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FRACTIONAL DIFFERENCE INEQUALITIES OF OPIAL TYPE AND INITIAL VALUE PROBLEM

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Abstract. In this paper some discrete Opial type inequalities of fractional order are established and using these inequalities of Opial type, a bound to the solution of a fractional initial value problem is obtained.

1. Introduction

The notions of fractional calculus may be traced back to the works of Euler, but the idea of fractional difference is very recent. Diaz and Osler [5] defined the fractional difference by the rather natural approach of allowing the index of differencing, in the standard expression for the nth difference, to be any real or complex number. Later, Hirota [8], defined the difference operator ∇^{α} as the first n terms of the Taylor series of $[\frac{1-B}{\varepsilon}]^{\alpha}$ where ε is interval length. Deekshitulu, G. V. S. R. and Jagan Mohan, J. [3] modified the definition of Atsushi Nagai [1] for $0 < \alpha \leqslant 1$ in such a way that the expression for ∇^{α} does not involve any difference operator.

The inequalities which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of differential, integral and finite difference equations. The landmark 1955 paper of Fan, Taussky and Todd [6] has brought about a lively interest in discrete inequalities. Many discrete inequalities involving functions and their sums and diferences have been found. Although some results in the discrete case are similar to those already known in the continuous case, the adaptation from continuous to discrete is not always direct, but often requires special devices. Discrete analogues of Opial-type inequalities started in 1967-69 with the papers of Lasota [9], Wong [16], Lee [12] and Beesack [2], which provided discrete versions of Opial's original inequality. Opial's inequality and its several generalizations, extensions and discretizations, play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations.

In this paper we obtain some discrete Opial type inequalities of fractional order. Further, we find a bound to the solution of a fractional initial value problem using the fractional difference inequalities of Opial type.

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2. Preliminaries

Let u_n be any function defined for $n \in \mathbb{N}_0^+$ where $\mathbb{N}_a^+ = \{a, a+1, a+2, ...\}$ for $a \in \mathbb{Z}$. Hirota [8] took the first n terms of Taylor series of $\Delta_{-n}^{\alpha} = \varepsilon^{-\alpha}(1-B)^{\alpha}$ and gave the following definition.

DEFINITION 2.1. Let $\alpha \in \mathbb{R}$. Then difference operator of order α is defined by

$$\Delta_{-n}^{\alpha} u_n = \begin{cases} \varepsilon^{-\alpha} \sum_{j=0}^{n-1} \binom{\alpha}{j} (-1)^j u_{n-j}, \ \alpha \neq 1, 2, ... \\ \varepsilon^{-m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j u_{n-j}, \ \alpha = m \in \mathbb{Z}_{>0}. \end{cases} \tag{2.1}$$

Here $\binom{a}{n}$, $(a \in \mathbb{R}, n \in \mathbb{Z})$ stands for a binomial coefficient defined by

$$\binom{a}{n} = \begin{cases} \frac{\Gamma(a+1)}{\Gamma(a-n+1)\Gamma(n+1)} & n > 0 \\ 1 & n = 0 \\ 0 & n < 0. \end{cases} \tag{2.2}$$

In 2002, Atsushi Nagai [1] introduced another definition of fractional difference which is a slight modification of Hirota's fractional difference operator.

DEFINITION 2.2. Let $\alpha \in \mathbb{R}$ and m be an integer such that $m - 1 < \alpha \leqslant m$. The difference operator $\Delta_{*,-n}$ of order α is defined as

$$\Delta_{*,-n}^{\alpha} u_n = \Delta_{-n}^{\alpha-m} \Delta_{-n}^{m} u_n = \varepsilon^{m-\alpha} \sum_{j=0}^{n-1} \binom{\alpha - m}{j} (-1)^j \Delta_{-(n-j)}^{m} u_{n-j}. \tag{2.3}$$

Deekshitulu, G. V. S. R. and Jagan Mohan, J. [3] gave the following definition.

DEFINITION 2.3. Let $\alpha \in \mathbb{R}$ such that $0 < \alpha \leqslant 1$. The difference operator ∇ of order α is defined as

$$\nabla^{\alpha} u_n = \sum_{j=0}^{n-1} \binom{j-\alpha}{j} \nabla u_{n-j}. \tag{2.4}$$

REMARK 1. For any $\alpha \in \mathbb{R}$ $(0 < \alpha \leqslant 1)$,

$$\nabla^{-\alpha} u_n = \sum_{j=0}^{n-1} \binom{j+\alpha}{j} \nabla u_{n-j}. \tag{2.5}$$

Further $\nabla^{\alpha} u_n$ and $\nabla^{-\alpha} u_n$ can be expressed in the terms of the arguments of u_n as, for any $\alpha \in \mathbb{R}$ $(0 < \alpha \leqslant 1)$,

$$\nabla^{\alpha} u_n = u_n - \binom{n-1-\alpha}{n-1} u_0 - \alpha \sum_{j=1}^{n-1} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} u_{n-j} \tag{2.6}$$

and

$$\nabla^{-\alpha}u_n = u_n - \binom{n-1+\alpha}{n-1}u_0 + \alpha\sum_{j=1}^{n-1}\frac{1}{(j+\alpha)}\binom{j+\alpha}{j}u_{n-j} \tag{2.7}$$

$$i.e. \qquad \nabla^{-\alpha}u_n = \sum_{j=1}^{n}\binom{n-j+\alpha-1}{n-j}u_j - \binom{n-1+\alpha}{n-1}u_0. \tag{2.8}$$

REMARK 2. The difference operator ∇ of order α satisfies the following properties.

1. For any real numbers α and β, $\nabla^{\alpha}\nabla^{\beta}u_n = \nabla^{\alpha+\beta}u_n$.
2. For any constant 'c', $\nabla^{\alpha}[cu_n + v_n] = c\nabla^{\alpha}u_n + \nabla^{\alpha}v_n$ where v_n be any function defined for $n \in \mathbb{N}_0^+$.
3. For $\alpha \in \mathbb{R}$, $\nabla^{\alpha}(u_n v_n)\sum_{m=0}^{n-1}\binom{\alpha}{m}[\nabla^{\alpha-m}u_{n-m}][\nabla^{m}v_m]$.
4. $\nabla^{\alpha}u_0 = 0$ and $\nabla^{\alpha}u_1 = u_1 - u_0 = \nabla u_1$.
5. $\nabla^{\alpha}\nabla^{-\alpha}u_n = \nabla^{-\alpha}\nabla^{\alpha}u_n = u_n - u_0$.
6. $\nabla^{\alpha}\nabla^{-\alpha}(u_n - u_0) = \nabla^{-\alpha}\nabla^{\alpha}(u_n - u_0) = u_n - u_0$.

REMARK 3. The following inequalities of discrete calculus will be used in proving main results.

1. Let x_n be any non negative function defined for $n \in \mathbb{N}_0^+$ and γ be any non negative real number then, for $0 \leqslant \gamma \leqslant 1$,

$$n^{\gamma-1}\sum_{i=1}^{n}x_i^{\gamma} \leqslant \Big(\sum_{i=1}^{n}x_i\Big)^{\gamma} \leqslant \sum_{i=1}^{n}x_i^{\gamma} \tag{2.9}$$

   and for $\gamma \geqslant 1$,

$$\sum_{i=1}^{n}x_i^{\gamma} \leqslant \Big(\sum_{i=1}^{n}x_i\Big)^{\gamma} \leqslant n^{\gamma-1}\sum_{i=1}^{n}x_i^{\gamma}. \tag{2.10}$$

2. Let $\{x_i\}_{i=0}^{n}$ and $\{y_i\}_{i=0}^{n}$ be sequences of numbers, then the following inequalities holds

$$\Big(\sum_{i=1}^{n}x_i y_i\Big) \leqslant \Big(\sum_{i=1}^{n}x_i\Big)\Big(\sum_{i=1}^{n}y_i\Big) \tag{2.11}$$

   and

$$x_i y_i \leqslant \frac{1}{2}[(x_i)^2 + (y_i)^2]. \tag{2.12}$$

3. (Youngs's Inequality) If a and b are nonnegative real numbers and p and q are positive real numbers such that $1/p + 1/q = 1$ then we have

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}. \tag{2.13}$$

   Equality holds if and only if $a^p = b^q$.

4. (Inequality of arithmetic and geometric means) If a, b, p and q are any non-negative real numbers then we have

$$a^p\, b^q \leqslant pa + qb. \tag{2.14}$$

THEOREM 2.1. (Discrete Gronwall's Inequality) *Let y_n, a_n and b_n be any three nonnegative functions defined for $n \in \mathbb{N}_0^+$. If for $n \in \mathbb{N}_0^+$,*

$$y_{n+1} \leqslant y_0 + \sum_{j=0}^{n} [a_j y_j + b_j]$$

then

$$y_n \leqslant y_0\, exp\left[\sum_{j=0}^{n-1} a_j\right] + \sum_{j=0}^{n-1} b_j\, exp\left[\sum_{k=j+1}^{n-1} a_k\right].$$

THEOREM 2.2. *Let $\{x_i\}_{i=0}^{n}$ be a sequence of non negative numbers, and $x_0 = 0$. Then, for $l \geqslant 1$, the following inequalities holds*

$$\sum_{i=1}^{n}\left[x_i\Big(\sum_{j=1}^{i} x_j\Big)^l\right] \leqslant \frac{(n+1)^l}{l+1}\sum_{i=1}^{n} x_i^{l+1} \tag{2.15}$$

and

$$\sum_{i=1}^{n}\left[x_i\Big(\sum_{j=0}^{i-1} x_j\Big)^l\right] \leqslant \frac{n^l}{l+1}\sum_{i=0}^{n} x_i^{l+1}. \tag{2.16}$$

Henry L Gray and Nien fan Zhang [7] gave the following definition:

DEFINITION 2.4. For any complex numbers α and β, let $(\alpha)_\beta$ be defined as follows.

$$(\alpha)_\beta = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} & \text{when } \alpha \text{ and } \alpha+\beta \text{ are neither zero nor negative integers,} \\ 1 & \text{when } \alpha=\beta=0, \\ 0 & \text{when } \alpha=0,\ \beta \text{ is neither zero nor negative integer,} \\ undefined & \text{otherwise.} \end{cases}$$

REMARK 4. For any complex numbers α and β, when α, β and $\alpha+\beta$ are neither zero nor negative integers,

$$(\alpha+\beta)_n = \sum_{k=0}^{n}\binom{n}{k}(\alpha)_{n-k}(\beta)_k \tag{2.17}$$

for any positive integer n.

3. Main Results

In this section we prove some fractional difference inequalities of Opial type.

LEMMA 3.1.

$$\sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} = \binom{n-1+\alpha}{n-1} \tag{3.1}$$

and

$$\sum_{j=1}^{n} \binom{j+\alpha-1}{j-1} = \binom{n+\alpha}{n-1}. \tag{3.2}$$

THEOREM 3.2. *Let* $\{x_i\}_{i=0}^{n}$ *be a sequence of numbers and* $x_0 = 0$. *Then, the following inequality holds*

$$\sum_{i=1}^{n} |x_i \nabla^{\alpha} x_i| \leqslant \frac{n+1}{2} \binom{n+\alpha}{n-1} \sum_{i=1}^{n} |\nabla^{\alpha} x_i|^2 \tag{3.3}$$

where ∇^{α} *is the fractional difference operator of order* α, $0 < \alpha \leqslant 1$.

Proof. Using (2.8), we have

$$\begin{aligned}
\nabla^{-\alpha}(\nabla^{\alpha} x_i) &= \sum_{j=1}^{i} \left[\binom{i-j+\alpha-1}{i-j} (\nabla^{\alpha} x_j) \right] - \binom{i-1+\alpha}{i-1} (\nabla^{\alpha} x_0) \\
x_i - x_0 &= \sum_{j=1}^{i} \left[\binom{i-j+\alpha-1}{i-j} (\nabla^{\alpha} x_j) \right] = \sum_{j=1}^{i} \left[\binom{i-j+\alpha-1}{i-j} (\nabla^{\alpha} x_j) \right] \qquad (3.4)
\end{aligned}$$

since $x_0 = 0$. Now consider

$$\begin{aligned}
|x_i| &= \left| \sum_{j=1}^{i} \left[\binom{i-j+\alpha-1}{i-j} (\nabla^{\alpha} x_j) \right] \right| \\
&\leqslant \sum_{j=1}^{i} \left[\left| \binom{i-j+\alpha-1}{i-j} \right| |\nabla^{\alpha} x_j| \right] \\
&\leqslant \sum_{j=1}^{i} \left| \binom{i-j+\alpha-1}{i-j} \right| \ \sum_{j=1}^{i} |\nabla^{\alpha} x_j| \quad \text{(using (2.11))}.
\end{aligned}$$

Since $\binom{i-j+\alpha-1}{i-j} \geqslant 0$ for j = 1, 2, ...i, $\left|\binom{i-j+\alpha-1}{i-j}\right| = \binom{i-j+\alpha-1}{i-j}$. Using Lemma 3.1,

$$|x_i| \leqslant \binom{i-1+\alpha}{i-1} \sum_{j=1}^{i} |\nabla^{\alpha} x_j|. \tag{3.5}$$

Now consider

$$\sum_{i=1}^{n}|x_i\nabla^{\alpha}x_i| \leqslant \sum_{i=1}^{n}\Big[\binom{i-1+\alpha}{i-1}\ (\sum_{j=1}^{i}|\nabla^{\alpha}x_j|)\ |\nabla^{\alpha}x_i|\Big] \quad \text{(using 3.5)}$$

$$\leqslant \sum_{i=1}^{n}\binom{i-1+\alpha}{i-1}\sum_{i=1}^{n}\Big[|\nabla^{\alpha}x_i|\ (\sum_{j=1}^{i}|\nabla^{\alpha}x_j|)\Big]$$

$$\leqslant \frac{n+1}{2}\binom{n+\alpha}{n-1}\sum_{i=1}^{n}|\nabla^{\alpha}x_i|^2 \quad \text{(using Lemma 3.1 and (2.15)).} \quad \square$$

THEOREM 3.3. *Let $\{x_i\}_{i=0}^{n}$ be a sequence of numbers and $x_0 = 0$. Then, the following inequality holds*

$$\sum_{i=1}^{n-1}|x_i\nabla^{\alpha}x_{i+1}| \leqslant \frac{n-1}{2}\binom{n+\alpha-1}{n-2}\sum_{i=0}^{n-1}|\nabla^{\alpha}x_{i+1}|^2. \tag{3.6}$$

Proof. Using (3.5), we have

$$|x_i| \leqslant \binom{i-1+\alpha}{i-1}\sum_{j=1}^{i}|\nabla^{\alpha}x_j|.$$

Take $(j-1)=k$, then k varies from 0 to $(i-1)$. Then

$$|x_i| \leqslant \binom{i-1+\alpha}{i-1}\sum_{k=0}^{i-1}|\nabla^{\alpha}x_{k+1}|. \tag{3.7}$$

Now consider

$$\sum_{i=1}^{n-1}|x_i\nabla^{\alpha}x_{i+1}| \leqslant \sum_{i=1}^{n-1}\Big[\binom{i-1+\alpha}{i-1}(\sum_{k=0}^{i-1}|\nabla^{\alpha}x_{k+1}|)\ |\nabla^{\alpha}x_{i+1}|\Big] \quad \text{(using (3.7))}$$

$$\leqslant \sum_{i=1}^{n-1}\binom{i-1+\alpha}{i-1}\sum_{i=1}^{n-1}\Big[|\nabla^{\alpha}x_{i+1}|\ (\sum_{k=0}^{i-1}|\nabla^{\alpha}x_{k+1}|)\Big] \quad \text{(using (2.11))}$$

$$\leqslant \frac{n-1}{2}\binom{n+\alpha-1}{n-2}\sum_{i=0}^{n-1}|\nabla^{\alpha}x_{i+1}|^2 \quad \text{(using Lemma 3.1 and (2.16)).} \quad \square$$

THEOREM 3.4. *Let $\{x_i\}_{i=0}^{n}$ be a nondecreasing sequence of nonnegative numbers and $x_0 = 0$. Then, for $l \geqslant 1$, the following inequality holds*

$$\sum_{i=1}^{n}(x_i^l\nabla^{\alpha}x_i) \leqslant \frac{(n+1)^l}{l+1}\Big[\binom{n+\alpha}{n-1}\Big]^l\sum_{i=1}^{n}(\nabla^{\alpha}x_i)^{l+1}. \tag{3.8}$$

Proof. Using (3.4), (2.11) and Lemma 3.1, we get

$$
\begin{aligned}
x_i^l &= \Big[\sum_{j=1}^{i}\binom{i-j+\alpha-1}{i-j}(\nabla^\alpha x_j)\Big]^l \\
&\leqslant \Big[\sum_{j=1}^{i}\binom{i-j+\alpha-1}{i-j}\sum_{j=1}^{i}(\nabla^\alpha x_j)\Big]^l \\
&= \Big[\sum_{j=1}^{i}\binom{i-j+\alpha-1}{i-j}\Big]^l\Big[\sum_{j=1}^{i}(\nabla^\alpha x_j)\Big]^l \\
&= \Big[\binom{i-1+\alpha}{i-1}\Big]^l\Big[\sum_{j=1}^{i}(\nabla^\alpha x_j)\Big]^l.
\end{aligned}
$$

Now consider

$$
\begin{aligned}
\sum_{i=1}^{n}(x_i^l\nabla^\alpha x_i) &\leqslant \sum_{i=1}^{n}\Big[\Big[\binom{i-1+\alpha}{i-1}\Big]^l\Big[\sum_{j=1}^{i}\nabla^\alpha x_j\Big]^l(\nabla^\alpha x_i)\Big] \\
&\leqslant \sum_{i=1}^{n}\Big[\binom{i-1+\alpha}{i-1}\Big]^l\sum_{i=1}^{n}\Big[(\sum_{j=1}^{i}\nabla^\alpha x_j)^l\ \ (\nabla^\alpha x_i)\Big] \quad \text{(using (2.11))} \\
&\leqslant \Big[\sum_{i=1}^{n}\binom{i-1+\alpha}{i-1}\Big]^l\sum_{i=1}^{n}\Big[(\nabla^\alpha x_i)\ (\sum_{j=1}^{i}\nabla^\alpha x_j)^l\Big] \quad \text{(using (2.10))} \\
&\leqslant \Big[\binom{n+\alpha}{n-1}\Big]^l\frac{(n+1)^l}{l+1}\sum_{i=1}^{n}(\nabla^\alpha x_i)^{l+1} \quad \text{(using Lemma 3.1 and (2.15)).} \quad \Box
\end{aligned}
$$

THEOREM 3.5. *Let $\{x_i\}_{i=0}^n$ be a sequence of numbers and $x_0 = 0$. Then, for $l \geqslant 1$, the following inequality holds*

$$
\sum_{i=1}^{n}|x_i^l\nabla^\alpha x_i| \leqslant \frac{(n+1)^l}{l+1}\Big[\binom{n+\alpha}{n-1}\Big]^l\sum_{i=1}^{n}|\nabla^\alpha x_i|^{l+1}. \tag{3.9}
$$

REMARK 5. It is clear from Theorem 3.5 that

(i) The inequality (3.8) does not hold for $l < 1$ except at $l = 0$. It is sufficient to consider $l = \frac{1}{2}, n = 1, x_0 = 0$ and $x_1 = 1$.

(ii) For all $l > 1$, strict inequality holds in (3.8).

(iii) For $\alpha = 1$ and $l = 0$, equality holds in (3.8).

THEOREM 3.6. *Let* $\{x_i\}_{i=0}^{n}$ *be a nondecreasing sequence of nonnegative numbers and* $x_0 = 0$. *Then,*

(i) *if* $l > 0,\ m > 0,\ l+m \geqslant 1$ *or* $l < 0,\ m < 0$,

$$\sum_{i=1}^{n} x_i^l (\nabla^\alpha x_i)^m \leqslant K_n \sum_{i=1}^{n} (\nabla^\alpha x_i)^{l+m}, \tag{3.10}$$

where $K_0 = mp,\ p = (l+m)^{-1}$, *and for n = 1, 2, ...*
$K_n = max\ \{K_{n-1} + lp\left[\binom{n+\alpha-1}{n-1}\right]^l n^{l-1}, mp\left[\binom{n+\alpha}{n}\right]^l (n+1)^l\}$.

(ii) *if* $l > 0,\ m < 0,\ l+m \leqslant 1,\ l+m \neq 0$ *or* $l < 0,\ m > 0,\ l+m \geqslant 1$,

$$\sum_{i=1}^{n} x_i^l (\nabla^\alpha x_i)^m \geqslant C_n \sum_{i=1}^{n} (\nabla^\alpha x_i)^{l+m}, \tag{3.11}$$

where $C_0 = mp$, *and for n = 1, 2, ...*
$C_n = min\ \{C_{n-1} + lp\left[\binom{n+\alpha-1}{n-1}\right]^l n^{l-1}, mp\left[\binom{n+\alpha}{n}\right]^l (n+1)^l\}$.

Proof. Let $y_i = (\nabla^\alpha x_i)^{l+m}$ for i = 1, 2, ... n, $l+m \neq 0$, so that $y_i^{mp} = (\nabla^\alpha x_i)^m$. Using (3.5), we have

$$x_i \leqslant \binom{i-1+\alpha}{i-1} \sum_{j=1}^{i} y_j^p.$$

By using (2.9), if $l+m \geqslant 1$ i.e. $p \leqslant 1$,

$$x_i \leqslant i^{1-p} \binom{i-1+\alpha}{i-1} [\sum_{j=1}^{i} y_j]^p$$

$$i.e.\ x_i \leqslant z_i,\ where\ z_i = i^{1-p} \binom{i-1+\alpha}{i-1} [\sum_{j=1}^{i} y_j]^p$$

and $x_i \geqslant z_i$, if $l+m < 0$ or $0 < l+m \leqslant 1$. Therefore, $x_i^l \leqslant z_i^l$ and hence

$$\sum_{i=1}^{n} x_i^l (\nabla^\alpha x_i)^m \leqslant \sum_{i=1}^{n} z_i^l y_i^{mp} \tag{3.12}$$

if $l > 0,\ m > 0,\ l+m \geqslant 1$ or $l < 0,\ m < 0$. Similarly $x_i^l \geqslant z_i^l$ and hence

$$\sum_{i=1}^{n} x_i^l (\nabla^\alpha x_i)^m \geqslant \sum_{i=1}^{n} z_i^l y_i^{mp} \tag{3.13}$$

if $l > 0,\ m < 0,\ l+m \leqslant 1,\ l+m \neq 0$ or $l < 0,\ m > 0,\ l+m \geqslant 1$. In order to prove (3.10) and (3.11) it is enough to prove

$$\sum_{i=1}^{n} z_i^l y_i^{mp} \leqslant K_n \sum_{i=1}^{n} y_i\ for\ lm > 0, \tag{3.14}$$

$$\sum_{i=1}^{n} z_i^l y_i^{mp} \geqslant C_n \sum_{i=1}^{n} y_i \ for \ lm < 0. \tag{3.15}$$

Now we prove (3.14) by using mathematical induction on n. Since $z_1 = y_1^p$ and $K_1 \geqslant 1$, it holds for n = 1. Assume that (3.14) holds for n = s. Then

$$\sum_{i=1}^{s} z_i^l y_i^{mp} \leqslant K_s \sum_{i=1}^{s} y_i.$$

Now consider

$$\sum_{i=1}^{s+1} z_i^l y_i^{mp} = \sum_{i=1}^{s} z_i^l y_i^{mp} + z_{s+1}^l y_{s+1}^{mp} \leqslant K_s \sum_{i=1}^{s} y_i + z_{s+1}^l y_{s+1}^{mp}.$$

Since $y_i \geqslant 0$ for $i \geqslant 1$ and $lp + mp = 1$, by the inequality of arithmetic and geometric means, for $lp > 0$, we obtain

$$\begin{aligned} z_{s+1}^l y_{s+1}^{mp} &= \left[\binom{s+\alpha}{s}\right]^l (s+1)^{l-lp} \left[\sum_{j=1}^{s+1} y_j\right]^{lp} y_{s+1}^{mp} \\ &= (s+1)^l \left[\binom{s+\alpha}{s}\right]^l \left[\frac{1}{s+1} \sum_{j=1}^{s+1} y_j\right]^{lp} y_{s+1}^{mp} \\ &\leqslant (s+1)^l \left[\binom{s+\alpha}{s}\right]^l \left[lp \frac{1}{s+1} \sum_{j=1}^{s+1} y_j + mp y_{s+1}\right] = w_{s+1}. \end{aligned}$$

Thus

$$\sum_{i=1}^{s+1} z_i^l y_i^{mp} \leqslant K_s \sum_{i=1}^{s} y_i + (s+1)^l \left[\binom{s+\alpha}{s}\right]^l \left[lp \frac{1}{s+1} \sum_{j=1}^{s+1} y_j + mp y_{s+1}\right].$$

Since $K_s \geqslant mp(s+1)^l \left[\binom{s+\alpha}{s}\right]^l$ and $K_{s+1} \geqslant K_s + lp(s+1)^{l-1} \left[\binom{s+\alpha}{s}\right]^l$,

$$\begin{aligned} \sum_{i=1}^{s+1} z_i^l y_i^{mp} &\leqslant K_s \sum_{i=1}^{s} y_i + K_s y_{s+1} + lp(s+1)^{l-1} \left[\binom{s+\alpha}{s}\right]^l \sum_{j=1}^{s+1} y_j \\ &= K_s \sum_{i=1}^{s+1} y_i + lp(s+1)^{l-1} \left[\binom{s+\alpha}{s}\right]^l \sum_{i=1}^{s+1} y_i \\ &\leqslant K_{s+1} \sum_{i=1}^{s+1} y_i. \end{aligned}$$

The statement (3.14) is true for n = s+1. Hence by mathematical induction, (3.14) is true for every $n \in Z^+$. For $lp < 0$, we get $z_{s+1}^l y_{s+1}^{mp} \geqslant w_{s+1}$ and the proof of (3.15) is similar to that of (3.14). Hence the proof. □

REMARK 6. The case (i) of Theorem (3.6) covers Theorem (3.5). Moreover, while inequality (3.10) holds even for $0 < l < 1$ and $m = 1$, the inequality (3.8) for this case fails.

COROLLARY 1. *Let $\{x_i\}_{i=0}^{n}$ be a sequence of numbers, and $x_0 = 0$. Then,*

(i) *if $l > 0,\ m > 0,\ l+m \geqslant 1$ or $l < 0,\ m < 0$,*

$$\sum_{i=1}^{n} |x_i|^l |\nabla^\alpha x_i|^m \leqslant K_n \sum_{i=1}^{n} |\nabla^\alpha x_i|^{l+m}, \tag{3.16}$$

where $K_0 = mp,\ p = (l+m)^{-1}$, and for n = 1, 2, ...
$K_n = max\ \{K_{n-1} + lp\left[\binom{n+\alpha-1}{n-1}\right]^l n^{l-1}, mp\left[\binom{n+\alpha}{n}\right]^l (n+1)^l\}.$

(ii) *if $l > 0,\ m < 0,\ l+m \leqslant 1,\ l+m \neq 0$ or $l < 0,\ m > 0,\ l+m \geqslant 1$,*

$$\sum_{i=1}^{n} |x_i|^l |\nabla^\alpha x_i|^m \geqslant C_n \sum_{i=1}^{n} |\nabla^\alpha x_i|^{l+m}, \tag{3.17}$$

where $C_0 = mp$, and for n = 1, 2, ...
$C_n = min\ \{C_{n-1} + lp\left[\binom{n+\alpha-1}{n-1}\right]^l n^{l-1}, mp\left[\binom{n+\alpha}{n}\right]^l (n+1)^l\}.$

THEOREM 3.7. *Let $\{x_i\}_{i=0}^{n}$ and $\{y_i\}_{i=0}^{n}$ be sequences of numbers, and $x_0 = y_0 = 0$. Then, the following inequality holds*

$$\sum_{i=1}^{n} |x_i \nabla^\alpha y_i + y_i \nabla^\alpha x_i| \leqslant \frac{n+1}{2} \binom{n+\alpha}{n-1} \sum_{i=1}^{n} \left[(\nabla^\alpha x_i)^2 + (\nabla^\alpha y_i)^2\right].$$

Proof. Using (3.5), we have

$$|x_i| \leqslant \binom{i-1+\alpha}{i-1} \sum_{j=1}^{i} |\nabla^\alpha x_j|.$$

Consider

$$\begin{aligned}
&\sum_{i=1}^{n} |x_i \nabla^\alpha y_i + y_i \nabla^\alpha x_i| \\
&\leqslant \sum_{i=1}^{n} \binom{i-1+\alpha}{i-1} \Big[(\sum_{j=1}^{i} |\nabla^\alpha x_j|)\ \ |\nabla^\alpha y_i| + (\sum_{j=1}^{i} |\nabla^\alpha y_j|)\ \ |\nabla^\alpha x_i| \Big] \\
&\leqslant \sum_{i=1}^{n} \binom{i-1+\alpha}{i-1} \sum_{i=1}^{n} \Big[(\sum_{j=1}^{i} |\nabla^\alpha x_j|)\ \ |\nabla^\alpha y_i| + (\sum_{j=1}^{i} |\nabla^\alpha y_j|)\ \ |\nabla^\alpha x_i| \Big] \quad \text{(using (2.11))} \\
&\leqslant \binom{n+\alpha}{n-1} \Big[\sum_{i=1}^{n}\sum_{j=1}^{i} |\nabla^\alpha x_j||\nabla^\alpha y_i| + \sum_{i=1}^{n}\sum_{j=1}^{i} |\nabla^\alpha y_j||\nabla^\alpha x_i| \Big] \quad \text{(using Lemma 3.1)}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{2}\binom{n+\alpha}{n-1}\Big[\sum_{i=1}^{n}\sum_{j=1}^{i}(|\nabla^\alpha x_j|^2+|\nabla^\alpha y_i|^2) \\
&\quad+\sum_{i=1}^{n}\sum_{j=1}^{i}(|\nabla^\alpha y_j|^2+|\nabla^\alpha x_i|^2)\Big] \quad \text{(using (2.12))} \\
&= \frac{1}{2}\binom{n+\alpha}{n-1}\Big[\sum_{i=1}^{n}\sum_{j=1}^{i}(|\nabla^\alpha x_j|^2+|\nabla^\alpha x_i|^2)+\sum_{i=1}^{n}\sum_{j=1}^{i}(|\nabla^\alpha y_j|^2+|\nabla^\alpha y_i|^2)\Big] \\
&= \frac{n+1}{2}\binom{n+\alpha}{n-1}\Big[\sum_{i=1}^{n}|\nabla^\alpha x_i|^2+\sum_{i=1}^{n}|\nabla^\alpha y_i|^2\Big] \\
&= \frac{n+1}{2}\binom{n+\alpha}{n-1}\sum_{i=1}^{n}\Big[(\nabla^\alpha x_i)^2+(\nabla^\alpha y_i)^2\Big]. \quad \square
\end{aligned}
$$

THEOREM 3.8. *Let $\{x_i\}_{i=0}^n$ and $\{y_i\}_{i=0}^n$ be sequences of numbers, and $x_0 = y_0 = 0$. Then, the following inequality holds*

$$
\sum_{i=1}^{n-1}|x_i\nabla^\alpha y_{i+1}+y_i\nabla^\alpha x_{i+1}| \leqslant \frac{n-1}{2}\binom{n+\alpha-1}{n-2}\sum_{i=1}^{n-1}\Big[(\nabla^\alpha x_{i+1})^2+(\nabla^\alpha y_{i+1})^2\Big]. \tag{3.18}
$$

Proof. Using (3.7), we have

$$
|x_i| \leqslant \binom{i-1+\alpha}{i-1}\sum_{k=0}^{i-1}|\nabla^\alpha x_{k+1}|.
$$

Using (2.11) and Lemma 3.1, we have

$$
\begin{aligned}
&\sum_{i=1}^{n-1}|x_i\nabla^\alpha y_{i+1}+y_i\nabla^\alpha x_{i+1}| \\
&\leqslant \sum_{i=1}^{n-1}\binom{i-1+\alpha}{i-1}\Big[(\sum_{k=0}^{i-1}|\nabla^\alpha x_{k+1}|)\ \ |\nabla^\alpha y_{i+1}|+(\sum_{k=0}^{i-1}|\nabla^\alpha y_{k+1}|)\ \ |\nabla^\alpha x_{i+1}|\Big] \\
&\leqslant \sum_{i=1}^{n-1}\binom{i-1+\alpha}{i-1}\sum_{i=1}^{n-1}\Big[(\sum_{k=0}^{i-1}|\nabla^\alpha x_{k+1}|)\ \ |\nabla^\alpha y_{i+1}|+(\sum_{k=0}^{i-1}|\nabla^\alpha y_{k+1}|)\ \ |\nabla^\alpha x_{i+1}|\Big] \\
&\leqslant \binom{n+\alpha-1}{n-2}\Big[\sum_{i=1}^{n-1}\sum_{k=0}^{i-1}|\nabla^\alpha x_{k+1}|\ \ |\nabla^\alpha y_{i+1}|+\sum_{i=1}^{n-1}\sum_{k=0}^{i-1}|\nabla^\alpha y_{k+1}|\ \ |\nabla^\alpha x_{i+1}|\Big] \\
&\leqslant \frac{1}{2}\binom{n+\alpha-1}{n-2}\Big[\sum_{i=1}^{n-1}\sum_{k=0}^{i-1}(|\nabla^\alpha x_{k+1}|^2+|\nabla^\alpha y_{i+1}|^2) \\
&\quad+\sum_{i=1}^{n-1}\sum_{k=0}^{i-1}(|\nabla^\alpha y_{k+1}|^2+|\nabla^\alpha x_{i+1}|^2)\Big] \quad \text{(using (2.12))}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\binom{n+\alpha-1}{n-2}\Big[\sum_{i=1}^{n-1}\sum_{k=0}^{i-1}(|\nabla^{\alpha}x_{k+1}|^2+|\nabla^{\alpha}x_{i+1}|^2)\\
&\quad+\sum_{i=1}^{n-1}\sum_{k=0}^{i-1}(|\nabla^{\alpha}y_{k+1}|^2+|\nabla^{\alpha}y_{i+1}|^2)\Big]\\
&= \frac{n-1}{2}\binom{n+\alpha-1}{n-2}\Big[\sum_{i=1}^{n-1}|\nabla^{\alpha}x_{i+1}|^2+\sum_{i=1}^{n-1}|\nabla^{\alpha}y_{i+1}|^2\Big]\\
&= \frac{n-1}{2}\binom{n+\alpha-1}{n-2}\sum_{i=1}^{n-1}\Big[(\nabla^{\alpha}x_{i+1})^2+(\nabla^{\alpha}y_{i+1})^2\Big]. \quad \square
\end{aligned}
$$

4. Application

In this section we will find a bound to the solution of a fractional initial value problem by using opial inequalities.

Consider an initial value problem of the form

$$\nabla^{\alpha+1}x_{n+1} = a_n x_n, \quad x(0) = 0, \tag{4.1}$$

where x_n and a_n be any two functions defined on $n \in \mathbb{N}_0^+$ such that $a_n \neq 0$ and $\sum_{i=0}^{n}|a_i| \leqslant M$, where $M > 0$ be any constant. We know that $\nabla x_{i+1}^2 = x_{i+1}\nabla x_{i+1} + x_i\nabla x_{i+1}$ for each $i = 0, 1, ...n$. Then

$$
\begin{aligned}
\sum_{i=0}^{n}(\nabla x_{i+1}^2) &= \sum_{i=0}^{n}(x_{i+1}\nabla x_{i+1}+x_i\nabla x_{i+1})\\
\Rightarrow x_{n+1}^2 &= \sum_{i=0}^{n}(x_{i+1}\nabla x_{i+1}+x_i\nabla x_{i+1}) \ (since\ x_0 = 0)\\
\Rightarrow |x_{n+1}^2| &= |\sum_{i=0}^{n}(x_{i+1}\nabla x_{i+1}+x_i\nabla x_{i+1})|\\
&\leqslant \sum_{i=0}^{n}|x_{i+1}\nabla x_{i+1}+x_i\nabla x_{i+1}|\\
&\leqslant \sum_{i=0}^{n}[|x_{i+1}\nabla x_{i+1}|+|x_i\nabla x_{i+1}|]\\
\Rightarrow |x_{n+1}|^2 &\leqslant \sum_{i=0}^{n}|x_{i+1}\nabla x_{i+1}|+\sum_{i=0}^{n}|x_i\nabla x_{i+1}|.
\end{aligned} \tag{4.2}
$$

Replacing x_i by $\nabla^{\alpha}x_i$ in (4.2), we get

$$
\begin{aligned}
&|\nabla^{\alpha}x_{n+1}|^2\\
&\leqslant \sum_{i=0}^{n}|\nabla^{\alpha}x_{i+1}\nabla^{\alpha+1}x_{i+1}|+\sum_{i=0}^{n}|\nabla^{\alpha}x_i\nabla^{\alpha+1}x_{i+1}|\\
&= \sum_{i=0}^{n}|a_ix_i\nabla^{\alpha}x_{i+1}|+\sum_{i=0}^{n}|a_ix_i\nabla^{\alpha}x_i| \quad \text{(using 4.1)}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{n} |a_i||x_i\nabla^{\alpha}x_{i+1}| + \sum_{i=0}^{n} |a_i||x_i\nabla^{\alpha}x_i| \\
&\leqslant \sum_{i=0}^{n} |a_i| \sum_{i=0}^{n} |x_i\nabla^{\alpha}x_{i+1}| + \sum_{i=0}^{n} |a_i| \sum_{i=0}^{n} |x_i\nabla^{\alpha}x_i| \quad \text{(using (2.11))} \\
&\leqslant M\sum_{i=0}^{n} |x_i\nabla^{\alpha}x_{i+1}| + M\sum_{i=0}^{n} |x_i\nabla^{\alpha}x_i| \\
&\leqslant M\frac{n}{2}\binom{n+\alpha}{n-1}\sum_{i=0}^{n} |\nabla^{\alpha}x_{i+1}|^2 + M\frac{n+1}{2}\binom{n+\alpha}{n-1}\sum_{i=1}^{n} |\nabla^{\alpha}x_i|^2 \quad \text{(using (3.3) and (3.6))} \\
&= M\frac{n}{2}\binom{n+\alpha}{n-1}\sum_{i=1}^{n+1} |\nabla^{\alpha}x_i|^2 + M\frac{n+1}{2}\binom{n+\alpha}{n-1}\sum_{i=1}^{n} |\nabla^{\alpha}x_i|^2 \\
&= M\left[\frac{n}{2}+\frac{n+1}{2}\right]\binom{n+\alpha}{n-1}\sum_{i=1}^{n} |\nabla^{\alpha}x_i|^2 + M\frac{n}{2}\binom{n+\alpha}{n-1}|\nabla^{\alpha}x_{n+1}|^2 \\
&= M\frac{2n+1}{2}\binom{n+\alpha}{n-1}\sum_{i=1}^{n} |\nabla^{\alpha}x_i|^2 + M\frac{n}{2}\binom{n+\alpha}{n-1}|\nabla^{\alpha}x_{n+1}|^2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\left[1-M\frac{n}{2}\binom{n+\alpha}{n-1}\right]|\nabla^{\alpha}x_{n+1}|^2 \leqslant M\frac{2n+1}{2}\binom{n+\alpha}{n-1}\sum_{i=1}^{n} |\nabla^{\alpha}x_i|^2 \\
\Rightarrow &\left[1-M\frac{n}{2}\binom{n+\alpha}{n-1}\right]^2|\nabla^{\alpha}x_{n+1}|^4 \leqslant \left[M\frac{2n+1}{2}\binom{n+\alpha}{n-1}\right]^2[\sum_{i=1}^{n} |\nabla^{\alpha}x_i|^2]^2 \\
&\qquad\leqslant n\left[M\frac{2n+1}{2}\binom{n+\alpha}{n-1}\right]^2[\sum_{i=1}^{n} |\nabla^{\alpha}x_i|^4] \quad \text{(using (2.9))} \\
&\qquad\leqslant n\left[M\frac{2n+1}{2}\binom{n+\alpha}{n-1}\right]^2[\sum_{i=0}^{n-1} |\nabla^{\alpha}x_{i+1}|^4].
\end{aligned}
$$

Thus

$$
\begin{aligned}
|\nabla^{\alpha}x_{n+1}|^4 &\leqslant b_n\sum_{i=0}^{n-1} |\nabla^{\alpha}x_{i+1}|^4 \\
&\leqslant |x_1|^4 + b_n\sum_{i=0}^{n-1} |\nabla^{\alpha}x_{i+1}|^4 \ (\textit{since } x_1 \geqslant 0)
\end{aligned}
$$

where

$$
b_n = \frac{n[M\frac{2n+1}{2}\binom{n+\alpha}{n-1}]^2}{[1-M\frac{n-1}{2}\binom{n+\alpha-1}{n-2}]^2} \tag{4.3}
$$

exists on $n \in \mathbb{N}_0^+$. Let $y_i = |\nabla^{\alpha}x_{i+1}|^4$ for $i = 0,1,2,....n$. Then $y_n \leqslant y_0 + b_n\sum_{i=0}^{n-1} y_i$, where $y_0 = |\nabla^{\alpha}x_1|^4 = |x_1|^4 \geqslant 0$. Using Gronwall's Inequality,

$$\begin{aligned} y_n &\leqslant y_0 \ \exp(b_n \sum_{i=0}^{n-1} 1) \\ \Rightarrow |\nabla^\alpha x_{n+1}|^4 &\leqslant |x_1|^4 \ \exp(nb_n) \\ \Rightarrow |\nabla^\alpha x_n| &\leqslant |x_1| \exp[(n-1)b_{n-1}]^{\frac{1}{4}}. \end{aligned} \tag{4.4}$$

Now using (3.5), for $n \geqslant 1$, we have

$$\begin{aligned} |x_n| &\leqslant \binom{n-1+\alpha}{n-1} \sum_{j=1}^{n} |\nabla^\alpha x_j| \\ &\leqslant \binom{n-1+\alpha}{n-1} \sum_{j=1}^{n} \{|x_1| \ \exp[(j-1)b_{j-1}]^{\frac{1}{4}}\}. \end{aligned}$$

Hence

$$|x_n| \leqslant |x_1| \binom{n-1+\alpha}{n-1} \sum_{j=1}^{n} \{\exp[(j-1)b_{j-1}]^{\frac{1}{4}}\}. \tag{4.5}$$

REFERENCES

[1] ATSUSHI NAGAI, *Fractional logistic map*, 2002.

[2] P. R. BEESACK, *On certain discrete inequalities involving partial sums*, Canad. J. Math. **21** (1969), 222–234.

[3] G. V. S. R. DEEKSHITULU AND J. JAGAN MOHAN, *Fractional difference inequalities*, Communications in Applied Analysis **14**, 1 (2010), 89–98.

[4] G. V. S. R. DEEKSHITULU AND J. JAGAN MOHAN, *Fractional difference inequalities of Bihari type*, Communications in Applied Analysis **14**, 4 (2010), 343–354.

[5] J. B. DIAZ AND T. J. OSLER, *Differences of fractional order*, Math. Comp. **28** (1974), 185–201.

[6] K. FAN, O. TAUSSKY AND J. TODD, *Discrete analogues of inequalities of Wirtinger*, Monatsh. Math. **59** (1955), 73–90.

[7] HENRY L. GRAY AND NIEN FAN ZHANG, *On a New Definition of the Fractional Difference*, Math.Comp. **50** (1988), 513–529.

[8] R. HIROTA, *Lectures on difference equations*, Science-sha, 2000 (in Japanese).

[9] A. LASOTA, *A discrete boundary value problem*, Ann. Polon. Math. **20** (1968), 183–190.

[10] V. LAKSHMIKANTHAM AND S. LEELA, *Differential and integral inequalities*, Vol. I, Academic Press, New York, 1969.

[11] V. LAKSHMIKANTHAM AND D. TRIGIANTE, *Theory of difference equations*, Academic Press, New York, 1988.

[12] C. M. LEE, *On a discrete analogue of inequalities of Opial and Yang*, Canad. Math. Bull. **11** (1968), 73–77.

[13] B. G. PACHPATTE, *Integral and finite difference inequalities and applications*, Elsevier, The Netherlands, 2006.

[14] I. PODLUBNY, *Fractional differential equations*, Academic Press, San Diego, 1999.
[15] RAVI P. AGARWAL AND PETER Y. H. PANG, *Opial inequalities with applications in differential and difference equations*, Vol. 320, Kluwer Academic Publishers, London, 1995.
[16] J. S. W. WONG, *A discrete analogue of Opial's inequality*, Canad. Math. Bull. **10** (1967), 115–118.

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