Fractional Differential Calculus
Volume 2, Number 2 (2012), 119–128
doi:10.7153/fdc-02-09

MULTIPLE POSITIVE SOLUTIONS FOR NONLINEAR FRACTIONAL DIFFERENTIAL SYSTEMS

NEMAT NYAMORADI AND TAHEREH BASHIRI

Abstract. In this paper, we study the existence of positive solutions to boundary value problem for fractional differential system

$$\begin{cases} D_{0+}^{\alpha}u(t)+f_1(t,u(t),v(t))=0, & t\in(0,1),\\ D_{0+}^{\alpha}v(t)+f_2(t,u(t),v(t))=0, & t\in(0,1),\ 1<\alpha\leqslant 2,\\ u(0)=0, \quad D_{0+}^{\beta}u(1)-\mu_1 D_{0+}^{\beta}u(\eta_1)=\lambda_1, \\ v(0)=0, \quad D_{0+}^{\beta}v(1)-\mu_2 D_{0+}^{\beta}v(\eta_2)=\lambda_2,\ 0<\beta<1, \end{cases}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative of order α. By using the Leggett-Williams fixed point theorem in a cone, the existence of three positive solutions for nonlinear singular boundary value problems is obtained.

1. Introduction

The purpose of this paper is to study the existence of positive solutions for the following boundary value problem for fractional differential system

$$\begin{cases} D_{0+}^{\alpha}u(t)+f_1(t,u(t),v(t))=0, & t\in(0,1),\\ D_{0+}^{\alpha}v(t)+f_2(t,u(t),v(t))=0, & t\in(0,1),\ 1<\alpha\leqslant 2,\\ u(0)=0, \quad D_{0+}^{\beta}u(1)-\mu_1 D_{0+}^{\beta}u(\eta_1)=\lambda_1, \\ v(0)=0, \quad D_{0+}^{\beta}v(1)-\mu_2 D_{0+}^{\beta}v(\eta_2)=\lambda_2,\ 0<\beta<1, \end{cases} \tag{1}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative of order α, $\eta_i\in(0,\frac{1}{2})$ and $\mu_i\in[0,\frac{1}{{\eta_i}^{\alpha-\beta-1}})$ are arbitrary constants, λ_i is a paramerer, $f_i:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ is continuous, $i=1,2$. Also, in this paper, we assume that $\mu_i\,{\eta_i}^{\alpha-\beta-2}\leqslant 1-\beta$.

Fractional differential equations have been of great interest recently. This is because of both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various scientific fields such as physics, mechanics, chemistry, engineering, etc. For details, see [1, 2, 3] and the references therein.

The existence of solutions of initial value problems for fractional order differential equations have been studied in the literature [4, 5, 6, 7] and the references therein.

Mathematics subject classification (2010): 34K15.
Keywords and phrases: Cone, multi point boundary value problem, fixed point theorem, Riemann-Liouville fractional derivative.

Abdelkader Saadi, Maamar Benbachir [8] considered the following boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) + a(t)g(u(t)) = 0, \quad & t \in (0,1),\ 2 < \alpha < 3, \\ u(0) = u'(0) = 0, \quad u'(1) - \mu u'(\eta) = \lambda, \end{cases} \tag{2}$$

where $\eta \in (0,1)$, $\mu \in \left[0, \frac{1}{\eta^{\alpha-2}}\right)$ are two arbitrary constants. They applied the Guo-Krasnosel'skii fixed point theorem and Schauder's fixed point theorem to establish some results on the existence, nonexistence and uniqueness of positive solutions (2).

Li, Luo and Zhou [9] considered the following boundary value problem of fractional order

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t,u(t)) = 0, \quad & t \in (0,1),\ 1 < \alpha \leqslant 2, \\ u(0) = 0, \quad D_{0+}^{\beta} u(1) = aD_{0+}^{\beta} u(\xi), \end{cases} \tag{3}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative of order α.

Motivated by the works mentioned above, our purpose in this paper is to show the existence and multiplicity of positive solutions to the problem (1) by using the Leggett-Williams fixed point theorem.

The rest of the article is organized as follows: in Section 2, we present some preliminaries that will be used in Section 3. The main result and proof will be given in Section 3. Finally, in Section 4, an example is given to demonstrate the application of our main result.

2. Preliminaries

In this section, we present some notations and preliminary lemmas that will be used in the proof of the main result.

First, we define

$$L^1([0,1]) := \left\{ u : [0,1] \to \mathbb{R};\ u \text{ is measurable on } [0,1] \text{ and } \int_0^1 |u(x)|dx < \infty \right\},$$

$$C([0,1]) := \left\{ u : [0,1] \to \mathbb{R};\ u \text{ is a continuous function on } [0,1] \right\}.$$

It is obvious that, $C([0,1])$, is a Banach space.

DEFINITION 1. Let X be a real Banach space. A non-empty closed set $P \subset X$ is called a cone of X if it satisfies the following conditions:

(1) $x \in P, \mu \geqslant 0$ implies $\mu x \in P$,

(2) $x \in P, -x \in P$ implies $x = 0$.

DEFINITION 2. ([4]) The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L^1([0,1])$ is defined as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

DEFINITION 3. ([4]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f:[0,1]\to\mathbb{R}$ is defined as

$$D_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{0}^{t}(t-s)^{n-\alpha-1}f(s)ds,$$

where $n=[\alpha]+1$.

LEMMA 1. ([10]) *The equality* $D_{0+}^{\gamma}I_{0+}^{\gamma}f(t)=f(t)$, $\gamma>0$ *holds for* $f\in L^{1}([0,1])$.

LEMMA 2. ([10]) *Let* $\alpha>0$. *Then the differential equation*

$$D_{0+}^{\alpha}u=0$$

has a unique solution $u(t)=c_{1}t^{\alpha-1}+c_{2}t^{\alpha-2}+\cdots+c_{n}t^{\alpha-n}$, $c_{i}\in\mathbb{R}$, $i=1,\ldots,n$, *there* $n-1<\alpha\leqslant n$, $u\in L^{1}([0,1])$.

LEMMA 3. ([10]) *Let* $\alpha>0$. *Then the following equality holds for* $u\in L^{1}([0,1])$, $D_{0+}^{\alpha}u\in L^{1}([0,1])$*;*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)=u(t)+c_{1}t^{\alpha-1}+c_{2}t^{\alpha-2}+\cdots+c_{n}t^{\alpha-n},$$

$c_{i}\in\mathbb{R}$, $i=1,\ldots,n$, *there* $n-1<\alpha\leqslant n$.

In the following, we present the Green function of fractional differential equation boundary value problem.

LEMMA 4. *Suppose that* $\Delta_{i}=1-\mu_{i}\ \eta_{i}^{\alpha-\beta-1}\neq 0$. *Let* $y(t)\in C[0,1]$, *then the boundary value problem*

$$\begin{cases} D_{0+}^{\alpha}u(t)+y(t)=0, & t\in(0,1),\\ u(0)=0, \quad D_{0+}^{\beta}u(1)-\mu_{i}D_{0+}^{\beta}u(\eta_{i})=\lambda_{i} \end{cases} \tag{4}$$

has a unique solution

$$u(t)=\int_{0}^{1}G_{i}(t,s)y(s)ds+\frac{\lambda_{i}\Gamma(\alpha-\beta)t^{\alpha-1}}{\Gamma(\alpha)\Delta_{i}}, \tag{5}$$

where

$$G_{i}(t,s)=\begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}-\mu_{i}t^{\alpha-1}(\eta_{i}-s)^{\alpha-\beta-1}-\Delta_{i}(t-s)^{\alpha-1}}{\Delta_{i}\Gamma(\alpha)} & 0\leqslant s\leqslant\min\{t,\eta_{i}\}<1,\\ \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}-\Delta_{i}(t-s)^{\alpha-1}}{\Delta_{i}\Gamma(\alpha)} & 0<\eta_{i}\leqslant s\leqslant t\leqslant 1,\\ \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}-\mu_{i}t^{\alpha-1}(\eta_{i}-s)^{\alpha-\beta-1}}{\Delta_{i}\Gamma(\alpha)} & 0\leqslant t\leqslant s<\eta_{i}<1,\\ \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Delta_{i}\Gamma(\alpha)} & \max\{t,\eta_{i}\}\leqslant s\leqslant 1. \end{cases} \tag{6}$$

Proof. In view of Lemma 3 and equation (4), we have

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y(s)\,ds - c_1 t^{\alpha-1} - c_2 t^{\alpha-2}, \tag{7}$$

for some arbitrary constants $c_1, c_2 \in \mathbb{R}$.

The boundary condition $u(0) = 0$ implies that $c_2 = 0$. Thus

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y(s)\,ds - c_1 t^{\alpha-1}.$$

On the other hand, by relations $D^{\beta}_{0^+} t^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)} t^{\alpha-\beta}$, $D^{\alpha}_{0^+} I^{\alpha}_{0^+} u(t) = u(t)$ and $I^{m}_{0^+} I^{n}_{0^+} u(t) = I^{m+n}_{0^+} u(t)$ for $m,n > 0$, $u \in L^1(0,1)$, we have

$$\begin{aligned}
D^{\beta}_{0^+} u(t) &= -D^{\beta}_{0^+} I^{\alpha}_{0^+} u(t) - c_1 D^{\beta}_{0^+} t^{\alpha-1} \\
&= -D^{\beta}_{0^+} I^{\beta}_{0^+} I^{\alpha-\beta}_{0^+} u(t) - c_1 D^{\beta}_{0^+} t^{\alpha-1} \\
&= -I^{\alpha-\beta}_{0^+} u(t) - c_1 D^{\beta}_{0^+} t^{\alpha-1} \\
&= -\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1} y(s)\,ds - c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1}.
\end{aligned}$$

In view of the boundary condition $D^{\beta}_{0^+} u(1) - \mu_i D^{\beta}_{0^+} u(\eta_i) = \lambda_i$, we conclude that

$$\begin{aligned}
c_1 = &-\frac{1}{\Gamma(\alpha)\Delta_i}\int_0^1 (1-s)^{\alpha-\beta-1} y(s)\,ds \\
&+ \frac{\mu_i}{\Gamma(\alpha)\Delta_i}\int_0^{\eta_i} (\eta_i - s)^{\alpha-\beta-1} y(s)\,ds - \frac{\lambda_i \Gamma(\alpha-\beta)}{\Gamma(\alpha)(\Delta_i)}.
\end{aligned}$$

Substituting the values of c_0 and c_1 in (7), we obtain the solution (4) as follow

$$\begin{aligned}
u(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds + \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)\Delta_i} y(s)\,ds \\
&\quad - \int_0^{\eta_i} \frac{\mu_i t^{\alpha-1}(\eta_i - s)^{\alpha-\beta-1}}{\Gamma(\alpha)\Delta_i} y(s)\,ds + \frac{\lambda_i \Gamma(\alpha-\beta) t^{\alpha-1}}{\Gamma(\alpha)\Delta_i} \\
&= \int_0^1 G_i(t,s) y(s) ds + \frac{\lambda_i \Gamma(\alpha-\beta) t^{\alpha-1}}{\Gamma(\alpha)\Delta_i},
\end{aligned}$$

where $G_i(t,s)$ is given in (6). Therefore, the proof is completed. □

LEMMA 5. ([9]) *The function $G_i(t,s)$, in lemma 4 satisfies the following conditions*

(i) $G_i(t,s)$ is continuous on $[0,1]\times[0,1]$,
(ii) $G_i(t,s) > 0$, for any $t \in (0,1)$.

LEMMA 6. ([9]) *Assume that* $\mu_i \eta_i^{\alpha-\beta-2} \leqslant 1-\beta$. *Then the function* $G_i(t,s)$ *satisfies the following conditions*

(i) $G_i(t,s) \leqslant G_i(s,s)$, *for* $s,t \in [0,1]$,

(ii) there exists a positive function $\gamma_i(s) \in C[0,1]$ *such that*

$$\min_{\eta_i \leqslant t \leqslant 1} G_i(t,s) \geqslant \gamma_i(s) \max_{0 \leqslant t \leqslant 1} G_i(t,s) = \gamma_i(s) G_i(s,s), \quad \text{for } 0 < s < 1.$$

Now, we consider the system (1). Obviously, $(u,v) \in C(0,1) \times C(0,1)$ is a solution of the system (1) if and only if $(u,v) \in C[0,1] \times C[0,1]$ is a solution of the following nonlinear system:

$$\begin{cases} u(t) = \displaystyle\int_0^1 G_1(t,s) f_1(s,u(s),v(s))ds + \frac{\lambda_1 \Gamma(\alpha-\beta) t^{\alpha-1}}{\Gamma(\alpha)\Delta_1}, \\ v(t) = \displaystyle\int_0^1 G_2(t,s) f_2(s,u(s),v(s))ds + \frac{\lambda_2 \Gamma(\alpha-\beta) t^{\alpha-1}}{\Gamma(\alpha)\Delta_2}. \end{cases} \tag{8}$$

To establish the existence three positive solutions of system (1), we will employ the following Leggett-Williams fixed point theorem.

For the convenience of the reader, we present here the Leggett-Williams fixed point theorem [11].

Given a cone K in a real Banach space E, a map α is said to be a nonnegative continuous concave (resp. convex) functional on K provided that $\alpha : K \to [0,+\infty)$ is continuous and

$$\alpha(tx+(1-t)y) \geqslant t\alpha(x) + (1-t)\alpha(y),$$
$$(resp.\ \alpha(tx+(1-t)y) \leqslant t\alpha(x) + (1-t)\alpha(y)),$$

for all $x,y \in K$ and $t \in [0,1]$.

Let $0 < a < b$ be given and let α be a nonnegative continuous concave functional on K. Define the convex sets P_r and $P(\alpha,a,b)$ by

$$P_r = \{x \in K \mid \|x\| < r\},$$

and

$$P(\alpha,a,b) = \{x \in K \mid a \leqslant \alpha(x), \|x\| \leqslant b\}.$$

THEOREM 1. ([11]) *Let* $A : \overline{P_c} \to \overline{P_c}$ *be a completely continuous operator and let* α *be a nonnegative continuous concave functional on* K *such that* $\alpha(x) \leqslant \|x\|$ *for all* $x \in \overline{P_c}$. *Suppose there exist* $0 < a < b < d \leqslant c$ *such that*

(A1) $\{x \in P(\alpha,b,d) \mid \alpha(x) > b\} \neq \emptyset$, *and* $\alpha(Ax) > b$ *for* $x \in P(\alpha,b,d)$,

(A2) $\|Ax\| < a$ *for* $\|x\| \leqslant a$, *and*

(A3) $\alpha(Ax) > b$ *for* $x \in P(\alpha,b,c)$ *with* $\|Ax\| > d$.

Then A *has at least three fixed points* x_1, x_2, *and* x_3 *and such that* $\|x_1\| < a, b < \alpha(x_2)$ *and* $\|x_3\| > a$, *with* $\alpha(x_3) < b$.

3. Main result

For convenience, we introduce the following notations. define

$$\tau = \min\{\eta_1, \eta_2\},$$

so $0 < \tau < \frac{1}{2}$. Let

$$M_i = \max_{0\leqslant t\leqslant 1}\Big[\int_0^1 G(t,s)ds\Big],$$
$$m_i = \min_{\tau\leqslant t\leqslant 1}\Big[\int_\tau^1 G(t,s)ds\Big].$$

Then $0 < m_i < M_i$, $i = 1,2$.

Define

$$\eta = \min_{\tau\leqslant s\leqslant 1}\{\gamma_1(s), \gamma_2(s)\}, \tag{9}$$

and let

$$\sigma = \frac{1}{2}\min\{\eta, 2\tau^{\alpha-1}\}. \tag{10}$$

The basic space used in this paper is a real Banach space $E = C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ with the norm $||(u,v)|| := ||u|| + ||v||$, where $\|u\| = \max_{t\in[0,1]} |u(t)|$.

Then, choose a cone $K \subset E$, by

$$K = \{(u,v) \in E \mid u(t) \geqslant 0,\ v(t) \geqslant 0,\ \min_{\tau\leqslant t\leqslant 1}(u(t)+v(t)) \geqslant \sigma\|(u,v)\|\}.$$

Define an operator T by

$$T(u,v)(t) = (A(u,v)(t), B(u,v)(t)), \quad \forall\, t \in (0,1), \tag{11}$$

where

$$\begin{cases} A(u,v)(t) = \int_0^1 G_1(t,s) f_1(s,u(s),v(s))ds + \frac{\lambda_1\Gamma(\alpha-\beta)t^{\alpha-1}}{\Gamma(\alpha)\Delta_1}, \\ B(u,v)(t) = \int_0^1 G_2(t,s) f_2(s,u(s),v(s))ds + \frac{\lambda_2\Gamma(\alpha-\beta)t^{\alpha-1}}{\Gamma(\alpha)\Delta_2}, \end{cases} \tag{12}$$

LEMMA 7. *The operator defined in (11) maps K into itself, i.e.,* $T : K \to K$.

Proof. For any $(u,v) \in K$, it follows from Lemma 6 that

$$\begin{aligned} ||A(u,v)|| &\leqslant \int_0^1 G_1(s,s) f_1(s,u(s),v(s))ds + \frac{\lambda_1\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Delta_1}, \\ &\leqslant 2\int_\tau^1 G_1(s,s) f_1(s,u(s),v(s))ds + \frac{\lambda_1\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Delta_1}, \end{aligned}$$

This implies that

$$\Big[\int_\tau^1 G_1(s,s)f_1(s,u(s),v(s))ds+\frac{\lambda_1\Gamma(\alpha-\beta)}{2\Gamma(\alpha)\Delta_1}\Big]\geqslant\frac{1}{2}||A(u,v)||.$$

Therefore, we can get

$$\begin{aligned}
&\min_{\tau\leqslant t\leqslant 1} A(u,v)(t)\\
&=\min_{\tau\leqslant t\leqslant 1}\Big[\int_0^1 G_1(t,s)f_1(s,u(s),v(s))ds+\frac{\lambda_1\Gamma(\alpha-\beta)t^{\alpha-1}}{\Gamma(\alpha)\Delta_1}\Big]\\
&\geqslant\int_\tau^1 \gamma_1(s)G_1(s,s)f_1(s,u(s),v(s))ds+\frac{\lambda_1\Gamma(\alpha-\beta)\tau^{\alpha-1}}{\Gamma(\alpha)\Delta_1}\\
&\geqslant\eta\int_\tau^1 G_1(s,s)f_1(s,u(s),v(s))ds+\frac{\lambda_1\Gamma(\alpha-\beta)\tau^{\alpha-1}}{\Gamma(\alpha)\Delta_1}\\
&\geqslant\min\{\eta,2\tau^{\alpha-1}\}\Big[\int_\tau^1 G_1(s,s)f_1(s,u(s),v(s))ds+\frac{\lambda_1\Gamma(\alpha-\beta)}{2\Gamma(\alpha)\Delta_1}\Big]\\
&\geqslant\frac{1}{2}\min\{\eta,2\tau^{\alpha-1}\}||A(u,v)||\\
&=\sigma||A(u,v)||.
\end{aligned}$$

In the same way, for any $(u,v)\in K$, we have

$$\min_{\tau\leqslant t\leqslant 1} B(u,v)(t)\geqslant\sigma||B(u,v)||.$$

Therefore

$$\min_{\tau\leqslant t\leqslant 1}(A(u,v)(t)+B(u,v)(t))\geqslant\sigma||A(u,v)||+\sigma||B(u,v)||=\sigma||(A(u,v),B(u,v))||.$$

From the above, we conclude that $T(u,v)(t)=(A(u,v)(t),B(u,v)(t))\in K$, that is, $T(K)\subset K$. Therefore, the proof is completed. □

It is clear that the existence of a positive solution for the system (1) is equivalent to the existence of a nontrivial fixed point of T in K.

Finally, we define the nonnegative continuous concave functional on K by

$$\alpha(u,v)=\min_{\tau\leqslant t\leqslant 1}(u(t)+v(t)).$$

It is obvious that, for each $(u,v)\in K,\alpha(u,v)\leqslant\|(u,v)\|$.

Throughout this section, we assume that p_i, q_i, $i=1,2$, are four positive numbers satisfying $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{q_1}+\frac{1}{q_2}\leqslant 1$.

Now, we can state our main result.

THEOREM 2. *Assume there exist nonnegative numbers a,b,c such that $0<a<b\leqslant\min\{\tau,\frac{m_1}{p_1M_1},\frac{m_2}{p_2M_2}\}c$, and $f_i(t,u,v)$ satisfy the following conditions:*

(H1) $f_i(t,u,v) \leqslant \frac{1}{p_i} \cdot \frac{c}{M_i}$, $\forall t \in [0,1], u+v \in [0,c],\ i=1,2,$
(H2) $f_i(t,u,v) \leqslant \frac{1}{p_i} \cdot \frac{a}{M_i}$, $\forall t \in [0,1], u+v \in [0,a],\ i=1,2,$
(H3) (i) $f_1(t,u,v) > \frac{b}{m_1}\ \forall t \in [\tau,1], u+v \in [b, \frac{b}{\sigma}]$, *or*
(ii) $f_2(t,u,v) > \frac{b}{m_2}\ \forall t \in [\tau,1], u+v \in [b, \frac{b}{\sigma}]$.
Then, for

$$\lambda_i \leqslant \frac{1}{q_i} \cdot \frac{c\Delta_i \Gamma(\alpha)}{\Gamma(\alpha-\beta)}, i=1,2, \tag{13}$$

the system (1) has at least three positive solutions $(u_1,v_1),(u_2,v_2),(u_3,v_3)$ *such that* $\|(u_1,v_1)\| < a, b < \min_{\tau\leqslant t\leqslant 1}(u_2(t)+v_2(t))$, *and* $\|(u_3,v_3)\| > a$, *with* $\min_{\tau\leqslant t\leqslant 1}(u_3(t)+v_3(t)) < b$.

Proof. First, we show that $T : \overline{P_c} \to \overline{P_c}$ is a completely continuous operator. If $(u,v) \in \overline{P_c}$, then $||(u,v)|| \leqslant c$ and by (H1) and (12), we have

$$\begin{aligned}
\|T(u,y)\| &= \max_{0\leqslant t\leqslant 1} |A(u,v)(t)| + \max_{0\leqslant t\leqslant 1} |B(u,v)(t)| \\
&= \max_{0\leqslant t\leqslant 1} \Big\{ \int_0^1 G_1(t,s) f_1(s,u(s),v(s))ds + \frac{\lambda_1 \Gamma(\alpha-\beta) t^{\alpha-1}}{\Gamma(\alpha)\Delta_1} \Big\}, \\
&\quad + \max_{0\leqslant t\leqslant 1} \Big\{ \int_0^1 G_2(t,s) f_2(s,u(s),v(s))ds + \frac{\lambda_2 \Gamma(\alpha-\beta) t^{\alpha-1}}{\Gamma(\alpha)\Delta_2} \Big\} \\
&\leqslant \frac{1}{p_1} \cdot \frac{c}{M_1} \max_{0\leqslant t\leqslant 1} \Big\{ \int_0^1 G_1(t,s)ds \Big\} + \frac{\lambda_1 \Gamma(\alpha-\beta)}{\Gamma(\alpha)\Delta_1} \\
&\quad + \frac{1}{p_2} \cdot \frac{c}{M_2} \max_{0\leqslant t\leqslant 1} \Big\{ \int_0^1 G_2(t,s)ds \Big\} + \frac{\lambda_2 \Gamma(\alpha-\beta)}{\Gamma(\alpha)\Delta_2} \\
&\leqslant \frac{1}{p_1} \cdot c + \frac{1}{q_1} \cdot c + \frac{1}{p_2} \cdot c + \frac{1}{q_2} \cdot c \leqslant c.
\end{aligned}$$

Therefore, $\|T(u,y)\| \leqslant c$, that is, $T : \overline{P_c} \to \overline{P_c}$. The operator T is completely continuous by an application of the Ascoli-Arzela theorem.

In the same way, the condition (H2) implies that the condition (A2) of Theorem 1 is satisfied. We now show that condition (A1) of Theorem 1 is satisfied. Clearly, $\{(u,v) \in P(\alpha, b, \frac{b}{\sigma}) | \alpha(u,v) > b\} \neq \emptyset$. If $(u,v) \in P(\alpha, b, \frac{b}{\sigma})$, then $b \leqslant u(s)+v(s) \leqslant \frac{b}{\sigma}, s \in [\tau,1]$.

By condition (H3)(i), we get

$$\begin{aligned}
\alpha(T(u,v)(t)) &= \min_{\tau\leqslant t\leqslant 1} (A(u,v)(t) + B(u,v)(t)) \\
&\geqslant \min_{\tau\leqslant t\leqslant 1} \Big\{ \int_\tau^1 G_1(t,s) f_1(s,u(s),v(s))ds + \frac{\lambda_1 \Gamma(\alpha-\beta) t^{\alpha-1}}{\Gamma(\alpha)\Delta_1} \Big\} \\
&\quad + \min_{\tau\leqslant t\leqslant 1} \Big\{ \int_\tau^1 G_2(t,s) f_2(s,u(s),v(s))ds + \frac{\lambda_2 \Gamma(\alpha-\beta) t^{\alpha-1}}{\Gamma(\alpha)\Delta_2} \Big\}
\end{aligned}$$

$$> \frac{b}{m_1} \min_{\tau \leqslant t \leqslant 1} \left\{ \int_\tau^1 G_1(t,s)ds \right\}$$
$$= \frac{b}{m_1} \cdot m_1 = b.$$

Similarly, by (H3)(ii), we get

$$\alpha(T(u,v)(t)) > \frac{b}{m_2} \min_{\tau \leqslant t \leqslant 1} \left\{ \int_\tau^1 G_2(t,s)ds \right\} = \frac{b}{m_2} \cdot m_2 = b.$$

Therefore, condition $A1$ of Theorem 1 is satisfied.

Finally, we show that the condition $A3$ of Theorem 1 is also satisfied.

If $(u,v) \in P(\alpha,b,c)$, and $\|T(u,v)\| > \frac{b}{\sigma}$, then

$$\alpha(T(u,v)(t)) = \min_{\tau \leqslant t \leqslant 1} (A(u,v)(t) + B(u,v)(t)) \geqslant \sigma\|A(u,v)\| + \sigma\|B(u,v)\|$$
$$= \sigma\|T(u,v)\| > \sigma \cdot \frac{b}{\sigma} = b.$$

Therefore, the condition $A3$ of Theorem 1 is also satisfied. By Theorem 1, there exist three positive solutions $(u_1,v_1),(u_2,v_2),(u_3,v_3)$ such that $\|(u_1,v_1)\| < a, b < \min_{\tau \leqslant t \leqslant 1}(u_2(t)+v_2(t))$, and $\|(u_3,v_3)\| > a$, with $\min_{\tau \leqslant t \leqslant 1}(u_3(t)+v_3(t)) < b$. we have the conclusion.

4. Application

EXAMPLE 3. Consider the following singular boundary value problem:

$$\begin{cases} D_{0^+}^{\frac{3}{2}} u(t) + f_1(t,u(t),v(t)) = 0, & t \in (0,1), \\ D_{0^+}^{\frac{3}{2}} v(t) + f_2(t,u(t),v(t)) = 0, & t \in (0,1), \\ u(0) = 0, \quad D_{0^+}^{\frac{1}{2}} u(1) - \frac{1}{8} D_{0^+}^{\frac{1}{2}} u(\frac{1}{4}) = \lambda_1, \\ v(0) = 0, \quad D_{0^+}^{\frac{1}{2}} v(1) - \frac{1}{8} D_{0^+}^{\frac{1}{2}} v(\frac{1}{4}) = \lambda_2, \end{cases} \tag{14}$$

where $\mu_1 = \mu_2 = \frac{1}{8}$, $\eta_1 = \eta_2 = \frac{1}{4}$ and

$$f_1(t,u,v) = \begin{cases} \frac{\sqrt{1-t^2}}{100} + \frac{1}{200}(u+v)^2, & t \in [0,1], \quad 0 \leqslant u+v \leqslant 1, \\ \frac{\sqrt{1-t^2}}{100} + 10[(u+v)^2 - (u+v)] + \frac{1}{200}, & t \in [0,1], \quad 1 < u+v < 2, \\ \frac{\sqrt{1-t^2}}{100} + 6[\log_2(u+v) + 2(u+v)] + \frac{1}{200}, & t \in [0,1], \quad 2 \leqslant u+v \leqslant 4 \\ \frac{\sqrt{1-t^2}}{100} + \frac{\sqrt{u+v}}{2} + 59 + \frac{1}{200}, & t \in [0,1], \quad 4 < u+v < +\infty, \end{cases}$$

and

$$f_2(t,u,v) = \begin{cases} \frac{\sqrt{1-t^2}}{1000} + \frac{1}{400}(u+v)^2, & t \in [0,1], \quad 0 \leqslant u+v \leqslant 1, \\ \frac{\sqrt{1-t^2}}{1000} + 20[(u+v)^2 - (u+v)] + \frac{1}{400}, & t \in [0,1], \quad 1 < u+v < 2, \\ \frac{\sqrt{1-t^2}}{1000} + 8[\log_2(u+v) + 2(u+v)] + \frac{1}{400}, & t \in [0,1], \quad 2 \leqslant u+v \leqslant 4 \\ \frac{\sqrt{1-t^2}}{1000} + \frac{\sqrt{u+v}}{2} + 79 + \frac{1}{400}, & t \in [0,1], \quad 4 < u+v < +\infty, \end{cases}$$

Choose $\tau = \frac{1}{4}$, $p_1 = 20$, $p_2 = 2$, $q_1 = 5$, $q_2 = 4$. Then by direct calculations, we can obtain that

$$M_1 = M_2 = 0.846500912998,$$
$$m_1 = m_2 = 0.427057586114.$$

So, we choose $a = 1$, $b = 2$, $c = 1300$. It is easy to check that f_i, $i = 1,2$ satisfy the conditions (H1)-(H3). So, for $\lambda_1 \leqslant 201.5655135$ and $\lambda_2 \leqslant 251.9568919$, system (14) has at least three positive solutions $(u_1,v_1),(u_2,v_2),(u_3,v_3)$ such that $\|(u_1,v_1)\| < 1, 2 < \min_{\frac{1}{4}\leqslant t\leqslant 1}(u_2(t)+v_2(t))$, and $\|(u_3,v_3)\| > 1$, with $\min_{\frac{1}{4}\leqslant t\leqslant 1}(u_3(t)+v_3(t)) < 2$.

Acknowledgement. We are very grateful to the reviewer for their useful comments and suggestions that led to improvement of our manuscript

REFERENCES

[1] A. M. A. El-Sayed, *Nonlinear functional differential equations of arbitrary orders*, Nonlinear Anal. 33 (1998) 181–186.
[2] A. A. Kilbas, J. J. Trujillo, *Differential equations of fractional order: Methods, results and problems I*, Appl. Anal. 78 (2001) 153–192.
[3] A. A. Kilbas, J. J. Trujillo, *Differential equations of fractional order: Methods, results and problems II*, Appl. Anal. 81 (2002) 435–493.
[4] G. Samko, A. Kilbas, O. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Amsterdam, 1993.
[5] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.
[6] R. P. Agarwal, M. Benchohra, S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math. 109 (2010) 973–1033.
[7] V. Lakshmikantham, A. S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Anal. TMA 69 (8) (2008) 2677–2682.
[8] A. Saadi, M. Benbachir, *Positive solutions for three-point nonlinear fractional boundary value problems*, Electron. J. Qual. Theory Differ. Equ. 2 (2011) 1–19.
[9] C. F. Li, X. N. Luo, Y. Zhou, *Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations*, Comput. Math. Apll. 59 (2010) 1363–1375.
[10] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
[11] R. W. Leggett, L. R. Williams, *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana Univ. Math. J. 28 (1979) 673–688.

(Received February 29, 2012)

Nemat Nyamoradi
Department of Mathematics Faculty of Sciences
Razi University
67149 Kermanshah, Iran
e-mail: nyamoradi@razi.ac.ir.

Tahereh Bashiri
Department of Mathematics Faculty of Sciences
Razi University
67149 Kermanshah, Iran