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A STUDY OF IMPULSIVE FRACTIONAL DIFFERENTIAL INCLUSIONS WITH ANTI–PERIODIC BOUNDARY CONDITIONS

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Abstract. In this paper, we prove the existence of solutions for impulsive fractional differential inclusions with anti-periodic boundary conditions by applying Bohnenblust-Karlin's fixed point theorem.

1. Introduction

The subject of fractional differential equations has recently gained much importance and attention. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electro dynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order. For details and examples, see [2, 7, 16, 18, 24–26, 28, 30, 36, 37, 39] and the references therein.

The theory of impulsive differential equations of integer order has found its extensive applications in realistic mathematical modelling of a wide variety of practical situations and has emerged as an important area of investigation in recent years. For the general theory and applications of impulsive differential equations, we refer the reader to the references [27, 38, 40, 42]. For some recent work on impulsive fractional differential equations, see [3, 9].

Anti-periodic boundary value problems have recently received considerable attention as anti-periodic boundary conditions appear in numerous situations, for instance, see [1, 6, 15, 18, 19, 32, 33].

Differential inclusions arise in the mathematical modelling of certain problems in economics, optimal control, etc. and are widely studied by many authors, see [12, 20, 34, 41] and the references therein. For some recent development on differential inclusions, we refer the reader to the references [5, 11, 13–14, 22, 31, 35].

Periodic solutions of fractional systems have been considered by some authors [8, 21] and in a recent paper, Ahmad and Otero-Espinar [4] discussed the existence of solutions for fractional differential inclusions with anti-periodic boundary conditions.

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In this paper, we study the following impulsive differential inclusions of fractional order $q \in (1,2]$ with anti-periodic boundary conditions

$$\begin{cases} {}^cD^q x(t) \in F(t,x(t)), \quad t \in J_1 = [0,T] \setminus \{t_1,t_2,...,t_p\}, \quad T > 0, \\ \Delta x(t_k) = \mathscr{I}_k(x(t_k^-)), \quad \Delta x'(t_k) = \mathscr{J}_k(x(t_k^-)), \quad t_k \in (0,T), \;\; k = 1,2,...,p, \\ x(0) = -x(T), \quad x'(0) = -x'(T), \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $F : [0,T] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$, where $\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $\mathscr{I}_k$, $\mathscr{J}_k : \mathbb{R} \to \mathbb{R}$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ with $x(t_k^+) = \lim_{h \to 0^+} x(t_k + h)$, $x(t_k^-) = \lim_{h \to 0^-} x(t_k + h)$, $k = 1,2,...,p$ for $0 = t_0 < t_1 < t_2 < ... < t_p < t_{p+1} = T$.

We define $PC(J,\mathbb{R}) = \{x : J \to \mathbb{R};\; x \in C((t_k,t_{k+1}],\mathbb{R}), k = 0,1,2,...,p+1$ and $x(t_k^+)$ and $x(t_k^-)$ exist with $x(t_k^-) = x(t_k),\; k = 1,2,...,p\}$. Notice that $PC(J,\mathbb{R})$ is a Banach space with the norm $\|x\| = \sup_{t \in J} |x(t)|$.

2. Preliminaries

Let $C([0,T],\mathbb{R})$ denote a Banach space of continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t \in [0,T]}\{|x(t)|\}$. Let $L^1([0,T],\mathbb{R})$ be the Banach space of functions $x : [0,T] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L_1} = \int_0^T |x(t)|dt$.

Now we recall some basic definitions on multi-valued maps [17, 23].

Let $(X, \|.\|)$ be a Banach space. Then a multi-valued map $G : X \to 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x \in \mathbb{B}} G(x)$ is bounded in X for any bounded set $\mathbb{B}$ of X (i.e. $\sup_{x \in \mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set $\mathbb{B}$ of X containing $G(x_0)$, there exists an open neighborhood $\mathscr{N}$ of x_0 such that $G(\mathscr{N}) \subseteq \mathbb{B}$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every bounded subset $\mathbb{B}$ of X. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

In the following study, let $P_{CC}(X)$ denotes the set of all compact and convex subsets of X. G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

The following lemma is necessary to define a solution of (1.1).

LEMMA 2.1. *For a given $\sigma \in PC[0,T]$, a function x is a solution of the following linear impulsive boundary value problem*

$$\begin{cases} {}^cD^q x(t) = \sigma(t), \quad 1 < q \leqslant 2, \quad t \in J_1 = [0,T] \setminus \{t_1,t_2,...,t_p\}, \\ \Delta x(t_k) = \mathscr{I}_k(x(t_k^-)), \quad \Delta x'(t_k) = \mathscr{J}_k(x(t_k^-)), \quad t_k \in (0,T), \quad k = 1,2,...,p, \\ x(0) = -x(T), \quad x'(0) = -x'(T), \end{cases} \tag{2.1}$$

if and only if x is a solution of the impulsive fractional integral equation

$$x(t)=\begin{cases} \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds-\frac{1}{2}\int_{t_k}^T \frac{(T-s)^{q-1}}{\Gamma(q)}\sigma(s)ds+\frac{(T-2t)}{4}\int_{t_k}^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds \\ \quad -\frac{1}{2}\sum\limits_{0<t_k<T}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}\sigma(s)ds+\mathscr{I}_k(x(t_k^-))\right) \\ \quad -\frac{1}{4}\sum\limits_{0<t_k<T}\left(T+2(t-t_k)\right)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds+\mathscr{J}_k(x(t_k^-))\right), \\ \qquad t\in[0,t_1], \\ \int_{t_k}^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds-\frac{1}{2}\int_{t_k}^T \frac{(T-s)^{q-1}}{\Gamma(q)}\sigma(s)ds+\frac{(T-2t)}{4}\int_{t_k}^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds \\ \quad -\frac{1}{2}\sum\limits_{0<t_k<T}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}\sigma(s)ds+\mathscr{I}_k(x(t_k^-))\right) \\ \quad -\frac{1}{4}\sum\limits_{0<t_k<T}\left(T+2(t-t_k)\right)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds+\mathscr{J}_k(x(t_k^-))\right) \\ \quad +\sum\limits_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}\sigma(s)ds+\mathscr{I}_k(x(t_k^-))\right) \\ \quad +\sum\limits_{0<t_k<t}(t-t_k)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds+\mathscr{J}_k(x(t_k^-))\right), \\ \qquad t\in(t_k,t_{k+1}]. \end{cases} \tag{2.2}$$

Proof. Suppose that x is a solution of (2.1). Then, for some constants $b_0,b_1\in\mathbb{R}$, we have

$$x(t)=I^q\sigma(t)-b_0-b_1t=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds-b_0-b_1t,\ \ t\in[0,t_1]. \tag{2.3}$$

For some constants $c_0,c_1\in\mathbb{R}$, we can write

$$x(t)=I^q\sigma(t)-c_0-c_1(t-t_1)=\int_{t_1}^t\frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds-c_0-c_1(t-t_1),\ \ t\in(t_1,t_2].$$

Using the impulse conditions $\Delta x(t_1)=x(t_1^+)-x(t_1^-)=\mathscr{I}_1(x(t_1^-))$ and $\Delta x'(t_1)=x'(t_1^+)-x'(t_1^-)=\mathscr{J}_1(x(t_1^-))$, we find that

$$\begin{aligned} -c_0 &= \int_0^{t_1}\frac{(t_1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds-b_0-b_1t_1+\mathscr{I}_1(x(t_1^-)), \\ -c_1 &= \int_0^{t_1}\frac{(t_1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds-b_1+\mathscr{J}_1(x(t_1^-)). \end{aligned}$$

Thus,

$$\begin{aligned} x(t) &= \int_{t_1}^t\frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds+\int_0^{t_1}\frac{(t_1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds-b_0-b_1t+\mathscr{I}_1(x(t_1^-)) \\ &\quad +(t-t_1)\left[\int_0^{t_1}\frac{(t_1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds+\mathscr{J}_1(x(t_1^-))\right],\ \ t\in(t_1,t_2]. \end{aligned}$$

Repeating the process in this way, the solution $x(t)$ for $t \in (t_k, t_{k+1}]$ cane be written as

$$\begin{aligned} x(t) = & \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s) ds - b_0 - b_1 t \\ & + \sum_{0<t_k<t} \Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} \sigma(s) ds + \mathscr{I}_k(x(t_k^-)) \Big) \\ & + \sum_{0<t_k<t} \Big[(t-t_k) \Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} \sigma(s) ds + \mathscr{J}_k(x(t_k^-)) \Big) \Big], \ t \in (t_k, t_{k+1}]. \end{aligned} \tag{2.4}$$

Applying the anti-periodic boundary conditions $x(0) = -x(T)$, $x'(0) = -x'(T)$, the values of b_0, b_1 are given by

$$\begin{aligned} b_0 = & \frac{1}{2} \int_{t_k}^{T} \frac{(T-s)^{q-1}}{\Gamma(q)} \sigma(s) ds - \frac{T}{4} \int_{t_k}^{T} \frac{(T-s)^{q-2}}{\Gamma(q-1)} \sigma(s) ds \\ & + \frac{1}{2} \sum_{0<t_k<T} \Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} \sigma(s) ds + \mathscr{I}_k(x(t_k^-)) \Big) \\ & + \frac{1}{2} \sum_{0<t_k<T} \Big(\frac{T}{2} - t_k \Big) \Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} \sigma(s) ds + \mathscr{J}_k(x(t_k^-)) \Big), \\ b_1 = & \frac{1}{2} \int_{t_k}^{T} \frac{(T-s)^{q-2}}{\Gamma(q-1)} \sigma(s) ds \\ & + \frac{1}{2} \sum_{0<t_k<T} \Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q)} \sigma(s) ds + \mathscr{J}_k(x(t_k^-)) \Big). \end{aligned}$$

Substituting the values of b_0, b_1 in (2.3) and (2.4), we obtain (2.2). Conversely, we assume that x is a solution of the impulsive fractional integral equation (2.2). It follows by a direct computation that x given by (2.2) satisfies the fractional linear anti-periodic boundary value problem (2.1). This completes the proof. □

REMARK 2.1. The first three terms of the solution (2.2) correspond to the solution for the problem without impulses [4, 6]. The solution for the associated homogeneous problem with impulses and anti-periodic boundary conditions can be obtained by taking $\sigma = 0$ in (2.2).

In view of Lemma 2.1, we define the solutions of (1.1) as follows.

DEFINITION 2.4. A function x is a solution of the problem (1.1) if there exists a function $f \in L^1([0,T], \mathbb{R})$ such that $f(t) \in F(t, x(t))$ a.e. on $[0,T]$ and

$$x(t)=\begin{cases}\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds-\frac{1}{2}\int_{t_k}^T \frac{(T-s)^{q-1}}{\Gamma(q)}f(s)ds+\frac{(T-2t)}{4}\int_{t_k}^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}f(s)ds\\ \quad -\frac{1}{2}\sum\limits_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}f(s)ds+\mathscr{I}_k(x(t_k^-))\Big)\\ \quad -\frac{1}{4}\sum\limits_{0<t_k<T}\big(T+2(t-t_k)\big)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f(s)ds+\mathscr{J}_k(x(t_k^-))\Big),\\ \qquad t\in[0,t_1],\\ \int_{t_k}^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds-\frac{1}{2}\int_{t_k}^T \frac{(T-s)^{q-1}}{\Gamma(q)}f(s)ds+\frac{(T-2t)}{4}\int_{t_k}^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}f(s)ds\\ \quad -\frac{1}{2}\sum\limits_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}f(s)ds+\mathscr{I}_k(x(t_k^-))\Big)\\ \quad -\frac{1}{4}\sum\limits_{0<t_k<T}\big(T+2(t-t_k)\big)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f(s)ds+\mathscr{J}_k(x(t_k^-))\Big)\\ \quad +\sum\limits_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}f(s)ds+\mathscr{I}_k(x(t_k^-))\Big)\\ \quad +\sum\limits_{0<t_k<t}(t-t_k)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f(s)ds+\mathscr{J}_k(x(t_k^-))\Big),\\ \qquad t\in(t_k,t_{k+1}].\end{cases}$$

For the forthcoming analysis, we need the following assumptions:

(A1) Let $F:[0,T]\times\mathbb{R}\to\mathscr{P}(\mathbb{R})$ be such that $F(.,.)$ is convex valued, $t\to F(t,x)$ is measurable for each $x\in\mathbb{R}$, $x\to F(t,x)$ is upper semicontinuous for a.e. $t\in[0,T]$, and for each $x\in C([0,T],\mathbb{R})$, the set $S_{F,x}:=\{f\in L^1([0,T],\mathbb{R}): f(t)\in F(t,x)$ for a.e. $t\in[0,T]\}$ is nonempty set;

(A2) For each $r>0$, there exists a function $m_r\in L^1([0,T],\mathbb{R}_+)$ such that $\|F(t,x)\|=\sup\{|v|: v(t)\in F(t,x)\}\leqslant m_r(t)$ for all $|x|\leqslant r$, and for a.e. $t\in[0,T]$, and

$$\liminf_{r\to+\infty}\Big(\frac{\int_0^T m_r(t)dt}{r}\Big)=\gamma<\infty,$$

where m_r depends on r;

(A3) $\mathscr{I}_k(x)$, $\mathscr{J}_k(x)\in C(\mathbb{R},\mathbb{R})$ and there exist continuous nondecreasing functions $M_k,N_k:\mathbb{R}_+\to\mathbb{R}_+$, $k=1,2,...,p$ such that $|\mathscr{I}_k(y)|\leqslant M_k(|y|)$, $|\mathscr{J}_k(y)|\leqslant N_k(|y|)$ for each $y\in\mathbb{R}$ with

$$\liminf_{r\to+\infty}M_k(r)/r=\alpha_k<+\infty,\quad \liminf_{r\to+\infty}N_k(r)/r=\beta_k<+\infty,$$

Now we state the following lemmas which provide a platform to establish the main result of the paper.

LEMMA 2.2. (Bohnenblust-Karlin [10]) *Let D be a nonempty subset of a Banach space X, which is bounded, closed, and convex. Suppose that $G:D\to 2^X\setminus\{0\}$ is u.s.c. with closed, convex values such that $G(D)\subset D$ and $\overline{G(D)}$ is compact. Then G has a fixed point.*

LEMMA 2.3. [29] *Let I be a compact real interval. Let F be a multi-valued map satisfying (A_1) and let Θ be linear continuous from $L^1(I,\mathbb{R}) \to C(I,\mathbb{R})$, then the operator $\Theta \circ S_F : C(I,\mathbb{R}) \to P_{CC}(C(I,\mathbb{R})), x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$ is a closed graph operator in $C(I,\mathbb{R}) \times C(I,\mathbb{R})$.*

3. Main result

THEOREM 3.1. *Assume that the assumptions $(A_1)-(A_3)$ hold and*

$$\frac{4\Gamma(q)\gamma}{(5+q+p(7q-1))T^{q-1}} + \sum_{k=1}^{p}\Big(\frac{3}{2}\alpha_k + \frac{T+6|T-t_k|}{4}\beta_k\Big) < 1. \tag{3.1}$$

Then the impulsive anti-periodic problem (1.1) has at least one solution on $[0,T]$.

Proof. To transform the problem (1.1) into a fixed point problem, we define a multi-valued map $\Omega : PC(J,\mathbb{R}) \to \mathscr{P}(PC(J,\mathbb{R}))$ as

$$\begin{aligned}
\Omega(x) = \Big\{h \in C([0,T]) : h(t) = &\int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \frac{1}{2}\int_{t_k}^{T} \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds \\
&+ \frac{(T-2t)}{4}\int_{t_k}^{T} \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds \\
&- \frac{1}{2}\sum_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} f(s)ds + \mathscr{I}_k(x(t_k^-))\Big) \\
&- \frac{1}{4}\sum_{0<t_k<T}\big(T+2(t-t_k)\big)\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} f(s)ds + \mathscr{J}_k(x(t_k^-))\Big) \\
&+ \sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} f(s)ds + \mathscr{I}_k(x(t_k^-))\Big) \\
&+ \sum_{0<t_k<t}(t-t_k)\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} f(s)ds + \mathscr{J}_k(x(t_k^-))\Big), \quad f \in S_{F,x}\Big\}.
\end{aligned}$$

Now we prove that the map Ω satisfies all the assumptions of Lemma 2.2, and thus Ω has a fixed point which is a solution of the problem (1.1). As a first step, we show that $\Omega(x)$ is convex for each $x \in PC(J,\mathbb{R})$. For that, let $h_1, h_2 \in \Omega(x)$. Then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in [0,T]$, we have

$$\begin{aligned}
h_i(t) = &\int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f_i(s)ds - \frac{1}{2}\int_{t_k}^{T} \frac{(T-s)^{q-1}}{\Gamma(q)} f_i(s)ds \\
&+ \frac{(T-2t)}{4}\int_{t_k}^{T} \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_i(s)ds \\
&- \frac{1}{2}\sum_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} f_i(s)ds + \mathscr{I}_k(x(t_k^-))\Big)
\end{aligned}$$

$$-\frac{1}{4}\sum_{0<t_k<T}\big(T+2(t-t_k)\big)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f_i(s)ds+\mathscr{J}_k(x(t_k^-))\Big)$$
$$+\sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}f_i(s)ds+\mathscr{I}_k(x(t_k^-))\Big)$$
$$+\sum_{0<t_k<t}(t-t_k)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f_i(s)ds+\mathscr{J}_k(x(t_k^-))\Big),\ \ i=1,2.$$

For $0\leqslant\lambda\leqslant 1$ and for each $t\in J$, we have

$$\begin{aligned}
&[\lambda h_1+(1-\lambda)h_2](t)\\
=&\int_{t_k}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}[\lambda f_1(s)+(1-\lambda)f_2(s)]ds-\frac{1}{2}\int_{t_k}^{T}\frac{(T-s)^{q-1}}{\Gamma(q)}[\lambda f_1(s)+(1-\lambda)f_2(s)]ds\\
&+\frac{(T-2t)}{4}\int_{t_k}^{T}\frac{(T-s)^{q-2}}{\Gamma(q-1)}[\lambda f_1(s)+(1-\lambda)f_2(s)]ds\\
&-\frac{1}{2}\sum_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}[\lambda f_1(s)+(1-\lambda)f_2(s)]ds+\mathscr{I}_k(x(t_k^-))\Big)\\
&-\frac{1}{4}\sum_{0<t_k<T}\big(T+2(t-t_k)\big)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}[\lambda f_1(s)+(1-\lambda)f_2(s)]ds+\mathscr{J}_k(x(t_k^-))\Big)\\
&+\sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}[\lambda f_1(s)+(1-\lambda)f_2(s)]ds+\mathscr{I}_k(x(t_k^-))\Big)\\
&+\sum_{0<t_k<t}(t-t_k)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}[\lambda f_1(s)+(1-\lambda)f_2(s)]ds+\mathscr{J}_k(x(t_k^-))\Big).
\end{aligned}$$

As $S_{F,x}$ is convex (F has convex values), therefore it follows that $\lambda h_1+(1-\lambda)h_2\in\Omega(x)$.

To show that $\Omega(x)$ is closed for each $x\in PC(J,\mathbb{R})$, let $\{u_n\}_{n\geqslant 0}\in\Omega(x)$ be such that $u_n\to u$ as $n\to\infty$ in $PC(J,\mathbb{R})$. Then $u\in PC(J,\mathbb{R})$ and there exists a $v_n\in S_{F,x}$ such that

$$\begin{aligned}
u_n(t)=&\int_{t_k}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}v_n(s)ds-\frac{1}{2}\int_{t_k}^{T}\frac{(T-s)^{q-1}}{\Gamma(q)}v_n(s)ds+\frac{(T-2t)}{4}\int_{t_k}^{T}\frac{(T-s)^{q-2}}{\Gamma(q-1)}v_n(s)ds\\
&-\frac{1}{2}\sum_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}v_n(s)ds+\mathscr{I}_k(x(t_k^-))\Big)\\
&-\frac{1}{4}\sum_{0<t_k<T}\big(T+2(t-t_k)\big)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}v_n(s)ds+\mathscr{J}_k(x(t_k^-))\Big)\\
&+\sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}v_n(s)ds+\mathscr{I}_k(x(t_k^-))\Big)\\
&+\sum_{0<t_k<t}(t-t_k)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}v_n(s)ds+\mathscr{J}_k(x(t_k^-))\Big).
\end{aligned}$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1([0,T],\mathbb{R}_+)$. Thus, $v \in S_{F,x}$ and

$$\begin{aligned}
u_n(t) \to u(t) = &\int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)ds - \frac{1}{2}\int_{t_k}^{T} \frac{(T-s)^{q-1}}{\Gamma(q)} v(s)ds \\
&+ \frac{(T-2t)}{4}\int_{t_k}^{T} \frac{(T-s)^{q-2}}{\Gamma(q-1)} v(s)ds \\
&- \frac{1}{2}\sum_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} v(s)ds + \mathscr{I}_k(x(t_k^-))\Big) \\
&- \frac{1}{4}\sum_{0<t_k<T}\big(T+2(t-t_k)\big)\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} v(s)ds + \mathscr{J}_k(x(t_k^-))\Big) \\
&+ \sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} v(s)ds + \mathscr{I}_k(x(t_k^-))\Big) \\
&+ \sum_{0<t_k<t}(t-t_k)\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} v(s)ds + \mathscr{J}_k(x(t_k^-))\Big).
\end{aligned}$$

Hence $u \in \Omega(x)$.

Next we show that there exists a positive number r such that $\Omega(B_r) \subseteq B_r$, where $B_r = \{x \in PC(J,\mathbb{R}) : \|x\| \leqslant r\}$. Clearly B_r is a bounded closed convex set in $PC(J,\mathbb{R})$ for each positive constant r. If it is not true, then for each positive number r, there exists a function $x_r \in B_r, h_r \in \Omega(x_r)$ with $\|\Omega(x_r)\| > r$, and

$$\begin{aligned}
h_r(t) = &\int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f_r(s)ds - \frac{1}{2}\int_{t_k}^{T} \frac{(T-s)^{q-1}}{\Gamma(q)} f_r(s)ds \\
&+ \frac{(T-2t)}{4}\int_{t_k}^{T} \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_r(s)ds \\
&- \frac{1}{2}\sum_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} f_r(s)ds + \mathscr{I}_k(x(t_k^-))\Big) \\
&- \frac{1}{4}\sum_{0<t_k<T}\big(T+2(t-t_k)\big)\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} f_r(s)ds + \mathscr{J}_k(x(t_k^-))\Big) \\
&+ \sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} f_r(s)ds + \mathscr{I}_k(x(t_k^-))\Big) \\
&+ \sum_{0<t_k<t}(t-t_k)\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} f_r(s)ds + \mathscr{J}_k(x(t_k^-))\Big), \quad \text{for some } f_r \in S_{F,x_r}.
\end{aligned}$$

On the other hand, in view of $(A_2)-(A_3)$, we have

$$
\begin{aligned}
r < \|\Omega(x_r)\| \leqslant & \int_{t_k}^{t} \frac{|t-s|^{q-1}}{\Gamma(q)}|f_r(s)|ds + \frac{1}{2}\int_{t_k}^{T} \frac{|T-s|^{q-1}}{\Gamma(q)}|f_r(s)|ds \\
& + \frac{|T-2t|}{4}\int_{t_k}^{T} \frac{|T-s|^{q-2}}{\Gamma(q-1)}|f_r(s)|ds \\
& + \frac{1}{2}\sum_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k} \frac{|t_k-s|^{q-1}}{\Gamma(q)}|f_r(s)|ds + |\mathscr{I}_k(x_r(t_k^-))|\Big) \\
& + \frac{1}{4}\sum_{0<t_k<T}\big(T+2|t-t_k|\big)\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)}|f_r(s)|ds + |\mathscr{J}_k(x_r(t_k^-))|\Big) \\
& + \sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k} \frac{|t_k-s|^{q-1}}{\Gamma(q)}|f_r(s)|ds + |\mathscr{I}_k(x_r(t_k^-))|\Big) \\
& + \sum_{0<t_k<t}|t-t_k|\Big(\int_{t_{k-1}}^{t_k} \frac{|t_k-s|^{q-2}}{\Gamma(q-1)}|f_r(s)|ds + |\mathscr{J}_k(x_r(t_k^-))|\Big) \\
\leqslant & \frac{T^{q-1}\big(5+q+p(7q-1)\big)}{4\Gamma(q)}\int_0^T m_r(s)ds + \sum_{k=1}^{p}\Big(\frac{3}{2}M_k(r) + \frac{T+6|T-t_k|}{4}N_k(r)\Big)
\end{aligned} \tag{3.2}
$$

Dividing both sides of (3.2) by r and taking the lower limit as $r \to \infty$, we find that

$$
\frac{4\Gamma(q)\gamma}{\big(5+q+p(7q-1)\big)T^{q-1}} + \sum_{k=1}^{p}\Big(\frac{3}{2}\alpha_k + \frac{T+6|T-t_k|}{4}\beta_k\Big) \geqslant 1,
$$

which contradicts (3.1). Hence there exists a positive number r' such that $\Omega(Br') \subseteq Br'$.

In order to show that $\Omega(B_{r'})$ is equi-continuous, we take $t', t'' \in [0,T]$ with $\tau_1 < \tau_2$. For $x \in B_{r'}$ and $h \in \Omega(x)$, there exists $f \in S_{F,x}$ such that for each $t \in [0,T]$, we have

$$
\begin{aligned}
h(t) = & \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds - \frac{1}{2}\int_{t_k}^{T} \frac{(T-s)^{q-1}}{\Gamma(q)}f(s)ds \\
& + \frac{(T-2t)}{4}\int_{t_k}^{T} \frac{(T-s)^{q-2}}{\Gamma(q-1)}f(s)ds \\
& - \frac{1}{2}\sum_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)}f(s)ds + \mathscr{I}_k(x(t_k^-))\Big) \\
& - \frac{1}{4}\sum_{0<t_k<T}\big(T+2(t-t_k)\big)\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f(s)ds + \mathscr{J}_k(x(t_k^-))\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}f(s)ds+\mathscr{I}_k(x(t_k^-))\Big)\\
&+\sum_{0<t_k<t}(t-t_k)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f(s)ds+\mathscr{J}_k(x(t_k^-))\Big).
\end{aligned}
\tag{3.3}
$$

Using (3.3), we obtain

$$
\begin{aligned}
&|h(\tau_2)-h(\tau_1)|\\
=&\Big|\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{q-1}}{\Gamma(q)}f(s)ds+\int_{t_k}^{\tau_1}\frac{(\tau_2-s)^{q-1}-(\tau_1-s)^{q-1}}{\Gamma(q)}f(s)ds\\
&-\frac{(\tau_2-\tau_1)}{2}\int_{t_k}^{T}\frac{(T-s)^{q-2}}{\Gamma(q-1)}f(s)ds\\
&-\sum_{0<t_k<T}\frac{(\tau_2-\tau_1)}{2}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f(s)ds+\mathscr{J}_k(x(t_k^-))\Big)\\
&+\sum_{0<t_k<\tau_2-\tau_1}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}f(s)ds+\mathscr{I}_k(x(t_k^-))\Big)\\
&+\sum_{0<t_k<\tau_2-\tau_1}(\tau_2-t_k)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f(s)ds+\mathscr{J}_k(x(t_k^-))\Big)\\
&+\sum_{0<t_k<\tau_1}(\tau_2-\tau_1)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f(s)ds+\mathscr{J}_k(x(t_k^-))\Big)\Big|\\
\leqslant&\int_{\tau_1}^{\tau_2}\frac{|\tau_2-s|^{q-1}}{\Gamma(q)}m_{r'}(s)ds+\int_{t_k}^{\tau_1}\frac{|(\tau_2-s)^{q-1}-(\tau_1-s)^{q-1}|}{\Gamma(q)}m_{r'}(s)ds\\
&+\frac{|\tau_2-\tau_1|}{2}\int_{t_k}^{T}\frac{|T-s|^{q-2}}{\Gamma(q-1)}m_{r'}(s)ds\\
&+\sum_{0<t_k<T}\frac{|\tau_2-\tau_1|}{2}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}m_{r'}(s)ds+N_k(r)\Big)\\
&+\sum_{0<t_k<\tau_2-\tau_1}\Big(\int_{t_{k-1}}^{t_k}\frac{|t_k-s|^{q-1}}{\Gamma(q)}m_{r'}(s)ds+M_k(r)\Big)\\
&+\sum_{0<t_k<\tau_2-\tau_1}|\tau_2-t_k|\Big(\int_{t_{k-1}}^{t_k}\frac{|t_k-s|^{q-2}}{\Gamma(q-1)}m_{r'}(s)ds+N_k(r)\Big)\\
&+\sum_{0<t_k<\tau_1}|\tau_2-\tau_1|\Big(\int_{t_{k-1}}^{t_k}\frac{|t_k-s|^{q-2}}{\Gamma(q-1)}m_{r'}(s)ds+N_k(r)\Big).
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x\in B_{r'}$ as $\tau_2\to\tau_1$. Thus, Ω is equi-continuous.

As Ω satisfies the above assumptions, therefore it follows by the Ascoli-Arzela theorem that Ω is a compact multi-valued map.

As a last step, we show that Ω has a closed graph. Let $x_n \to x_*, h_n \in \Omega(x_n)$ and $h_n \to h_*$. We will show that $h_* \in \Omega(x_*)$. By the relation $h_n \in \Omega(x_n)$, we mean that there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0,T]$,

$$
\begin{aligned}
h_n(t) = & \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f_n(s)ds - \frac{1}{2}\int_{t_k}^{T} \frac{(T-s)^{q-1}}{\Gamma(q)} f_n(s)ds \\
& + \frac{(T-2t)}{4}\int_{t_k}^{T} \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_n(s)ds \\
& - \frac{1}{2}\sum_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} f_n(s)ds + \mathscr{I}_k(x_n(t_k^-))\Big) \\
& - \frac{1}{4}\sum_{0<t_k<T}\big(T+2(t-t_k)\big)\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} f_n(s)ds + \mathscr{J}_k(x_n(t_k^-))\Big) \\
& + \sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} f_n(s)ds + \mathscr{I}_k(x_n(t_k^-))\Big) \\
& + \sum_{0<t_k<t}(t-t_k)\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} f_n(s)ds + \mathscr{J}_k(x_n(t_k^-))\Big).
\end{aligned}
$$

Thus we need to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,T]$,

$$
\begin{aligned}
h_*(t) = & \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)ds - \frac{1}{2}\int_{t_k}^{T} \frac{(T-s)^{q-1}}{\Gamma(q)} f_*(s)ds \\
& + \frac{(T-2t)}{4}\int_{t_k}^{T} \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_*(s)ds \\
& - \frac{1}{2}\sum_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} f_*(s)ds + \mathscr{I}_k(x_*(t_k^-))\Big) \\
& - \frac{1}{4}\sum_{0<t_k<T}\big(T+2(t-t_k)\big)\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} f_*(s)ds + \mathscr{J}_k(x_*(t_k^-))\Big) \\
& + \sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} f_*(s)ds + \mathscr{I}_k(x_*(t_k^-))\Big) \\
& + \sum_{0<t_k<t}(t-t_k)\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} f_*(s)ds + \mathscr{J}_k(x_*(t_k^-))\Big).
\end{aligned}
$$

Let us consider the continuous linear operator $\Theta : L^1([0,T],\mathbb{R}) \to PC([0,T])$ so that

$$
\begin{aligned}
f \mapsto \Theta(f)(t) = & \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \frac{1}{2}\int_{t_k}^{T} \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds \\
& + \frac{(T-2t)}{4}\int_{t_k}^{T} \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds \\
& - \frac{1}{2}\sum_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} f(s)ds + \mathscr{I}_k(x(t_k^-))\Big)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{4}\sum_{0<t_k<T}\big(T+2(t-t_k)\big)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f(s)ds+\mathscr{J}_k(x(t_k^-))\Big)\\
&+\sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}f(s)ds+\mathscr{I}_k(x(t_k^-))\Big)\\
&+\sum_{0<t_k<t}(t-t_k)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f(s)ds+\mathscr{J}_k(x(t_k^-))\Big).
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\|h_n(t)-h_*(t)\|\\
=&\int_{t_k}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}\big(f_n(s)-f_*(s)\big)ds-\frac{1}{2}\int_{t_k}^{T}\frac{(T-s)^{q-1}}{\Gamma(q)}\big(f_n(s)-f_*(s)\big)ds\\
&+\frac{(T-2t)}{4}\int_{t_k}^{T}\frac{(T-s)^{q-2}}{\Gamma(q-1)}\big(f_n(s)-f_*(s)\big)ds\\
&-\frac{1}{2}\sum_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}\big(f_n(s)-f_*(s)\big)ds+\big(\mathscr{I}_k(x_n(t_k^-))-\mathscr{I}_k(x_*(t_k^-))\big)\Big)\\
&-\frac{1}{4}\sum_{0<t_k<T}\big(T+2(t-t_k)\big)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\big(f_n(s)-f_*(s)\big)ds+\big(\mathscr{J}_k(x_n(t_k^-))-\mathscr{J}_k(x_*(t_k^-))\big)\Big)\\
&+\sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}\big(f_n(s)-f_*(s)\big)ds+\big(\mathscr{I}_k(x_n(t_k^-))-\mathscr{I}_k(x_*(t_k^-))\big)\Big)\\
&+\sum_{0<t_k<t}(t-t_k)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\big(f_n(s)-f_*(s)\big)ds+\big(\mathscr{J}_k(x_n(t_k^-))-\mathscr{J}_k(x_*(t_k^-))\big)\Big)\Big\|\\
&\to 0 \ \text{ as } \ n\to\infty.
\end{aligned}
$$

Thus, it follows by Lemma 2.3 that $\Theta\circ S_F$ is a closed graph operator. Further, we have $h_n(t)\in\Theta(S_{F,x_n})$. Since $x_n\to x_*$, therefore, Lemma 2.3 yields

$$
\begin{aligned}
h_*(t)=&-\int_{t_k}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}f_*(s)ds-\frac{1}{2}\int_{t_k}^{T}\frac{(T-s)^{q-1}}{\Gamma(q)}f_*(s)ds\\
&+\frac{(T-2t)}{4}\int_{t_k}^{T}\frac{(T-s)^{q-2}}{\Gamma(q-1)}f_*(s)ds\\
&-\frac{1}{2}\sum_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}f_*(s)ds+\mathscr{I}_k(x_*(t_k^-))\Big)\\
&-\frac{1}{4}\sum_{0<t_k<T}\big(T+2(t-t_k)\big)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f_*(s)ds+\mathscr{J}_k(x_*(t_k^-))\Big)\\
&+\sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}f_*(s)ds+\mathscr{I}_k(x_*(t_k^-))\Big)\\
&+\sum_{0<t_k<t}(t-t_k)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f_*(s)ds+\mathscr{J}_k(x_*(t_k^-))\Big),\ \text{for some } f_*\in S_{F,x_*}.
\end{aligned}
$$

Hence, we conclude that Ω is a compact multi-valued map, u.s.c. with convex closed values. Thus, all the assumptions of Lemma 2.2 are satisfied. Consequently, by the conclusion of Lemma 2.2, it follows that Ω has a fixed point x which is a solution of the problem (1.1). $\square$

EXAMPLE. Consider the following impulsive differential inclusions with anti-periodic boundary conditions.

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) \in F(t,x(t)), \quad t \in [0,1], \quad t \neq \frac{1}{3}, \\ \Delta x(\frac{1}{3}) = \frac{|x(\frac{1}{3}^-)|}{(7+|x(\frac{1}{3}^-)|)}, \quad \Delta x'(\frac{1}{3}) = \frac{|x(\frac{1}{3}^-)|}{(11+|x(\frac{1}{3}^-)|)}, \\ x(0) = -x(1), \quad x'(0) = -x'(1). \end{cases} \tag{3.4}$$

Clearly $q = \frac{3}{2}$, $T = 1$ and $p = 1$. Let us Choose $F(t,x)$ satisfying the conditions $\|F(t,x)\| \leqslant \frac{1}{4(1+t)^2}|x| + e^{-t}$. Clearly the condition (3.1) is satisfied:

$$\frac{4\Gamma(q)\gamma}{\big(5+q+p(7q-1)\big)T^{q-1}} + \sum_{k=1}^{p}\Big(\frac{3}{2}\alpha_k + \frac{T+6|T-t_k|}{4}\beta_k\Big) = \frac{\sqrt{\pi}}{32} < 1.$$

As all the assumptions of Theorem 3.1 are satisfied, the anti-periodic impulsive problem (3.4) has at least one solution on $[0,1]$.

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