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A NEW CLASS OF MULTIVALENTLY ANALYTIC FUNCTIONS ASSOCIATED WITH FRACTIONAL q–CALCULUS OPERATORS

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Abstract. Making use of a certain operator of fractional q-derivative, we introduce a new class of multivalently analytic functions in the open unit disk. Among the results investigated for this class of functions include the coefficient inequalities and some distortion theorems. The results provide q-extensions of various known results in the theory of analytic functions. Special cases of the results are also pointed out in the concluding section of this paper.

1. Introduction, preliminaries and definitions

The q-shifted factorial is defined for $\alpha, q \in \mathbb{C}$ as a product of n factors by

$$(\alpha;q)_n = \begin{cases} 1 & , \quad n=0 \\ (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}) & , \quad n \in \mathbb{N}, \end{cases} \tag{1.1}$$

and in terms of the basic analogue of the gamma function

$$(q^{\alpha};q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}, \qquad (n>0), \tag{1.2}$$

where the q-gamma function is defined by ([3, p. 16, eqn. (1.10.1)])

$$\Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty}, \qquad (0<q<1). \tag{1.3}$$

If $|q|<1$, the definition (1.1) remains meaningful for $n=\infty$ as a convergent infinite product:

$$(\alpha;q)_\infty = \prod_{j=0}^{\infty}(1-\alpha q^j).$$

We recall here the following q-analogue definitions given by Gasper and Rahman [3]. The recurrence relation for q-gamma function is given by

$$\Gamma_q(1+x) = \frac{(1-q^x)\,\Gamma_q(x)}{1-q} \tag{1.4}$$

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and the q-binomial expansion is given by

$$(x-y)_\nu = x^\nu(-y/x;q)_\nu = x^\nu \prod_{n=0}^{\infty}\left[\frac{1-(y/x)q^n}{1-(y/x)q^{\nu+n}}\right] = x^\nu_1\Phi_0[q^{-\nu};-;q,yq^\nu/x]. \quad (1.5)$$

Also, the Jackson's q-derivative and q-integral of a function f defined on a subset of $\mathbb{C}$ are, respectively, given by (see Gasper and Rahman [3, pp. 19-22])

$$D_{q,z}f(z) = \frac{f(z)-f(zq)}{z(1-q)}, \qquad (z \neq 0,\ q \neq 0) \quad (1.6)$$

and

$$\int_0^z f(t)d(t;q) = z(1-q)\sum_{k=0}^{\infty} q^k f(zq^k). \quad (1.7)$$

In view of the relation

$$\lim_{q\to 1^-}\frac{(q^\alpha;q)_n}{(1-q)^n} = (\alpha)_n, \quad (1.8)$$

we observe that the q-shifted factorial (1.1) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$.

2. Fractional q-calculus operators

The fractional calculus operators has gained importance and popularity, mainly due to its vast potential of demonstrated applications in various fields of applied sciences, engineering and also in the geometric function theory of complex analysis (see, for example [5] and [10]).

The fractional q-calculus is the extension of the ordinary fractional calculus in the q-theory. The theory of q-calculus operators in recent past have been applied in the areas of ordinary fractional calculus, optimal control problems and in finding solutions of the q-difference and q-integral equations, and in q-transform analysis. One may refer to the book [3], and the recent papers [1], [2], [4] and [8] on the subject. In a recent paper Purohit and Raina [7], investigated applications of fractional q-calculus operators to defined certain new classes of functions which are analytic in the open disk.

We now define the fractional q-calculus operators of a complex-valued function $f(z)$, which were recently studied by Purohit and Raina [7].

DEFINITION 1. (Fractional q-integral operator) The fractional q-integral operator $I^\alpha_{q,z}$ of a function $f(z)$ of order α is defined by

$$I^\alpha_{q,z}f(z) \equiv D^{-\alpha}_{q,z}f(z) = \frac{1}{\Gamma_q(\alpha)}\int_0^z (z-tq)_{\alpha-1}f(t)d(t;q), \quad (\alpha > 0), \quad (2.1)$$

where $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin. In view of relation (1.5), the q-binomial function $(z-tq)_{\alpha-1}$ can be expressed as

$$(z-tq)_{\alpha-1} = z^{\alpha-1}_1\Phi_0[q^{-\alpha+1};-;q,tq^\alpha/z]. \quad (2.2)$$

Following Gasper and Rahman [3], the series ${}_1\Phi_0[\alpha;-;q,z]$ (which is a special case of the series ${}_2\Phi_1[\alpha,\beta;\gamma;q,z]$ for $\gamma=\beta$) is single-valued when $|\arg(z)|<\pi$ and $|z|<1$ (see for details [3, pp.104-106]), therefore, the function $(z-tq)_{\alpha-1}$ in (2.1) is single-valued when $|\arg(-tq^{\alpha}/z)|<\pi, |tq^{\alpha}/z)|<1$ and $|\arg z|<\pi$.

DEFINITION 2. (Fractional q-derivative operator) The fractional q-derivative operator $D^{\alpha}_{q,z}$ of a function $f(z)$ of order α is defined by

$$D^{\alpha}_{q,z}f(z)=D_{q,z}I^{1-\alpha}_{q,z}f(z)=\frac{1}{\Gamma_q(1-\alpha)}D_{q,z}\int_0^z(z-tq)_{-\alpha}f(t)d(t;q) \qquad (0\leqslant\alpha<1), \tag{2.3}$$

where $f(z)$ is suitably constrained and the multiplicity of $(z-tq)_{-\alpha}$ is removed as in Definition 1 above. Also for $\alpha=1$, we have

$$D^{1}_{q,z}f(z)=D_{q,z}f(z). \tag{2.4}$$

DEFINITION 3. (Extended fractional q-derivative operator) Under the hypotheses of Definition 2, the fractional q-derivative for a function $f(z)$ of order α is defined by

$$D^{\alpha}_{q,z}f(z)=D^{m}_{q,z}I^{m-\alpha}_{q,z}f(z), \tag{2.5}$$

where $m-1\leqslant\alpha<m$, $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and $\mathbb{N}$ denotes the set of natural numbers.

The object of this paper is to introduce a new class of functions defined by using fractional q-calculus operators which is multivalently analytic in the open unit disk. We also derive some results giving coefficient inequalities and distortion theorems involving the fractional q-calculus operators. Special cases of the results are also pointed out in the concluding section of this paper.

3. A new class of functions

Let $\mathscr{A}_p$ denote the class of functions of the form

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k, \qquad (p\in\mathbb{N}), \tag{3.1}$$

which are analytic and p-valent in the open disc $\mathbb{U}=\{z:z\in\mathbb{C},|z|<1\}$. Also, let $\mathscr{A}_p^-$ denote the subclass of $\mathscr{A}_p$ consisting of analytic and p-valent functions expressed in the form

$$f(z)=z^p-\sum_{k=p+1}^{\infty}a_kz^k, \qquad (a_k\geqslant 0,\ k>p,\ p\in\mathbb{N}). \tag{3.2}$$

For the purpose of this paper, we define a fractional q-differintegral operator $\Omega^{\alpha,p}_{q,z}$ for a function $f(z)$ by

$$\Omega^{\alpha,p}_{q,z}f(z)=\frac{\Gamma_q(p+1-\alpha)}{\Gamma_q(p+1)}z^{\alpha-p}D^{\alpha}_{q,z}f(z) \tag{3.3}$$

where $\alpha < p+1,\ p \in \mathbb{N},\ 0 < q < 1$ and $z \in \mathbb{U}$. Here $D^{\alpha}_{q,z} f(z)$ in (3.3) represents, respectively, a fractional q-integral of $f(z)$ of order α when $-\infty < \alpha < 0$, and a fractional q-derivative of $f(z)$ of order α when $0 \leqslant \alpha < p+1$.

We denote by $\mathscr{J}_{q,p}(\delta,\beta,\alpha)$ the subclass of functions in $\mathscr{A}_p^-$, which also satisfy the condition

$$\left| \frac{\Omega^{\alpha,p}_{q,z} f(z) - 1}{\Omega^{\alpha,p}_{q,z} f(z) - 2\delta + 1} \right| < \beta \tag{3.4}$$

where $\alpha < p+1$, $0 \leqslant \delta < 1$, $0 < \beta \leqslant 1$, $0 < q < 1$, $z \in \mathbb{U}$, and the operator $\Omega^{\alpha,p}_{q,z}$ is given by (3.3).

In view of the relationships in (2.4) and (1.4), we find from (3.3) that

$$\Omega^{1,p}_{q,z} f(z) = \frac{(1-q)}{(1-q^p)} z^{1-p} D_{q,z} f(z) \tag{3.5}$$

where $p \in \mathbb{N}$, $0 < q < 1$, $z \in \mathbb{U}$. Thus the condition (3.4) reduce, when $\alpha = 1$, to the inequality

$$\left| \frac{(1-q)z^{1-p} D_{q,z} f(z) - (1-q^p)}{(1-q)z^{1-p} D_{q,z} f(z) + (1-q^p)(1-2\delta)} \right| < \beta \tag{3.6}$$

where $p \in \mathbb{N}$, $0 \leqslant \delta < 1$, $0 < \beta \leqslant 1$, $0 < q < 1$, $z \in \mathbb{U}$, and we have

$$\mathscr{J}_{q,p}(\delta,\beta,1) = \mathscr{J}_{q,p}(\delta,\beta), \tag{3.7}$$

where $\mathscr{J}_{q,p}(\delta,\beta)$ is a subclass of analytic and p-valent functions, which satisfy the condition (3.6).

We now obtain the following coefficient bounds for functions of the form (3.2) belong to the classes $\mathscr{J}_{q,p}(\delta,\beta,\alpha)$ and $\mathscr{J}_{q,p}(\delta,\beta)$ (defined above).

THEOREM 1. *A function f of the form (3.2) belongs to the class $\mathscr{J}_{q,p}(\delta,\beta,\alpha)$ if and only if*

$$\sum_{k=p+1}^{\infty} \frac{\Gamma_q(k+1)\Gamma_q(p+1-\alpha)}{\Gamma_q(p+1)\Gamma_q(k-\alpha+1)} (1+\beta) a_k \leqslant 2\beta(1-\delta). \tag{3.8}$$

The result is sharp.

Proof. Assume that the inequality (3.8) holds true and let $|z| = 1$, then on using (3.2) and (3.3), we find that

$$\left|\Omega^{\alpha,p}_{q,z} f(z) - 1\right| - \beta \left|\Omega^{\alpha,p}_{q,z} f(z) - 2\delta + 1\right| = \left| - \sum_{k=p+1}^{\infty} \frac{\Gamma_q(p+1-\alpha)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\alpha)} a_k z^{k-p} \right|$$

$$-\beta \left| 2(1-\delta) - \sum_{k=p+1}^{\infty} \frac{\Gamma_q(p+1-\alpha)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\alpha)} a_k z^{k-p} \right|$$

$$\leqslant \sum_{k=p+1}^{\infty} \frac{\Gamma_q(p+1-\alpha)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\alpha)} (1+\beta) a_k - 2\beta(1-\delta) \leqslant 0,$$

by our hypothesis. This implies that $f(z) \in \mathcal{J}_{q,p}(\delta,\beta,\alpha)$.

To prove the converse, assume that $f(z)$ is defined by (3.2) and is in the class $\mathcal{J}_{q,p}(\delta,\beta,\alpha)$, then it follows that

$$\left|\frac{\Omega_{q,z}^{\alpha,p} f(z)-1}{\Omega_{q,z}^{\alpha,p} f(z)-2\delta+1}\right| = \left|-\sum_{k=p+1}^{\infty}\frac{\Gamma_q(p+1-\alpha)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\alpha)}a_k z^{k-p}\right|$$

$$\times\left|2(1-\delta)-\sum_{k=p+1}^{\infty}\frac{\Gamma_q(p+1-\alpha)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\alpha)}a_k z^{k-p}\right|^{-1} < \beta. \tag{3.9}$$

Since $|\Re(z)| \leqslant |z|$ for any z, therefore, choosing values of z on the real axis so that $\Omega_{q,z}^{\alpha,p} f(z)$ is real, and letting $z \to 1^-$ through real values, we obtain from (3.9) the following inequality:

$$\sum_{k=p+1}^{\infty}\frac{\Gamma_q(p+1-\alpha)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\alpha)}a_k \leqslant 2\beta(1-\delta)-\beta\sum_{k=p+1}^{\infty}\frac{\Gamma_q(p+1-\alpha)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\alpha)}a_k, \tag{3.10}$$

which yields the desired result (3.8).

Finally, we note that the assertion (3.8) of Theorem 1 is sharp and the extremal function is given by

$$f(z) = z^p - \frac{2\beta(1-\delta)\Gamma_q(p+1)\Gamma_q(n-\alpha+p+1)}{(1+\beta)\Gamma_q(n+p+1)\Gamma_q(p+1-\alpha)}z^{p+n}, \qquad (n \in \mathbb{N}). \tag{3.11}$$

□

COROLLARY 1. *Let the function $f(z)$ defined by (3.2) belong to the class $\mathcal{J}_{q,p}(\delta,\beta,\alpha)$, then*

$$a_{p+n} \leqslant \frac{2\beta(1-\delta)\Gamma_q(p+1)\Gamma_q(n-\alpha+p+1)}{(1+\beta)\Gamma_q(n+p+1)\Gamma_q(p+1-\alpha)}, \tag{3.12}$$

for every integer $n \in \mathbb{N}$.

If we put $\alpha = 1$ and make use of the relation (3.7), Theorem 1 yields the following coefficient inequality for the class $\mathcal{J}_{q,p}(\delta,\beta)$:

COROLLARY 2. *A function f of the form (3.2) belongs to the class $\mathcal{J}_{q,p}(\delta,\beta)$ if and only if*

$$\sum_{k=p+1}^{\infty}(1-q^k)(1+\beta)a_k \leqslant 2\beta(1-\delta)(1-q^p). \tag{3.13}$$

The result is sharp.

4. Distortion theorems

In this section, we prove distortion theorems for the function $f(z)$ of the form (3.2) involving the fractional q-calculus operators.

THEOREM 2. *Let the function $f(z)$ defined by (3.2) be in the class $\mathscr{J}_{q,p}(\delta,\beta,\alpha)$, then*

$$|f(z)| \geqslant |z|^p - 2\beta\frac{(1-\delta)(1-q^{p+1-\alpha})}{(1+\beta)(1-q^{p+1})}|z|^{p+1}, \tag{4.1}$$

and

$$|f(z)| \leqslant |z|^p + 2\beta\frac{(1-\delta)(1-q^{p+1-\alpha})}{(1+\beta)(1-q^{p+1})}|z|^{p+1}, \tag{4.2}$$

for $z \in \mathbb{U}$. Furthermore

$$|z|^p - 2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|^{p+1} \leqslant \left|z^p\Omega_{q,z}^{\alpha,p}f(z)\right| \leqslant |z|^p + 2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|^{p+1}, \quad (z\in\mathbb{U}). \tag{4.3}$$

Proof. Since $f(z)\in\mathscr{J}_{q,p}(\delta,\beta,\alpha)$, then in view of Theorem 1, we first show that the function

$$\phi(k) = \frac{\Gamma_q(k+1)\Gamma_q(p+1-\alpha)}{\Gamma_q(p+1)\Gamma_q(k+1-\alpha)}, \qquad (k\geqslant p+1;\ p\in\mathbb{N})$$

is a decreasing function of k for $\alpha < p+1$, $0<q<1$.

It follows that

$$\frac{\phi(k+1)}{\phi(k)} = \frac{\Gamma_q(k+2)\Gamma_q(k+1-\alpha)}{\Gamma_q(k+1)\Gamma_q(k+2-\alpha)}, \qquad (k\geqslant p+1;\ p\in\mathbb{N}),$$

and it is sufficient to consider here the value $k=p+1$, so that on using (1.4), we get

$$\frac{\phi(p+2)}{\phi(p+1)} = \frac{1-q^{p+2}}{1-q^{p+2-\alpha}}, \qquad (0<q<1).$$

The function $\phi(k)$ is a decreasing function of k if $\frac{\phi(p+2)}{\phi(p+1)}\leqslant 1$, $(p\in\mathbb{N})$, and this gives

$$\frac{1-q^{p+2}}{1-q^{p+2-\alpha}} \leqslant 1, \qquad (0<q<1).$$

Multiplying the above inequality both sides by $1-q^{p+2-\alpha}$ (provided that $\alpha<p+1$), we are at once lead to the inequality $\alpha\leqslant 0$. Thus, $\phi(k)$ $(k\geqslant p+1; p\in\mathbb{N})$ is a decreasing function of k for $-\infty<\alpha<p+1, 0<q<1$.

Now (3.8) gives the following inequality:

$$\frac{\Gamma_q(p+2)\Gamma_q(p+1-\alpha)}{\Gamma_q(p+1)\Gamma_q(p+2-\alpha)}\sum_{k=p+1}^{\infty}a_k \leqslant \sum_{k=p+1}^{\infty}\frac{\Gamma_q(k+1)\Gamma_q(p+1-\alpha)}{\Gamma_q(p+1)\Gamma_q(k-\alpha+1)}(1+\beta)a_k \leqslant 2\beta(1-\delta),$$

which in view of (1.4), implies that

$$\sum_{k=p+1}^{\infty} a_k \leqslant \frac{2\beta(1-\delta)(1-q^{p+1-\alpha})}{(1+\beta)(1-q^{p+1})},$$

and this last inequality in conjunction with the following inequality (easily obtainable from (3.2)):

$$|z|^p - |z|^{p+1} \sum_{k=p+1}^{\infty} a_k \leqslant |f(z)| \leqslant |z|^p + |z|^{p+1} \sum_{k=p+1}^{\infty} a_k \tag{4.4}$$

yields the assertions (4.1) and (4.2) of Theorem 2.

Next, on using (3.2) and (3.3), we observe that

$$\left|z^p \Omega_{q,z}^{\alpha,p} f(z)\right| \geqslant |z|^p - \sum_{k=p+1}^{\infty} \frac{\Gamma_q(p+1-\alpha)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\alpha)} a_k |z|^k$$

$$\geqslant |z|^p - |z|^{p+1} \sum_{k=p+1}^{\infty} \frac{\Gamma_q(p+1-\alpha)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\alpha)} a_k,$$

which on using Theorem 1 give rise to

$$\left|z^p \Omega_{q,z}^{\alpha,p} f(z)\right| \geqslant |z|^p - 2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|^{p+1}. \tag{4.5}$$

Similarly, it follows that

$$\left|z^p \Omega_{q,z}^{\alpha,p} f(z)\right| \leqslant |z|^p + 2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|^{p+1}, \tag{4.6}$$

which establishes the assertion (4.3) of Theorem 2. □

In view of (3.3), Theorem 2 gives the following distortion inequality for the function $f(z) \in \mathscr{A}_p^-$ involving fractional q-derivative operator $D_{q,z}^{\alpha}$:

COROLLARY 3. *Let the function $f(z)$ defined by (3.2) be in the class $\mathscr{J}_{q,p}(\delta,\beta,\alpha)$, then*

$$\frac{\Gamma_q(p+1)}{\Gamma_q(p+1-\alpha)}|z|^{p-\alpha}\left\{1-2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|\right\} \leqslant \left|D_{q,z}^{\alpha} f(z)\right|$$

$$\leqslant \frac{\Gamma_q(p+1)}{\Gamma_q(p+1-\alpha)}|z|^{p-\alpha}\left\{1+2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|\right\}, \tag{4.7}$$

where $0 \leqslant \alpha < p+1,\ z \in \mathbb{U}$.

Also, in view of (3.3), Theorem 2 gives the following inequality involving fractional q-integral operator $I_{q,z}^{\alpha}$:

COROLLARY 4. *Let the function* $f(z)$ *be in the class* $\mathscr{J}_{q,p}(\delta,\beta,\alpha)$*, then*

$$\frac{\Gamma_q(p+1)}{\Gamma_q(p+1+\alpha)}|z|^{p+\alpha}\left\{1-2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|\right\}\leqslant\left|I_{q,z}^{\alpha}f(z)\right|$$

$$\leqslant\frac{\Gamma_q(p+1)}{\Gamma_q(p+1+\alpha)}|z|^{p+\alpha}\left\{1+2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|\right\}, \tag{4.8}$$

where $\alpha>0$, $z\in\mathbb{U}$.

On setting $\alpha=1$ and making use of relationship (3.7), Theorem 2 gives the following distortion inequality for the function $f(z)\in\mathscr{A}_p^-$ involving q-derivative operator $D_{q,z}$:

COROLLARY 5. *Let the function* $f(z)$ *defined by (3.2) be in the class* $\mathscr{J}_{q,p}(\delta,\beta)$*, then*

$$|f(z)|\geqslant|z|^p-2\beta\frac{(1-\delta)(1-q^p)}{(1+\beta)(1-q^{p+1})}|z|^{p+1}, \tag{4.9}$$

and

$$|f(z)|\leqslant|z|^p+2\beta\frac{(1-\delta)(1-q^p)}{(1+\beta)(1-q^{p+1})}|z|^{p+1}, \tag{4.10}$$

for $z\in\mathbb{U}$.

Furthermore

$$\frac{(1-q^p)}{(1-q)}|z|^{p-1}\left\{1-2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|\right\}\leqslant\left|D_{q,z}f(z)\right|$$

$$\leqslant\frac{(1-q^p)}{(1-q)}|z|^{p-1}\left\{1+2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|\right\}(z\in\mathbb{U}). \tag{4.11}$$

5. Concluding observations and remarks

In this section we briefly consider some consequences of the results derived in the preceding sections. If we let $q\to1^-$, and make use of the limit formula (1.8), we observe that the function class $\mathscr{J}_{q,p}(\delta,\beta,\alpha)$, and the inequalities (3.8) of Theorem 1, (3.12) of Corollary 1, (4.1) to (4.3) of Theorem 2 provide, respectively, the q-extensions of the known class and the related inequalities due to Srivastava and Aouf [9]. Also, the function class $\mathscr{J}_{q,p}(\delta,\beta)$ defined with condition (3.6), Corollary 2 and the distortion inequalities (4.19) to (4.11) of Corollary 5 are the q-extensions of the corresponding known function class and related results due to Owa [6, p. 42, Theorem 1; p. 45, Theorems 6].

Further, if we set $p=1$, the operator (3.3) reduces to

$$\Omega_{q,z}^{\alpha,1}f(z)\equiv\Omega_{q,z}^{\alpha}f(z)=\frac{\Gamma_q(2-\alpha)}{\Gamma_q(2)}z^{\alpha-1}D_{q,z}^{\alpha}f(z). \tag{5.1}$$

Thus the condition (3.4) reduce, when $p=1$, to the inequality

$$\left|\frac{\Omega_{q,z}^{\alpha}f(z)-1}{\Omega_{q,z}^{\alpha}f(z)-2\delta+1}\right|<\beta \tag{5.2}$$

where $\alpha<2$, $0\leqslant\delta<1$, $0<\beta\leqslant 1$, $0<q<1$, $z\in\mathbb{U}$, and we have

$$\mathscr{J}_{q,1}(\delta,\beta,\alpha)=\mathscr{J}_{q,\delta}^{\alpha}, \tag{5.3}$$

where $\mathscr{J}_{q,\delta}^{\alpha}$ is precisely the subclass of analytic and univalent functions studied recently by Purohit and Raina [7].

Now, if we set $p=1$ and taking relation (5.3) into account, Theorems 1 and 2 yields the following results obtained by Purohit and Raina [7] in a slightly different form:

COROLLARY 6. *A function*

$$f(z)=z-\sum_{k=2}^{\infty}a_k z^k, \qquad (a_k\geqslant 0), \tag{5.4}$$

is in the class $\mathscr{J}_{q,\delta}^{\alpha}$ *if and only if*

$$\sum_{k=2}^{\infty}\frac{\Gamma_q(k+1)\Gamma_q(2-\alpha)}{\Gamma_q(2)\Gamma_q(k-\alpha+1)}(1+\beta)a_k\leqslant 2\beta(1-\delta). \tag{5.5}$$

The result is sharp.

COROLLARY 7. *Let the function* $f(z)$ *defined by (5.4) be in the class* $\mathscr{J}_{q,\delta}^{\alpha}$, *then*

$$|f(z)|\geqslant|z|-2\beta\frac{(1-\delta)(1-q^{2-\alpha})}{(1+\beta)(1-q^2)}|z|^2, \tag{5.6}$$

and

$$|f(z)|\leqslant|z|+2\beta\frac{(1-\delta)(1-q^{2-\alpha})}{(1+\beta)(1-q^2)}|z|^2, \tag{5.7}$$

for $z\in\mathbb{U}$.

Furthermore

$$|z|-2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|^2\leqslant\left|z\Omega_{q,z}^{\alpha}f(z)\right|\leqslant|z|+2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|^2, \quad (z\in\mathbb{U}). \tag{5.8}$$

We conclude this paper by remarking that the results presented in this paper giving various properties of different classes of analytic functions yield several new and known results. Also, the fractional q-calculus operators defined in Section 2 can fruitfully be used in investigating several other analytic, multivalent (or meromorphic) function classes and their various geometric properties like, the coefficient estimates, distortion bounds, radii of starlikeness, convexity and close to convexity etc.

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