \tag{31}$$

On the other hand, let $\varepsilon=\min\{(8\lambda\ell_2\int_a^b\sigma_\alpha(s)\mathscr{P}_1(s)\frac{ds}{s^{1-\rho}})^{-1},(8\lambda\ell_2\int_a^b\sigma_\beta(s)\mathscr{P}_2(s)\frac{ds}{s^{1-\rho}})^{-1}\}$. It follows from $\overline{\mathscr{F}_\star}(\infty)=\overline{\mathscr{G}_\star}(\infty)=0$ that there exists a positive constant L_2 such that for any $t\in[a,b]$, we have for $\phi,\psi\geqslant 0$ and $\phi+\psi\geqslant L_2$,

$$\mathscr{F}_\star(t,\phi,\psi)\leqslant\varepsilon(\phi+\psi)\ \text{ and }\ \mathscr{G}_\star(t,\phi,\psi)\leqslant\varepsilon(\phi+\psi),$$

which yields the following results

$$\mathscr{F}_\star(t,\phi,\psi)\leqslant\mathscr{F}_\star^{L_2}+\varepsilon(\phi+\psi),\ \mathscr{G}_\star(t,\phi,\psi)\leqslant\mathscr{G}_\star^{L_2}+\varepsilon(\phi+\psi)\ \text{for}\ \phi,\psi\geqslant 0\ \text{and}\ t\in[a,b]. \tag{32}$$

Set $R_2=\max\{2R_1,8\lambda(\mathscr{F}_\star^{L_2}+1)\vartheta_\alpha,8\lambda(\mathscr{G}_\star^{L_2}+1)\vartheta_\beta\}$ and $\Omega_2=\{(\Phi,\Psi)\in\mathbb{E}:\|(\Phi,\Psi)\|_1<R_2\}$. For any $(\Phi,\Psi)\in\mathbb{P}\cap\partial\Omega_2$, by (20) and (32), we have

$$
\begin{aligned}
&\mathbf{T}_1(\Phi,\Psi)(t)\\
&\leqslant\lambda\ell_2\bigg(\int_a^b\sigma_\alpha(s)(\mathscr{P}_1(s)\mathscr{F}_\star(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\varpi_2(s)]^*)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}\\
&\quad+\int_a^b\sigma_\beta(s)(\mathscr{P}_2(s)\mathscr{G}_\star(s,[\Phi(s)-\varpi_1(s)]^*,[\Psi(s)-\omega_2(s)]^*)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\bigg)\\
&\leqslant\lambda\ell_2\bigg(\int_a^b\sigma_\alpha(s)(\mathscr{P}_1(s)(\mathscr{F}_\star^{L_2}+\varepsilon[\Phi(s)-\varpi_1(s)]^*+\varepsilon[\Psi(s)-\varpi_2(s)]^*)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}\\
&\quad+\int_a^b\sigma_\beta(s)(\mathscr{P}_2(s)(\mathscr{G}_\star^{L_2}+\varepsilon[\Phi(s)-\varpi_1(s)]^*+\varepsilon[\Psi(s)-\varpi_2(s)]^*)+\mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}}\bigg)\\
&\leqslant\lambda(\mathscr{F}_\star^{L_2}+1)\ell_2\int_a^b\sigma_\alpha(s)(\mathscr{P}_1(s)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}\\
&\quad+\lambda\ell_2\varepsilon(\|\Phi\|+\|\Psi\|)\int_a^b\sigma_\alpha(s)\mathscr{P}_1(s)\frac{ds}{s^{1-\rho}}\\
&\quad+\lambda(\mathscr{G}_\star^{L_2}+1)\ell_2\int_a^b\sigma_\beta(s)(\mathscr{P}_1(s)+\mathscr{Q}_1(s))\frac{ds}{s^{1-\rho}}\\
&\quad+\lambda\ell_2\varepsilon(\|\Phi\|+\|\Psi\|)\int_a^b\sigma_\beta(s)\mathscr{P}_2(s)\frac{ds}{s^{1-\rho}}\\
&\leqslant\lambda(\mathscr{F}_\star^{L_2}+1)\vartheta_\alpha+\lambda\ell_2\varepsilon R_2\int_a^b\sigma_\alpha(s)\mathscr{P}_1(s)\frac{ds}{s^{1-\rho}}\\
&\quad+\lambda(\mathscr{G}_\star^{L_2}+1)\vartheta_\beta+\lambda\ell_2\varepsilon R_2\int_a^b\sigma_\beta(s)\mathscr{P}_2(s)\frac{ds}{s^{1-\rho}}\\
&=\frac{R_2}{8}+\frac{R_2}{8}+\frac{R_2}{8}+\frac{R_2}{8}=\frac{R_2}{2}=\frac{\|(\Phi,\Psi)\|_1}{2}.
\end{aligned}
$$

Similarly, we derive $\|\mathbf{T}_2(\Phi,\Psi)\|\leqslant\|(\Phi,\Psi)\|_1/2$. Thus we obtain

$$\|\mathbf{T}(\Phi,\Psi)\|_1=\|\mathbf{T}_1(\Phi,\Psi)\|+\|\mathbf{T}_2(\Phi,\Psi)\|\leqslant\|(\Phi,\Psi)\|_1\ \text{ for }\ \forall(x,y)\in\mathbb{P}\cap\partial\Omega_2.\quad(33)$$

It follows from (31), (33) and Lemma 11 that $\mathbf{T}$ has a fixed point $(\widetilde{\Phi},\widetilde{\Psi})\in\mathbb{P}$ satisfying $R_1\leqslant\|(\widetilde{\Phi},\widetilde{\Psi})\|_1\leqslant R_2$. Given the condition $\|(\widetilde{\Phi},\widetilde{\Psi})\|_1\geqslant R_1$, then there exists a component $\widetilde{\Phi}$ or $\widetilde{\Psi}$ such that $\|\widetilde{\Phi}\|\geqslant R_1/2$ or $\|\widetilde{\Psi}\|\geqslant R_1/2$. We can assume, preserving

generality, that $\|\widetilde{\Phi}\| \geqslant R_1/2$, we observe

$$\widetilde{\Phi}(t) - \varpi_1(t) \geqslant \frac{L_1}{\varepsilon_\alpha^\beta}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} = \tilde{m}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} \quad \text{for } t \in [a,b],$$

where $\tilde{w} = L_1/\varepsilon_\alpha^\beta > 0$. From the boundary conditions of singular BVP (17), we have $\widetilde{\Psi}(e) = q\widetilde{\Phi}(\eta) > 0$, and so we have

$$\|\widetilde{\Psi}\| \geqslant \widetilde{\Psi}(b) = q\widetilde{\Phi}(\eta) \geqslant q\ell\Big(\frac{\eta^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1}\|\widetilde{\Phi}\| \geqslant \frac{1}{2}q\ell\Big(\frac{\eta^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1}R_1.$$

Combining this result with (16) yields

$$\begin{aligned}
\widetilde{\Psi}(t) - \varpi_2(t) \geqslant& \widetilde{\Psi}(t) - \lambda\ell_2\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\beta-1}\int_a^b (\mathscr{Q}_1(s) + \mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}} \\
\geqslant& \frac{1}{2}q\ell^2\Big(\frac{\eta^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\beta-1}R_1 \\
&- \lambda\ell_2\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\beta-1}\int_a^b (\mathscr{Q}_1(s) + \mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}} \\
\geqslant& \frac{1}{2}q\ell^2\Big(\frac{\eta^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\beta-1}\frac{4\lambda\ell_2}{q\ell^2}\Big(\frac{b^\rho - a^\rho}{\eta^\rho - a^\rho}\Big)^{\alpha-1} \\
&\cdot\int_a^b (\mathscr{Q}_1(s) + \mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}} - \lambda\ell_2\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\beta-1}\int_a^b (\mathscr{Q}_1(s) + \mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}} \\
\geqslant& \lambda\ell_2\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\beta-1}\int_a^b (\mathscr{Q}_1(s) + \mathscr{Q}_2(s))\frac{ds}{s^{1-\rho}} \\
\geqslant& \frac{L_1}{\varepsilon_\alpha^\beta}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\beta-1} = \tilde{w}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\beta-1}, \quad t \in [a,b].
\end{aligned}$$

Let $\tilde{\phi}(t) = \widetilde{\Phi}(t) - \varpi_1(t)$ and $\tilde{\psi}(t) = \widetilde{\Psi}(t) - \varpi_2(t)$, then we acquire

$$\tilde{\phi}(t) \geqslant \tilde{w}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} > 0 \text{ and } \tilde{\psi}(t) \geqslant \tilde{w}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\beta-1} > 0 \text{ for } t \in (a,b].$$

It follows Lemma 11 that we can quickly observe that the NEP (1) has at least one positive solution $(\tilde{\phi}, \tilde{\psi})$ satisfying $\tilde{\phi}(t) \geqslant \tilde{w}(\frac{t^\rho - a^\rho}{b^\rho - a^\rho})^{\alpha-1}$ and $\tilde{\psi}(t) \geqslant \tilde{w}(\frac{t^\rho - a^\rho}{b^\rho - a^\rho})^{\beta-1}$ for any $t \in [a,b]$. The proof Theorem 2 is completed. □

Combined with the proof of Theorem 2, we have the following corollary.

COROLLARY 1. *The conclusion of Theorem 2 holds if* (C_2) *is exchanged for the condition* (C_2')*:* $\overline{\mathscr{F}_\star}(\infty) = \overline{\mathscr{G}_\star}(\infty) = 0$ *and* $\widetilde{\mathscr{F}}(\infty) = +\infty$ *or* $\widetilde{\mathscr{G}}(\infty) = +\infty$.

4. Two examples

EXAMPLE 1. Consider the following singular NEP

$$\begin{cases} \mathscr{D}_{4+}^{\frac{5}{2},\frac{1}{2}}\phi(t)+\lambda\left(\dfrac{2(\phi+\psi)^2}{\sqrt{(\sqrt{t}-2)(3-\sqrt{t})}}+\log(\sqrt{t}-2)\right)=0, \quad t\in(4,9), \\ \mathscr{D}_{4+}^{\frac{5}{2},\frac{1}{2}}\psi(t)+\lambda\left(\dfrac{2\log(1+\phi+\psi)}{\sqrt{(\sqrt{t}-2)(3-\sqrt{t})}}+\log(3-\sqrt{t})\right)=0, \quad t\in(4,9), \\ \gamma^i\phi(4)=\gamma^i\psi(4)=0,\ i,j=1,2,\ \ \phi(9)=\dfrac{2}{3}\psi\Big(\dfrac{81}{16}\Big),\ \psi(9)=4\sqrt{3}\phi\Big(\dfrac{121}{16}\Big), \end{cases} \tag{34}$$

where λ is a positive parameter. We take $\alpha=\beta=5/2$, $\rho=1/2$, $a=4$, $b=9$, $p=2/3$, $\xi=81/16$, $q=4\sqrt{3}$, $\eta=121/16$, then $C(\alpha,\beta)=(b^\rho-a^\rho)^{\alpha+\beta-2}-pq(\eta^\rho-a^\rho)^{\alpha-1}(\xi^\rho-a^\rho)^{\beta-1}=5/8>0$. For $\phi,\psi\geqslant 0$ and $t\in(4,9)$, let

$$\begin{aligned} \mathscr{F}(t,\phi,\psi)=&\frac{2(\phi+\psi)^2}{\sqrt{(\sqrt{t}-2)(3-\sqrt{t})}}+\log(\sqrt{t}-2), \\ \mathscr{G}(t,\phi,\psi)=&\frac{2\log(1+\phi+\psi)}{\sqrt{(\sqrt{t}-2)(3-\sqrt{t})}}+\log(3-\sqrt{t}). \end{aligned}$$

Taking $\mathscr{F}_\star(t,\phi,\psi)=(\phi+\psi)^2$, $\mathscr{G}_\star(t,\phi,\psi)=\log(1+\phi+\psi)$, $\mathscr{P}_1(t)=\mathscr{P}_2(t)=\frac{2}{\sqrt{(\sqrt{t}-2)(3-\sqrt{t})}}$, $\mathscr{Q}_1(t)=-\log(\sqrt{t}-2)$, $\mathscr{Q}_2(t)=-\log(3-\sqrt{t})$, then, for $\phi,\psi\geqslant 0$ and $t\in(4,9)$, we can observe

$$-\mathscr{Q}_1(t)\leqslant\mathscr{F}(t,\phi,\psi)\leqslant\mathscr{P}_1(t)\mathscr{F}_\star(t,\phi,\psi),\ \ -\mathscr{Q}_2(t)\leqslant\mathscr{G}(t,\phi,\psi)\leqslant\mathscr{P}_2(t)\mathscr{G}_\star(t,\phi,\psi).$$

Through straightforward computation, we obtain

$$\int_4^9\mathscr{P}_1(s)\frac{ds}{\sqrt{s}}=\int_4^9\mathscr{P}_2(s)\frac{ds}{\sqrt{s}}=4\pi, \quad \int_4^9\mathscr{Q}_1(s)\frac{ds}{\sqrt{s}}=\int_4^9\mathscr{Q}_2(s)\frac{ds}{\sqrt{s}}=2.$$

The previous results confirm the validity of condition (H). Furthermore, we choose $[\varepsilon_1,\varepsilon_2]\in(4,9)$. It is straightforward to verify that $\underline{\mathscr{F}}(\infty)=+\infty$, that is, the condition (C_1) of Theorem 1 is fulfilled. Then by applying Theorem 1, the NEP (34) has at least one positive solution whenever $\lambda>0$ is sufficiently small.

EXAMPLE 2. Consider the following singular NEP

$$\begin{cases} \mathscr{D}_{4+}^{\frac{9}{4},\frac{1}{2}}\phi(t)+\lambda\left(\dfrac{3\sqrt{\phi+\psi}\mathrm{e}^{2(\phi+\psi)}}{\sqrt[4]{(\sqrt{t}-2)^3(3-\sqrt{t})}[1+(\sqrt{t}-2)(3-\sqrt{t})\mathrm{e}^{\phi+\psi}+\mathrm{e}^{2(\phi+\psi)}]}\right. \\ \qquad\qquad \left.-\dfrac{2}{\sqrt{\sqrt{t}-2}}\right)=0, \quad t\in(4,9), \\ \mathscr{D}_{4+}^{\frac{9}{4},\frac{1}{2}}\psi(t)+\lambda\left(\dfrac{2\sqrt{\phi+\psi}[(3-\sqrt{t})^2+\tanh(\phi+\psi)]}{\sqrt[4]{(\sqrt{t}-2)(3-\sqrt{t})^3}}-\dfrac{3}{\sqrt{3-\sqrt{t}}}\right)=0, \\ \gamma^i\phi(4)=\gamma^i\psi(4)=0,\ i,j=1,2,\ \ \phi(9)=2\psi\Big(\dfrac{81}{16}\Big),\ \psi(9)=\sqrt[4]{27}\phi\Big(\dfrac{121}{16}\Big), \end{cases} \tag{35}$$

where λ is a positive parameter. We take $\alpha=\beta=9/4$, $\rho=1/2$, $a=4$, $b=9$, $p=2$, $\xi=81/16$, $q=\sqrt[4]{27}$, $\eta=121/16$, then $C(\alpha,\beta)=(b^\rho-a^\rho)^{\alpha+\beta-2}-pq(\eta^\rho-a^\rho)^{\alpha-1}(\xi^\rho-a^\rho)^{\beta-1}=7/16>0$. For $\phi,\psi\geqslant 0$ and $t\in(4,9)$, let

$$\begin{aligned} \mathscr{F}(t,\phi,\psi)=&\frac{3\sqrt{\phi+\psi}\mathrm{e}^{2(\phi+\psi)}}{\sqrt[4]{(\sqrt{t}-2)^3(3-\sqrt{t})}[1+(\sqrt{t}-2)(3-\sqrt{t})\mathrm{e}^{\phi+\psi}+\mathrm{e}^{2(\phi+\psi)}]}-\frac{2}{\sqrt{\sqrt{t}-2}},\\ \mathscr{G}(t,\phi,\psi)=&\frac{2\sqrt{\phi+\psi}[(3-\sqrt{t})^2+\tanh(\phi+\psi)]}{\sqrt[4]{(\sqrt{t}-2)(3-\sqrt{t})^3}}-\frac{3}{\sqrt{3-\sqrt{t}}}. \end{aligned}$$

Taking $\mathscr{F}_\star(t,\phi,\psi)=\frac{3\sqrt{\phi+\psi}\mathrm{e}^{2(\phi+\psi)}}{1+(\sqrt{t}-2)(3-\sqrt{t})\mathrm{e}^{\phi+\psi}+\mathrm{e}^{2(\phi+\psi)}}$, $\mathscr{G}_\star(t,\phi,\psi)=\sqrt{\phi+\psi}[(3-\sqrt{t})^2+\tanh(\phi+\psi)]$, $\mathscr{P}_1(t)=\frac{3}{\sqrt[4]{(\sqrt{t}-2)^3(3-\sqrt{t})}}$, $\mathscr{P}_2(t)=\frac{2}{\sqrt[4]{(\sqrt{t}-2)(3-\sqrt{t})^3}}$, $\mathscr{Q}_1(t)=\frac{2}{\sqrt{\sqrt{t}-2}}$, $\mathscr{Q}_2(t)=\frac{3}{\sqrt{3-\sqrt{t}}}$, then, for $\phi,\psi\geqslant 0$ and $t\in(4,9)$, we have

$$-\mathscr{Q}_1(t)\leqslant\mathscr{F}(t,\phi,\psi)\leqslant\mathscr{P}_1(t)\mathscr{F}_\star(t,\phi,\psi),\ \ -\mathscr{Q}_2(t)\leqslant\mathscr{G}(t,\phi,\psi)\leqslant\mathscr{P}_2(t)\mathscr{G}_\star(t,\phi,\psi).$$

A direct computational approach reveals that

$$\int_4^9\mathscr{P}_1(s)\frac{ds}{\sqrt{s}}=6\sqrt{2}\pi,\ \int_4^9\mathscr{P}_2(s)\frac{ds}{\sqrt{s}}=4\sqrt{2}\pi,\ \int_4^9\mathscr{Q}_1(s)\frac{ds}{\sqrt{s}}=8,\ \int_4^9\mathscr{Q}_2(s)\frac{ds}{\sqrt{s}}=12.$$

The preceding results demonstrate that condition (H) holds. Moreover, we choose $[\varepsilon_1,\varepsilon_2]\in(4,9)$. It can be easily verified that $\overline{\mathscr{F}_\star}(\infty)=\overline{\mathscr{G}_\star}(\infty)=0$, $\widetilde{\mathscr{F}}(\infty)=+\infty$ or $\widetilde{\mathscr{G}}(\infty)=+\infty$, that is, the condition (C_2') of Corollary 1 is fulfilled. Then, from Corollary 1, the NEP (35) has at least one positive solution whenever $\lambda>0$ is sufficiently large.

5. Conclusion

In this article, the NEPs have been investigated for a class of singular Katugampola fractional differential systems with four-point coupled boundary conditions. First and foremost, the Green's functions of the above-mentioned NEPs and their fundamental

analytic properties have been systematically established. Subsequently, through the application of classical fixed-point theorems, some explicit eigenvalue interval-dependent criteria have been developed to guarantee the existence of at least one positive solution for the addressed NEPs. Finally, numerical examples have been implemented to demonstrate both the theoretical validity and computational effectiveness of these proposed criteria. Positive solutions to other BVPs of nonlinear Katugampola FDEs will be our future research topics.

REFERENCES

[1] B. Ahmad, S. K. Ntouyas, *Nonlinear systems of fractional differential equations*, Springer Cham, 2024.

[2] B. Basti, Y. Arioua, N. Benhamidouche, *Existence and uniqueness of solutions for nonlinear Katugampola fractional differential equations*, J. Math. Appl., **42** (2019), 35–61.

[3] S. Batik, F. Y. Deren, *Semipositone fractional boundary value problems with n point fractional integral boundary conditions*, Miskolc Math. Notes, **23** (1) (2022), 93–104.

[4] A. Bouchra, B. Tellab, K. Zennir, *Positive solutions for integral nonlinear boundary value problem in fractional Sobolev spaces*, Math. Meth. Appl. Sci., **46** (3) (2023), 3115–3131.

[5] H. Djourdem, S. Benaicha, *Positive solutions for fractional differential equations with non-separated type nonlocal multi-point and multi-term integral boundary conditions*, Studia Univer. Babes-Bolyai Mat., **66** (4) (2021), 691–708.

[6] D. Guo, V. Lakshmikantham, *Nonlinear problems in abstract cones*, New York: Academic Press, 1988.

[7] X. Hao, L. Zhang, L. Liu, *Positive solutions of higher order fractional integral boundary value problem with a parameter*, Nonlinear Anal. Model. Control, **24** (2) (2019), 210–223.

[8] J. Henderson, R. Luca, A. Tudorache, *On a system of Riemann-Liouville fractional boundary value problems with ρ-laplacian operators and positive parameters*, Fractal Fract., **6** (6) (2022), 299.

[9] F. Jarad, T. Abdeljawad, D. Baleanu, *On the generalized fractional derivatives and their Caputo modification*, J. Nonlinear Sci. Appl., **10** (2017), 2607–2619.

[10] J. Jiang, L. Liu, Y. Wu, *Positive solutions to singular fractional differential system with coupled boundary conditions*, Commun. Nonlinear Sci. Numer. Simul., **18** (11) (2013), 3061–3074.

[11] U. N. Katugampola, *New approach to a generalized fractional integral*, Appl. Math. Comput., **218** (2011), 860–865.

[12] U. N. Katugampola, *A new approach to generalized fractional derivatives*, Bull. Math. Anal. Appl., **6** (2014), 1–15.

[13] Y. Li, W. Cheng, J. Xu, *Monotone iterative schemes for positive solutions of a fractional differential system with integral boundary conditions on an infinite interval*, Filomat, **34** (13) (2020), 4399–4417.

[14] R. Luca, A. Tudorache, *On a system of Hadamard fractional differential equations with nonlocal boundary conditions on an infinite interval*, Fractal Fract., **7** (6) (2023), 458.

[15] B. Łupińska, T. Odzijewicz, *A Lyapunov-type inequality with the Katugampola fractional derivative*, Math. Meth. Appl. Sci., **41** (2018), 8985–8996.

[16] B. Łupińska, E. Schmeidel, *Analysis of some Katugampola fractional differential equations with fractional boundary conditions*, Math. Biosci. Eng., **18** (6) (2021), 7269–7279.

[17] K. Neamprem, T. Muensawat, S. K. Ntouyas, J. Tariboon, *Positive solutions for fractional differential systems with nonlocal Riemann-Liouville fractional integral boundary conditions*, Positivity, **21** (3) (2016), 825–845.

[18] S. N. Rao, *Multiplicity of positive solutions for fractional differential equation with p-Laplacian boundary value problems*, Int. J. Differ. Equ., **2016** (2016), 6906049.

[19] A. Souahi, A. Guezane-Lakoud, A. Hitta, *Positive Solutions for Higher-Order Nonlinear Fractional Differential Equations*, Vietnam J. Math., **45** (3) (2016), 441–450.

[20] S. N. Srivastava, S. Pati, J. R. Graef, A. Domoshnitsky, S. Padhi, *Existence of solution for a Katugampola fractional differential equation using coincidence degree theory*, Mediterr. J. Math., **21** (2024), 123.

[21] M. SUBRAMANIAN, K. ABUASBEH, M. MANIGANDAN, *On the generalized Liouville-Caputo type fractional differential equations supplemented with Katugampola integral boundary conditions*, Symmetry, **14** (2022), 2273.
[22] J. TARIBOON, S.K. NTOUYAS, W. SUDSUTAD, *Positive solutions for fractional differential equations with three-point multi-term fractional integral boundary conditions*, Adv. Differ. Equ., **2014** (2014), 28.
[23] A. TUDORACHE, R. LUCA, *Positive solutions for a fractional differential equation with sequential derivatives and nonlocal boundary conditions*, Symmetry, **14** (9) (2022), 1779.
[24] Y. WANG, L. LIU, Y. WU, *Positive solutions for a class of higher-order singular semipositone fractional differential systems with coupled integral boundary conditions and parameters*, Adv. Differ. Equ., 2014 (2014), 268.
[25] W. YANG, *Monotone iterative technique for a coupled system of nonlinear Hadamard fractional differential equations*, J. Appl. Math. Comput., **59** (1) (2019), 585–596.
[26] W. YANG, *Positive solutions for a singular coupled system of nonlinear higher-order fractional q-difference boundary value problems with two parameters*, Differ. Equ. Appl., **11** (4) (2019), 509–529.
[27] W. YANG, *Positive solutions for a class of nonlinear p-Laplacian Hadamard fractional differential systems with coupled nonlocal Riemann-Stieltjes integral boundary conditions*, Filomat, **36** (19) (2022), 6631–6654,
[28] H. ZHANG, Z. HAO, M. BOHNER, *Multiple and nonexistence of positive solutions for a class of fractional differential equations with p-Laplacian operator*, Mathematics, **12** (23) (2024), 3869.
[29] W. ZHANG, C. ZHAI, *Positive solutions for a Riemann-Liouville fractional system with ρ-Laplacian operators*, Differ. Equ. Appl., **16** (4) (2024), 305–319.

(Received May 15, 2025)

Wengui Yang
Normal School
Sanmenxia Polytechnic
Sanmenxia 472000, China
e-mail: yangwg8088@163.com