Fractional Differential Calculus
Volume 4, Number 1 (2014), 1–30 doi:10.7153/fdc-04-01

WELL–POSEDNESS AND REGULARITY OF THE CAUCHY PROBLEM FOR NONLINEAR FRACTIONAL IN TIME AND SPACE EQUATIONS

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Abstract. The purpose is to study the Cauchy problem for non-linear in time and space pseudo-differential equations. These include the fractional in time versions of Hamilton-Jacobi-Bellman (HJB) equations governing the limits of controlled scaled Continuous Time Random Walks (CTRWs). As a preliminary step which is of independent interest we analyse the corresponding linear equation proving its well-posedness and smoothing properties.

Introduction

The purpose of this paper is to study well-posedness of the Cauchy problem for the fractional in time and space pseudo-differential equation

$$D_{0,t}^{*\beta} f(t,y) = -a(-\Delta)^{\alpha/2} f(t,y) + H(t,y,\nabla f(t,y)) \tag{0.1}$$

where $y \in \mathbb{R}^d$, $t \geqslant 0$, $\beta \in (0,1), \alpha \in (1,2]$, $H(t,y,p)$ is a Lipschitz function in all of its variables, and $f(0,y) = f_0(y)$ is known and bounded, and a is a constant, $a > 0$. Here ∇ denotes the gradient with respect to the spatial variable. For a function dependent on several spatial variables, say x, y, we may occasionally indicate the variable with respect to which the gradient is taken, by a subscript, as in ∇_x. The extension of our results for (0.1) to the case where $H = H(t,y,f(t,y),\nabla f(t,y))$ is straightforward and we omit it here. We denote by $D_{0,t}^{*\beta}$ the Caputo derivative:

$$D_{0,t}^{*\beta} f(t,y) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{df(s,y)}{ds}(t-s)^{-\beta} ds, \tag{0.2}$$

whilst $-(-\Delta)^{\alpha/2}$ is the fractional Laplacian

$$-(-\Delta)^{\alpha/2} f(t,y) = p.v.\ C_{d,\alpha} \int_{\mathbb{R}^d} \frac{f(t,y) - f(t,x)}{|y-x|^{d+\alpha}} dx, \tag{0.3}$$

where "*p.v.*" stands for "*principal value*" and $C_{d,\alpha}$ is a normalizing constant.

Mathematics subject classification (2010): 35R11.
Keywords and phrases: fractional calculus, Caputo derivative, Mittag-Leffler functions, fractional Hamilton-Jacobi-Bellman type equations.
[1] Associate member of IPI RAN, Russia. Partially supported by the IPI RAN grants RFBR 11-01-12026 and 12-07-00115, and by the grant 4402 of the Ministry of Education and Science of Russia.
[2] Supported by EPSRC grant EP/HO23364/1 through MASDOC, University of Warwick, UK.

As a preliminary analysis we establish the regularity properties of the linear equations of the form

$$D_{0,t}^{*\beta} f(t,y) = -a(-\Delta)^{\alpha/2} f(t,y) + h(t,y), \tag{0.4}$$

with a given function h, an initial condition $f(0,y) = f_0(y)$, $\beta \in (0,1)$, $\alpha \in (1,2]$, a constant $a > 0$ and $-(-\Delta)^{\alpha/2}$ as defined in (0.3). This allows one to reduce the analysis of (0.1) to a fixed point problem. Section 3 is devoted to the linear problem (0.4) and in section 4 we formulate and prove our main results for equation (0.1).

As will be shown in [6] the results of this paper are needed for fully justifying the heuristic derivation of the fractional HJB equation for the limit of controlled scaled Continuous Time Random Walks (CTRWs) in [5]. Among other researchers who studied solutions to fractional differential equations (FDEs) are [9], [3], [10], [11], [13], [1], [12], [16], [21], [31]. More results and reviews can be found in references therein. FDEs appear for example in modelling processes with memory or with a random time change, see [28], [27], [30], [29], [4], [5].

Several authors solve FDEs using Laplace transforms in time, see [11], [21] and [13] for example.

The book [1] covers analysis for Caputo time-fractional differential equations with the parameter $\beta > 0$, for example

$$D_{0,t}^{*\beta} y(x) = -\mu y(x) + q(x), \tag{0.5}$$

with $y(0) = y_0^{(0)}$, $Dy(0) = y_0^{(1)}$, $\beta \in (1,2)$, $\mu > 0$.

In [10] the theory for FDEs in L^p spaces is developed. Well-posedness of (0.4) in L^p may be deduced from there.

In [9] the authors consider classical solutions for fractional Cauchy problems in bounded domains $D \subset \mathbb{R}^d$ with Dirichlet boundary conditions.

In [17] one may find the analysis for the non-local Cauchy problem in a Banach space, where instead of $-(-\Delta)^{\alpha/2}$ there is a general infinitesimal generator of a strongly continuous semigroup of bounded linear operators. The authors present conditions that should be satisfied to ensure existence of mild forms of the FDE.

The paper [18] establishes asymptotic estimates of solutions to the following fractional equation and its similar versions:

$$D_{0,t}^{*\alpha} u(x,t) = a^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \tag{0.6}$$

for $t > 0$, $x \in \mathbb{R}$, $\alpha \in (0,1)$, $u(x,0) = \phi(x)$, $\lim_{|x|\to+\infty} u(x,t) = 0$, however the case of the fractional Laplacian is not included and there is no $h(x,t)$ term on the right hand side (RHS).

For solvability of linear FDEs in Banach spaces one may see [22], where

$$D_{0,t}^{*\alpha} x(t) = Ax(t), \text{ for } m-1 < \alpha \leqslant m \in \mathbb{N}, \tag{0.7}$$

and $\frac{d^k}{dt^k} x(t)|_{t=0} = \xi_k$, for $k = 0, \ldots, m-1$. The authors give sufficient conditions under which the set of initial data ξ_k for $k = 0, \ldots, m-1$ provides a solution to (0.7) of the form $\sum_{k=0}^{m-1} t^k E_{\alpha,k+1}(At^\alpha)\xi_k$. In particular, these conditions depend on Roumieu, Gevrey and Beurling spaces related to the operator A.

In [16] there is a construction and investigation of a fundamental solution for the Cauchy problem with a regularised fractional derivative $D^\alpha_{0,t,reg}$, and $\alpha \in (0,1)$ defined by

$$D^\alpha_{0,t,reg}u(t,x) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\partial}{\partial t}\int_0^t (t-\tau)^{-\alpha}u(\tau,x)d\tau - t^{-\alpha}u(0,x)\right]. \tag{0.8}$$

Note that

$$D^\alpha_{0,t}u(t,x) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{-\alpha}u(\tau,x)d\tau \tag{0.9}$$

is the definition of the Riemann-Liouville fractional derivative. Since $D^{*\alpha}_{0,t}f(t,x) = D^\alpha_{0,t}f(t,x) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)}f(0,x)$, the regularised derivative in (0.8) is in fact identical to our definition of the Caputo derivative in (0.2). The problem studied there is

$$D^\alpha_{0,t,reg}u(t,x) - Bu(t,x) = f(t,x), \tag{0.10}$$

$t \in (0,T], x \in \mathbb{R}^n$, where

$$B = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^n b_j(x)\frac{\partial}{\partial x_j} + c(x) \tag{0.11}$$

with bounded real-valued coefficients.

Our analysis goes beyond to include $B = -a(-\Delta)^{\alpha/2}$, with $a > 0$. Theorems 3, 6 and 7 concerning the case $\alpha = 2$ are obtainable from the results in [16] by slightly different arguments.

Denote a bounded domain by D. Taking $\alpha \in (0,2)$, $\beta \in (0,1)$ the paper [25] develops strong solutions to the equation

$$D^{*\beta}_{0,t}u(t,x) = \Delta_x^{\alpha/2}u(t,x), \tag{0.12}$$

for $x \in D$, $t > 0$, $u(0,x) = f(x)$ for $x \in D$ and $u(t,x) = 0$ for $x \in D^c$, $t > 0$.

Our approach to the non-linear FDE seems to be different and includes the fractional Laplacian $-(-\Delta)^{\alpha/2}$ instead of the standard one Δ_y. We extend this to the scenario with the RHS term including $H(t,y,\nabla f(t,y))$. We concentrate on the case with only one fractional time derivative $D^{*\beta}_{0,t}$.

1. The mild form for the linear fractional dynamics

Our analysis of equation (0.4) is based on the Fourier transform in space, where for a function $g(y)$ its Fourier transform will be defined in the following way

$$\hat{g}(p) = \int_{R^d} e^{-ipy}g(y)dy. \tag{1.1}$$

Applying the Fourier transform in y to (0.4) yields

$$D_{0,t}^{*\beta}\hat{f}(t,p) = -a|p|^{\alpha}\hat{f}(t,p) + \hat{h}(t,p). \tag{1.2}$$

This is a standard linear equation with the Caputo fractional derivative. For continuous h its solution is given by

$$\begin{aligned}\hat{f}(t,p) &= \hat{f}_0(p)E_{\beta,1}(-a|p|^{\alpha}t^{\beta}) \\ &\quad + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-a(t-s)^{\beta}|p|^{\alpha}))\hat{h}(s,p)ds,\end{aligned} \tag{1.3}$$

where $E_{\beta,1}$ and $E_{\beta,\beta}$ are Mittag-Leffler functions, see formulas $(7.3)-(7.4)$ in [1].

Let us recall that the Mittag-Leffler functions are defined for $Re(\beta) > 0$, and $\gamma, z \in \mathbb{C}$:

$$E_{\beta,\gamma}(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(\beta k + \gamma)}. \tag{1.4}$$

We will use the following connection between $E_{\beta,\beta}$ and $E_{\beta,1}$:

$$x^{\beta-1}E_{\beta,\beta}(-a|p|^{\alpha}x^{\beta}) = -\frac{1}{a|p|^{\alpha}}\frac{d}{dx}E_{\beta,1}(-a|p|^{\alpha}x^{\beta}). \tag{1.5}$$

To prove (1.5) one may use the representation of $E_{\beta,1}(-a|p|^{\alpha}x^{\beta})$ in (1.4) and differentiate with respect to x term by term. Now we present two convenient notations for further analysis. Let us denote

$$S_{\beta,1}(t,y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ipy}E_{\beta,1}(-a|p|^{\alpha}t^{\beta})dp \tag{1.6}$$

and

$$G_{\beta}(t,y) = \frac{t^{\beta-1}}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ipy}E_{\beta,\beta}(-a|p|^{\alpha}t^{\beta})dp. \tag{1.7}$$

Using (1.5) we can re-write (1.7) as

$$G_{\beta}(t,y) = -\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ipy}\frac{1}{a|p|^{\alpha}}\frac{d}{dt}E_{\beta,1}(-a|p|^{\alpha}t^{\beta})dp. \tag{1.8}$$

Applying the inverse Fourier transform to (1.3) we obtain:

$$\begin{aligned}f(t,y) &= \int_{\mathbb{R}^d} S_{\beta,1}(t,y-x)f_0(x)dx \\ &\quad + \int_0^t\int_{\mathbb{R}^d} G_{\beta}(t-s,y-x)h(s,x)dxds.\end{aligned} \tag{1.9}$$

It is natural to call this integral equation the mild form of the fractional linear equation (0.4). In particular we see that the function $S_{\beta,1}(t,y-y_0)$ is the solution of equation (0.4) with $f_0(y) = \delta(y - y_0)$ and $h(t,y) = 0$. On the other hand the function $G_\beta(t - t_0, y - y_0)$ is the solution of (0.4) with $f_0(y) = 0$ and $h(t,y) = \delta(t - t_0, y - y_0)$. Thus the functions $S_{\beta,1}$ and G_β may be called Green functions of the corresponding Cauchy problems. Notice the crucial difference with the usual evolution corresponding to $\beta = 1$ where G_β and $S_{\beta,1}$ coincide.

In order to clarify the properties of f in (1.9) we are now going to carefully analyse the asymptotic properties of the integral kernels $S_{\beta,1}(t,y)$ and $G_\beta(t,y)$.

2. Regularity properties for $S_{\beta,1}$ and G_β

For $d \geqslant 1$ let us define the symmetric stable density g in $\mathbb{R}^d$ as

$$g(y;\alpha,\sigma,\gamma=0) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\exp\{-ipy - a\sigma|p|^\alpha\}dp, \tag{2.1}$$

where α is the stability parameter, σ is the scaling parameter and γ is the skewness parameter which is $\gamma = 0$ for symmetric stable densities. In $d = 1$ and $\alpha \neq 1$ we define the fully skewed density with $\gamma = 1$ and without scaling:

$$w(x;\alpha,1) = \frac{1}{2\pi}Re\int_{-\infty}^{\infty}\exp\left\{-ipx - |p|^\alpha \exp\left\{-i\frac{\pi}{2}K(\alpha)\right\}\right\}dp, \tag{2.2}$$

where $K(\alpha) = \alpha - 1 + \text{sign}(1-\alpha)$. The function $w(x;\alpha,1)$ is infinitely differentiable and vanishes identically for $x < 0$, see [8], theorem C.3 and §2.2, equation (2.2.1a).

The starting point of the analysis of $S_{\beta,1}, G_\beta$ is the following representation of the Mittag-Leffler function due to [8], see chapter 2.10, Theorem 2.10.2, equations $(2.10.8 - 2.10.9)$. For $\beta \in (0,1)$

$$E_{\beta,1}(-a\lambda) = \frac{1}{\beta}\int_0^\infty \exp(-a\lambda x)x^{-1-1/\beta}w(x^{-1/\beta},\beta,1)dx. \tag{2.3}$$

Substitute $\lambda = |p|^\alpha t^\beta$:

$$E_{\beta,1}(-a|p|^\alpha t^\beta) = \frac{1}{\beta}\int_0^\infty \exp(-a|p|^\alpha t^\beta x)x^{-1-1/\beta}w(x^{-1/\beta},\beta,1)dx. \tag{2.4}$$

So then

$$\begin{aligned}t^{\beta-1}E_{\beta,\beta}(-a|p|^\alpha t^\beta) &= \frac{-1}{a|p|^\alpha}\frac{d}{dt}E_{\beta,1}(-a|p|^\alpha t^\beta)\\ &= t^{\beta-1}\int_0^\infty x^{-1/\beta}\exp(-a|p|^\alpha t^\beta x)w(x^{-1/\beta},\beta,1)dx, \end{aligned}\tag{2.5}$$

implying

$$
\begin{aligned}
G_\beta(t,y) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ipy}E_{\beta,\beta}(-a|p|^\alpha t^\beta)t^{\beta-1}dp \\
&= \frac{t^{\beta-1}}{(2\pi)^d}\int_0^\infty\int_{\mathbb{R}^d} e^{ipy}\exp\{-a|p|^\alpha t^\beta x\}x^{-1/\beta}w(x^{-1/\beta},\beta,1)dpdx \\
&= t^{\beta-1}\int_0^\infty x^{-1/\beta}w(x^{-1/\beta},\beta,1)g(-y,\alpha,t^\beta x)dx, \qquad (2.6)
\end{aligned}
$$

where g is as in (2.1) and w is as in (2.2).

Throughout this paper we shall denote by C various constants that may be different from formula to formula and line to line.

THEOREM 1. *For G_β defined in (1.7), in the case $\beta \in (0,1)$ and $\alpha \in (1,2)$*

$$
\int_{\mathbb{R}^d} |G_\beta(t,y)|dy \leqslant Ct^{\beta-1}, \qquad (2.7)
$$

where $C > 0$ is a constant.

Proof. Let us split the integral representing $G_{\beta,1}(t,y)$ in the sum of two, so that

$$
G_\beta(t,y) = I_A + I_B, \qquad (2.8)
$$

where

$$
\begin{aligned}
I_A &= \frac{t^{\beta-1}}{(2\pi)^d}\int_{|y|^\alpha t^{-\beta}}^\infty\int_{\mathbb{R}^d} e^{ipy}e^{-a|p|^\alpha t^\beta x}x^{-1/\beta}w(x^{-1/\beta},\beta,1)dpdx \\
&= t^{\beta-1}\int_{|y|^\alpha t^{-\beta}}^\infty x^{-1/\beta}w(x^{-1/\beta},\beta,1)g(-y,\alpha,t^\beta x)dx \qquad (2.9)
\end{aligned}
$$

and

$$
\begin{aligned}
I_B &= \frac{t^{\beta-1}}{(2\pi)^d}\int_0^{|y|^\alpha t^{-\beta}}\int_{\mathbb{R}^d} e^{ipy}e^{-a|p|^\alpha t^\beta x}x^{-1/\beta}w(x^{-1/\beta},\beta,1)dpdx \\
& t^{\beta-1}\int_0^{|y|^\alpha t^{-\beta}} x^{-1/\beta}w(x^{-1/\beta},\beta,1)g(-y,\alpha,t^\beta x)dx. \qquad (2.10)
\end{aligned}
$$

To estimate $|I_A|$ and $|I_B|$, let us examine cases $|y| > t^{\beta/\alpha}$ and $|y| \leqslant t^{\beta/\alpha}$ and start with $|y| > t^{\beta/\alpha}$. Note that the asymptotic expansions for $g(y,\alpha,\sigma)$ and $g(-y,\alpha,\sigma)$, namely, (5.2) and (5.6) appearing in the Appendix, are the same, by inspection. Since $x > |y|^\alpha t^{-\beta}$ in I_A we may use the asymptotic for $|y|/x^{1/\alpha}t^{\beta/\alpha} \to 0$, see (5.2). We also use that for $x \to \infty$, $x^{-1/\beta} \to 0$, so for $x \to \infty$ we have $w(x^{-1/\beta},\beta,1) \sim C$, where $C \geqslant 0$

is a constant. Thus we have

$$
\begin{aligned}
|I_A| &\leqslant \left| \int_{|y|^\alpha t^{-\beta}}^{\infty} x^{-1/\beta-d/\alpha} w(x^{-1/\beta},\beta,1) A_0 t^{\beta-1-d\beta/\alpha} dx \right| \\
&\leqslant C t^{\beta-1-d\beta/\alpha} |A_0| \frac{(|y|^\alpha t^{-\beta})^{1-1/\beta-d/\alpha}}{|1-1/\beta-d/\alpha|} \\
&\leqslant C t^{\beta-1-d\beta/\alpha} \frac{|A_0|}{|1-1/\beta-d/\alpha|} (|y|^\alpha t^{-\beta})^{1-1/\beta-d/\alpha} \\
&\leqslant C t^{\beta-1+1-\beta} |y|^{\alpha-\alpha/\beta-d}. \qquad (2.11)
\end{aligned}
$$

Now, let us study I_B in the case $|y| > t^{\beta/\alpha}$. Here we use the asymptotic expansion for $|y|/x^{1/\alpha} t^{\beta/\alpha} \to \infty$ as it appears in (5.6) in the Appendix and take the first term only. Here we use the change of variables $z = x^{-1/\beta}$.

$$
\begin{aligned}
|I_B| &\leqslant \left| A_1 t^{\beta-1} \int_0^{|y|^\alpha t^{-\beta}} x^{-1/\beta} w(x^{-1/\beta},\beta,1) |y|^{-d-\alpha} t^\beta x dx \right| \\
&\leqslant C t^{2\beta-1} |y|^{-d-\alpha} \int_{|y|^{-\alpha/\beta} t}^{\infty} z^{-2\beta} w(z,\beta,1) dz. \qquad (2.12)
\end{aligned}
$$

We split this integral into two parts: $z \in [1,\infty)$ and $z \in (|y|^{-\alpha/\beta} t, 1)$. In the case $z \in [1,\infty)$

$$
\begin{aligned}
& t^{2\beta-1} |y|^{-d-\alpha} \int_1^\infty z^{-2\beta} w(z,\beta,1) dz \\
&\qquad \leqslant t^{2\beta-1} |y|^{-d-\alpha} \int_0^\infty w(z,\beta,1) dz \leqslant C t^{2\beta-1} |y|^{-d-\alpha}, \qquad (2.13)
\end{aligned}
$$

In the case $z \in (|y|^{-\alpha/\beta} t, 1)$ we may use that z is small and so

$$
z^{-2\beta} w(z,\beta,1) < C z^{-2\beta+q-3},
$$

for any $q > 1$. So

$$
\begin{aligned}
& t^{2\beta-1} |y|^{-d-\alpha} \int_{|y|^{-\alpha/\beta} t}^{1} z^{-2\beta} w(z,\beta,1) dz \\
&\qquad \leqslant C t^{2\beta-1} |y|^{-d-\alpha} \int_{|y|^{-\alpha/\beta}}^{1} z^{-2\beta+q-3} dz \\
&\qquad = C t^{2\beta-1} |y|^{-d-\alpha} \left(1 - (|y|^{-\alpha/\beta} t)^{-2\beta+q-2} \right). \qquad (2.14)
\end{aligned}
$$

Now let us study the case $|y| \leqslant t^{\beta/\alpha}$. For I_A we use that x is large, so $x^{-1/\beta}$ is small, and that for $q \geqslant 4$ we have $x^{-d/\alpha-1/\beta} w(x^{-1/\beta}) < C x^{-d/\alpha-(\frac{q-2}{\beta})}$. Here $|y|^\alpha \leqslant t^\beta$

and we obtain

$$
\begin{aligned}
|I_A| &\leqslant Ct^{\beta-1-d\beta/\alpha}\left|\int_{|y|^\alpha t^{-\beta}}^{\infty} A_0 x^{-d/\alpha-\frac{q-2}{\beta}}dx\right| \\
&\leqslant t^{\beta-1-\beta d/\alpha}\left(y^\alpha t^{-\beta}\right)^{-d/\alpha-\frac{q-2}{\beta}+1}C \\
&\leqslant t^{\beta-1-d\beta/\alpha}t^{-d\beta/\alpha-(q-2)+\beta+d\beta/\alpha+(q-2)-\beta}C \\
&\leqslant t^{\beta-1-d\beta/\alpha}C.
\end{aligned}
\tag{2.15}
$$

As for I_B in yhe case $|y| \leqslant t^{\beta/\alpha}$,

$$
\begin{aligned}
|I_B| &\leqslant C\int_0^{|y|^\alpha t^{-\beta}} x^{1-1/\beta}w(x^{-1/\beta},\beta,1)t^{2\beta-1}|y|^{-d-\alpha}dx \\
&\leqslant C|y|^{-d-\alpha}t^{2\beta-1}\int_0^{|y|^\alpha t^{-\beta}} x^{1-1/\beta}(x^{-1/\beta})^{-1-\beta}dx \\
&\leqslant C|y|^{2\alpha-d}t^{-\beta-1}.
\end{aligned}
\tag{2.16}
$$

Integrating (2.11) in polar coordinates gives

$$
\begin{aligned}
\int_{|y|>t^{\beta/\alpha}}|I_A|dy &\leqslant C\int_{|r|>t^{\beta/\alpha}}|r|^{\alpha-\alpha/\beta-d+d-1}d|r| \\
&\leqslant (t^{\beta/\alpha})^{\alpha-\alpha/\beta}C = t^{\beta-1}C,
\end{aligned}
\tag{2.17}
$$

Integration of (2.14) in polar coordinates gives

$$
\begin{aligned}
\int_{|y|>t^{\beta/\alpha}}|I_B|dy &\leqslant Ct^{2\beta-1}\int_{|r|>t^{\beta/\alpha}}|r|^{-d-\alpha+d-1}dr \\
&\quad + Ct^{2\beta-1}\int_{|r|>t^{\beta/\alpha}}|r|^{d-1-d-\alpha}|r|^{2\alpha}t^{-2\beta}dr \\
&= Ct^{2\beta-1}(t^{\beta/\alpha})^{-\alpha}+Ct^{2\beta-1-2\beta}(t^{\beta/\alpha})^{\alpha} = Ct^{\beta-1}.
\end{aligned}
\tag{2.18}
$$

Integration of (2.15) gives

$$
\begin{aligned}
\int_{|y|\leqslant t^{\beta/\alpha}}|I_A|dy &\leqslant Ct^{\beta-1-d\beta/\alpha}\int_{|r|\leqslant t^{\beta/\alpha}}|r|^{d-1}d|r| \\
&\leqslant t^{d\beta/\alpha-d\beta/\alpha+\beta-1}\frac{C|A_0|}{|d|} \leqslant t^{\beta-1}\frac{|A_0|C}{d}.
\end{aligned}
\tag{2.19}
$$

Integration of (2.16) yields

$$
\begin{aligned}
\int_{|y|\leqslant t^{\beta/\alpha}}|I_B|dy &\leqslant C\int_{|r|\leqslant t^{\beta/\alpha}}t^{-\beta-1}|r|^{-d+2\alpha+d-1}dr \\
&\leqslant Ct^{-\beta-1}(t^{\beta/\alpha})^{2\alpha} = Ct^{\beta-1}.
\end{aligned}
\tag{2.20}
$$

Combining (2.17)–(2.20) yields (2.7). □

THEOREM 2. *For G_β defined in (1.7) in the case $\beta \in (0,1)$ and for $\alpha \in (1,2)$*

$$\int_{\mathbb{R}^d} |\nabla G_\beta(t,y)|dy \leqslant t^{\beta-1-\beta/\alpha}C. \tag{2.21}$$

Proof. In the case $|y| > t^{\beta/\alpha}$, we have $|y|^{-1} < t^{-\beta/\alpha}$ and so differentiation with respect to y yields

$$|\nabla I_A| \leqslant Ct^{-\beta/\alpha}|I_A| \tag{2.22}$$

and

$$|\nabla I_B| \leqslant Ct^{-\beta/\alpha}|I_B|. \tag{2.23}$$

In the case $|y| \leqslant t^{\beta/\alpha}$ we need to take into account the second term of the asymptotic expansion, since the first term is independent of $|y|$. Consequently,

$$\begin{aligned}|\nabla I_A| &\leqslant C\int_{|y|^\alpha t^{-\beta}}^\infty x^{-d/\alpha}t^{-d\beta/\alpha}|y|(xt^\beta)^{-2/\alpha}x^{-1/\beta}t^{\beta-1}dx\\ &\leqslant C\int_{|y|^\alpha t^{-\beta}}^\infty x^{-d/\alpha-2/\alpha-1/\beta}t^{-d\beta/\alpha-2\beta/\alpha+\beta-1}dx\\ &\leqslant Ct^{-d\beta/\alpha-2\beta/\alpha+\beta-1}|y| - C(|y|^\alpha t^{-\beta})^{-d/\alpha-2/\alpha-1/\beta+1}t^{\beta/\alpha}\\ &= Ct^{-d\beta/\alpha-2\beta/\alpha+\beta-1}|y| - Ct^{\beta/\alpha}|y|^{-d-2-\alpha/\beta+\alpha}.\end{aligned} \tag{2.24}$$

Integration of the first term in (2.24) yields

$$\begin{aligned}&C\int_{|r|<t^{\beta/\alpha}} t^{-d\beta/\alpha-2\beta/\alpha+\beta-1}|r|^{d-1+1}dr\\ &\qquad\leqslant Ct^{-d\beta/\alpha-\beta/\alpha+\beta-1+d\beta/\alpha} \leqslant Ct^{\beta-1-\beta/\alpha}.\end{aligned} \tag{2.25}$$

Integration of the second term in (2.24) gives

$$\begin{aligned}&\int_{|y|<t^{\beta/\alpha}} t^{\beta/\alpha}|y|^{-d+d-3-\alpha/\beta+\alpha}dy\\ &\qquad\leqslant t^{\beta/\alpha}(t^{\beta/\alpha})^{-2-\alpha/\beta+\alpha} \leqslant t^{\beta-1-\beta/\alpha}.\end{aligned} \tag{2.26}$$

Combining (2.25) and (2.26)

$$\int_{|y|\leqslant t^{\beta/\alpha}} |\nabla I_A|dy \leqslant Ct^{\beta-1-\beta/\alpha}. \tag{2.27}$$

As for I_B for $|y| \leqslant t^{\beta/\alpha}$

$$\begin{aligned}|\nabla I_B| &\leqslant Ct^{2\beta-1}|y|^{-d-\alpha-1}\int_0^{|y|^\alpha t^{-\beta}} \xi^{1-1/\beta}w(\xi^{-1/\beta},\beta,1)d\xi\\ &\leqslant Ct^{2\beta-1}|y|^{-d-\alpha-1}\int_0^{|y|^\alpha t^{-\beta}} \xi^{1-1/\beta}(\xi^{-1/\beta})^{-1-\beta}d\xi\\ &\leqslant Ct^{2\beta-1}|y|^{-d-\alpha-1}(|y|^\alpha t^{-\beta})^3 \leqslant Ct^{-\beta-1}|y|^{-d+2\alpha-1}.\end{aligned} \tag{2.28}$$

Integration gives

$$\int_{|y|\leqslant t^{\beta/\alpha}} |\nabla I_B| dy \leqslant C \int_{|y|\leqslant t^{\beta/\alpha}} t^{-\beta-1}|y|^{-d+d+2\alpha-2} dy \leqslant C t^{-\beta-1-\beta/\alpha}. \tag{2.29}$$

So

$$\int_{|y|\leqslant t^{\beta/\alpha}} |\nabla I_B| dy \leqslant C t^{\beta-1-\beta/\alpha}. \tag{2.30}$$

Since

$$\int_{\mathbb{R}^d} |\nabla G_\beta(t,y)| dy \leqslant \int_{\mathbb{R}^d} |\nabla I_A| dy + \int_{\mathbb{R}^d} |\nabla I_B| dy \tag{2.31}$$

combining results (2.22), (2.23), (2.27) and (2.30) we obtain

$$\int_{\mathbb{R}^d} |\nabla G_\beta(t,y)| dy \leqslant C t^{\beta-1-\beta/\alpha}. \tag{2.32}$$

which proves (2.21). □

Now let us consider the case $\alpha = 2$.

THEOREM 3. *Let $G_{\beta,1}(t,y)$ be as in (1.7) and (2.6). For $\alpha = 2$ and any $\beta \in (0,1)$:*

- $\int_0^t \int_{\mathbb{R}^d} |G_\beta(t,y)| dy ds \leqslant C t^\beta$,
- $\int_0^t \int_{\mathbb{R}^d} |\nabla G_\beta(t,y)| dy ds \leqslant C t^{\beta/2}$.

Proof. Note that

$$\int_{\mathbb{R}^d} \exp\{-a\sigma p^2 - iyp\} dp = \left(\frac{\sqrt{\pi}}{\sqrt{\sigma}}\right)^d \exp\left\{-\frac{y^2}{4a\sigma}\right\}, \tag{2.33}$$

where in our case $\sigma = xt^\beta$. Substitute this into (2.6) to obtain

$$G_\beta(t,y) = \frac{t^{\beta-1-\beta d/2}\pi^{d/2}}{(2\pi)^d} \int_0^\infty x^{-1/\beta-d/2} w(x^{-1/\beta},\beta,1) e^{-y^2/4at^\beta x} dx \tag{2.34}$$

where $y^2 = y_1^2 + y_2^2 + \ldots + y_d^2$. We are interested in $\int_0^t \int_{\mathbb{R}^d} |G_\beta(t,y)| dy ds$. Integrating y-dependent terms in G_β with respect to y gives

$$\int_{\mathbb{R}^d} \exp\left\{-|y|^2/4axt^\beta\right\} dy = (4\pi x t^\beta)^{d/2} = C x^{d/2} t^{\beta d/2}. \tag{2.35}$$

The term $x^{d/2}t^{\beta d/2}$ cancels out with $\left(\frac{1}{\sqrt{t^\beta x}}\right)^d$ and we obtain

$$I(t) := \int_{\mathbb{R}^d} |G_\beta(t,y)| dy = C \int_0^\infty x^{-1/\beta} w(x^{-1/\beta},\beta,1) t^{\beta-1} dx. \tag{2.36}$$

Now we split the integral $I(t)$ into 2 parts: $I_a(t)$ for $x>1$ and $I_b(t)$ for $0\leqslant x\leqslant 1$. In $I_a(t)$, $x>1$ and so $x^{-1/\beta}<1$ and $w(x^{-1/\beta},\beta,1)\sim C$, so we have

$$\begin{aligned} I_a(t) &= \int_1^\infty Ct^{\beta-1}x^{-1/\beta}w(x^{-1/\beta},\beta,1)dx \\ &\leqslant t^{\beta-1}\int_1^\infty x^{-1/\beta}Cdx \leqslant C1^{1-1/\beta}t^{\beta-1} = Ct^{\beta-1}. \end{aligned} \tag{2.37}$$

Integrating with respect to s gives

$$\int_0^t |I_a(t-s)|ds \leqslant C\int_0^t C(t-s)^{\beta-1}ds = Ct^\beta. \tag{2.38}$$

For $I_b(t)$, $x\leqslant 1$, so $x^{-1/\beta}\geqslant 1$ and $w(x^{-1/\beta},\beta,1)\sim (x^{-1/\beta})^{-1-\beta}=x^{1+1/\beta}$ and

$$\begin{aligned} I_b(t) &= \int_0^1 Cx^{-1/\beta}w(x^{-1/\beta},\beta,1)t^{\beta-1}dx \\ &\leqslant Ct^{\beta-1}\int_0^1 x^{-1/\beta+1/\beta+1}dx \leqslant C. \end{aligned} \tag{2.39}$$

with a constant $C_2>0$. Now we integrate with respect to s

$$\int_0^t |I_b(t-s)|ds = \int_0^t C(t-s)^{\beta-1}ds = Ct^\beta. \tag{2.40}$$

Together with (2.37) and (2.38) this yields the first statement of the theorem.

Differentiating G_β with respect to y gives us

$$\begin{aligned} I_1(t) &= \int_{\mathbb{R}^d} |\nabla G_\beta(t,y)|dy \\ &= \int_0^\infty \int_{\mathbb{R}^d} t^{-1-\beta d/2}x^{-1-1/\beta-d/2}|y|e^{-|y|^2/4axt^\beta}w(x^{-1/\beta},\beta,1)dydx. \end{aligned} \tag{2.41}$$

Since

$$\int_{\mathbb{R}^d} |y|\exp\{-|y|^2/4axt^\beta\}dy = Cxt^\beta(\sqrt{xt^\beta})^{d-1} = Cx^{\frac{d+1}{2}}t^{\frac{\beta(d+1)}{2}}, \tag{2.42}$$

we have

$$\begin{aligned} I_1(t) &= \int_0^\infty \int_{\mathbb{R}^d} t^{-1-\beta d/2}x^{-1-1/\beta-d/2}|y|e^{-|y|^2/4axt^\beta}w(x^{-1/\beta},\beta,1)dydx \\ &= C\int_0^\infty t^{-1+\beta/2}x^{-1/2-1/\beta}w(x^{-1/\beta},\beta,1)dx. \end{aligned} \tag{2.43}$$

Now we split the integral $I_1(t)$ into parts corresponding to $x\in(0,1)$ and $x\in[1,\infty)$:

$$I_2(t) = \int_0^1 t^{-1+\beta/2}x^{-1/2-1/\beta}w(x^{-1/\beta},\beta,1)dx \tag{2.44}$$

and

$$I_3(t) = \int_1^\infty t^{-1+\beta/2} x^{-1/2-1/\beta} w(x^{-1/\beta},\beta,1)dx. \tag{2.45}$$

Let us examine $I_2(t)$. Since $x \in (0,1)$, we have $w(x^{-1/\beta},\beta,1) \sim (x^{-1/\beta})^{-1-\beta}$, so

$$\begin{aligned} I_2(t) &= \int_0^1 t^{-1+\beta/2} x^{-1/2-1/\beta} w(x^{-1/\beta},\beta,1)dx \\ &= \int_0^1 t^{-1+\beta/2} x^{-1/2-1/\beta+1+1/\beta} dx = 2t^{-1+\beta/2}/3. \end{aligned} \tag{2.46}$$

Integrating

$$\int_0^t |I_2(t-s)|ds \leqslant C\int_0^t (t-s)^{-1+\beta/2} ds = Ct^{\beta/2}. \tag{2.47}$$

Now, for $I_3(t)$ we use that $x^{-1/\beta} \leqslant 1$ and so $w(x^{-1/\beta},\beta,1) \sim C$.

$$\begin{aligned} |I_3(t)| &\leqslant \left| \int_1^\infty t^{-1+\beta/2} x^{-1/2-1/\beta} w(x^{-1/\beta},\beta,1)dx \right| \\ &\leqslant Ct^{-1+\beta/2} \left| \int_1^\infty x^{-1/2-1/\beta} dx \right| = Ct^{-1+\beta/2}. \end{aligned} \tag{2.48}$$

Integrating with respect to s

$$\int_0^t |I_3(t-s)|ds \leqslant \int_0^t (t-s)^{\beta/2-1} ds = Ct^{\beta/2}. \tag{2.49}$$

Note that $\beta/2 = \beta - \beta/\alpha$ for $\alpha = 2$. So for $\alpha = 2$ the form of the estimate is the same as for $\alpha \in (1,2)$. □

The following corollary is a consequence of the previous theorem.

COROLLARY 1. *For G_β as defined in (1.7) in the case $\alpha = 2$ and $\beta \in (0,1)$*

$$\int_0^t \int_{\mathbb{R}^d} \left(|\nabla G_\beta(t,y)| + |G_\beta(t,y)| \right) dyds \leqslant Ct^{\beta/2}. \tag{2.50}$$

Proof. Since $\beta/2 < \beta$, we take the minimum power, $\beta/2$, to write the common estimate of the terms $\int_{\mathbb{R}^d} |\nabla G_\beta(t,y)|dy$ and $\int_{\mathbb{R}^d} |G_\beta(t,y)|dy$, obtaining

$$\int_{\mathbb{R}^d} \left(|\nabla G_\beta(t,y)| + |G_\beta(t,y)| \right) dy \leqslant Ct^{\beta/2-1}, \tag{2.51}$$

substitute t by $t-s$ and we use that

$$\int_0^t (t-s)^{\beta/2-1} ds = Ct^{\beta/2}, \tag{2.52}$$

which yields the result (2.50). □

Here we present several theorems regarding $S_{\beta,1}(t,y)$ which are particularly useful for the well-posedness analysis of (0.4) and (0.1).

THEOREM 4. *For* $\alpha \in (1,2)$, $\beta \in (0,1)$ *the first term from the RHS of (1.9) satisfies*

$$\left|\int_{\mathbb{R}^d} S_{\beta,1}(t,y-x)f_0(x)dx\right| \leqslant Ct^0. \tag{2.53}$$

Proof. Using (1.6) and (2.4) we represent $S_{\beta,1}(t,y)$ as

$$I = \frac{1}{\beta(2\pi)^d}\int_{\mathbb{R}^d}\int_0^\infty e^{ipy}e^{-a|p|^\alpha t^\beta \xi}\xi^{-1-1/\beta}w(\xi^{-1/\beta},\beta,1)d\xi dp \tag{2.54}$$

and use the assumption $|f_0(y)| < C$. We split the integral I into two parts: I_A for $\xi \in [|y|^\alpha t^{-\beta},\infty)$ and I_B for $\xi \in (0,|y|^\alpha t^\beta)$. There are 2 cases for each of the integrals: $|y| \leqslant t^{\beta/\alpha}$ and $|y| > t^{\beta/\alpha}$. Let us study $|I_B|$ in the case $|y| \leqslant t^{\beta/\alpha}$.

$$\begin{aligned}
|I_B| &\leqslant C\int_0^{|y|^\alpha t^{-\beta}} \xi^{-1-1/\beta}w(\xi^{-1/\beta},\beta,1)|y|^{-d-\alpha}t^\beta \xi d\xi \\
&\leqslant C\int_0^{|y|^\alpha t^{-\beta}} \xi^{-1/\beta}(\xi^{-1/\beta})^{-1-\beta}t^\beta|y|^{-d-\alpha}d\xi \\
&\leqslant C\int_0^{|y|^\alpha t^{-\beta}} \xi t^\beta|y|^{-d-\alpha}d\xi \\
&= C(|y|^\alpha t^{-\beta})^2|y|^{-d-\alpha}t^\beta = Ct^{-\beta}|y|^{-d+\alpha}.
\end{aligned} \tag{2.55}$$

Now, integrating gives

$$\int_{|y|\leqslant t^{\beta/\alpha}}|I_B|dy \leqslant C\int_{|y|\leqslant Ct^{\beta/\alpha}} t^{-\beta}|y|^{-d+\alpha+d-1}dy = Ct^{-\beta}(t^{\beta/\alpha})^\alpha = Ct^0. \tag{2.56}$$

Let us study $|I_B|$ in the case $|y| > t^{\beta/\alpha}$. Here we split the integral I_B into 2 parts: when $\xi \in (0,1]$ and when $\xi \in (1,|y|^\alpha t^{-\beta})$.

$$|I_B| \leqslant C\int_0^{|y|^\alpha t^{-\beta}} \xi^{-1/\beta}w(\xi^{-1/\beta},\beta,1)t^\beta|y|^{-d-\alpha}d\xi, \tag{2.57}$$

so since for $\xi \leqslant 1$, $w(\xi^{-1/\beta},\beta,1) \sim (\xi^{-1/\beta})^{-1-\beta}$, we have

$$\int_0^1 \xi^{-1/\beta}w(\xi^{-1/\beta},\beta,1)t^\beta|y|^{-d-\alpha}d\xi \leqslant C|y|^{-d-\alpha}t^\beta. \tag{2.58}$$

Integration yields

$$\int_{|y|>t^{\beta/\alpha}} t^\beta|y|^{-d-\alpha+d-1}dy = Ct^\beta(t^{\beta/\alpha})^{-\alpha} = Ct^0. \tag{2.59}$$

When $\xi \in (1, |y|^\alpha t^{-\beta})$

$$\begin{aligned}
\int_1^{|y|^\alpha t^{-\beta}} & \xi^{-1/\beta} w(\xi^{-1/\beta}, \beta, 1) t^\beta |y|^{-d-\alpha} d\xi \\
&= \int_1^{|y|^\alpha t^{-\beta}} \xi^{-1/\beta} \xi^{-q/\beta} t^\beta |y|^{-d-\alpha} d\xi \\
&= t^\beta |y|^{-d-\alpha} \left((|y|^\alpha t^{-\beta})^{-1/\beta - q/\beta + 1} - 1 \right) \\
&= t^{1+q+\beta-\beta} |y|^{-d-\alpha-\alpha/\beta - q\alpha/\beta+\alpha} - t^\beta |y|^{-d-\alpha}.
\end{aligned} \tag{2.60}$$

Integration gives

$$\int_{|y|>t^{\beta/\alpha}} t^{1+q} |y|^{-\alpha/\beta - q\alpha/\beta - 1} dy = t^{1+q} (t^{\beta/\alpha})^{-\alpha/\beta - q\alpha/\beta} = t^0, \tag{2.61}$$

and

$$\int_{|y|>t^{\beta/\alpha}} |y|^{-d-\alpha+d-1} t^\beta dy = t^\beta (t^{\beta/\alpha})^{-\alpha} = Ct^0. \tag{2.62}$$

Combining (2.59), (2.61) and (2.62) gives

$$\int_{\mathbb{R}^d} |I_B| dy \leqslant Ct^0. \tag{2.63}$$

Let us study $|I_A|$ case $|y| > t^{\beta/\alpha}$. Here $\xi^{-1/\beta}$ is small, so $w(\xi^{-1/\beta}, \beta, 1) \sim C$, where C is a constant.

$$\begin{aligned}
|I_A| &\leqslant C \int_{|y|^\alpha t^{-\beta}}^\infty \xi^{-1-1/\beta} w(\xi^{-1/\beta}, \beta, 1) t^{-d\beta/\alpha} \xi^{-d/\alpha} d\xi \\
&= C \int_{|y|^\alpha t^{-\beta}}^\infty \xi^{-1-1/\beta} t^{-\beta d/\alpha} \xi^{-d/\alpha} d\xi \\
&\leqslant Ct^{-\beta d/\alpha} (|y|^\alpha t^{-\beta})^{-1/\beta - d/\alpha} \leqslant C|y|^{-\alpha/\beta - d} t,
\end{aligned} \tag{2.64}$$

Integrating gives

$$\begin{aligned}
\int_{|y|>t^{\beta/\alpha}} |I_A| dy &\leqslant C \int_{|y|>t^{\beta/\alpha}} |y|^{-d-\alpha/\beta+d-1} t dy \\
&= Ct (t^{\beta/\alpha})^{-\alpha/\beta} = Ct^0.
\end{aligned} \tag{2.65}$$

Let us study $|I_A|$, case $|y| \leqslant t^{\beta/\alpha}$. Here we need to split the integral I_A into 2 parts. The first one is

$$\int_1^\infty \xi^{-d/\alpha} \xi^{-1-1/\beta} t^{-\beta d/\alpha} w(\xi^{-1/\beta}, \beta, 1) d\xi. \tag{2.66}$$

Here ξ is large, so $\xi^{-1/\beta}$ is small, so $w(\xi^{-1/\beta},\beta,1)\sim(\xi^{-1/\beta})^q$, for all $q>1$, which enables us to write

$$\begin{aligned}\int_1^\infty \xi^{-d/\alpha}t^{-\beta d/\alpha}\xi^{-1-1/\beta}\xi^{-q/\beta}d\xi\\ &= t^{-\beta d/\alpha}\int_1^\infty \xi^{-d/\alpha-1-1/\beta-q/\beta}d\xi\\ &\leqslant Ct^{-\beta d/\alpha}\left(\lim_{K\to\infty}K^{-d/\alpha-1/\beta-q/\beta}-1\right)=Ct^{-\beta d/\alpha}.\end{aligned}\tag{2.67}$$

Integrating gives

$$\int_{|y|\leqslant t^{\beta/\alpha}} t^{-\beta d/\alpha}|y|^{d-1}dy = t^{-\beta d/\alpha}(t^{\beta/\alpha})^d = t^0.\tag{2.68}$$

The second part of I_A is

$$\int_{|y|^\alpha t^{-\beta}}^1 \xi^{-1-1/\beta}w(\xi^{-1/\beta},\beta,1)\xi^{-d/\alpha}t^{-\beta d/\alpha}d\xi.\tag{2.69}$$

Since $\xi<1$, $\xi^{-1/\beta}>1$, so $w(\xi^{-1/\beta},\beta,1)\sim(\xi^{-1/\beta})^{-1-\beta}$, so we re-write (2.69) as

$$\begin{aligned}\int_{|y|^\alpha t^{-\beta}}^1 \xi^{-d/\alpha}t^{-\beta d/\alpha}\xi^{-1-1/\beta+1+1/\beta}d\xi\\ &\leqslant \int_{|y|^\alpha t^{-\beta}}^1 \xi^{-d/\alpha}t^{-\beta d/\alpha}d\xi = Ct^{-\beta d/\alpha}(1-|y|^\alpha t^{-\beta}).\end{aligned}\tag{2.70}$$

Integrating (2.70) in polar coordinates

$$\begin{aligned}C\int_{|y|\leqslant t^{\beta/\alpha}}|y|^{d-1}\left(t^{-\beta d/\alpha}-|y|^\alpha t^{-\beta}t^{-\beta d/\alpha}\right)dy\\ &\leqslant Ct^{-\beta d/\alpha}t^{\beta d/\alpha}-Ct^{\beta+d\beta/\alpha-\beta-\beta d/\alpha}=Ct^0.\end{aligned}\tag{2.71}$$

Combining (2.71) and (2.68) gives that for $|y|\leqslant t^{\beta/\alpha}$

$$\int_{\mathbb{R}^d}|I_A|dy\leqslant Ct^0.\tag{2.72}$$

Using the assumption $|f_0(y)|<C$ and putting together estimates (2.63) and (2.72) yields the theorem statement (2.53). $\square$

THEOREM 5. *For $\alpha\in(1,2)$, $\beta\in(0,1)$*

$$\int_{\mathbb{R}^d}\nabla S_{\beta,1}(t,y)f_0(x-y)dy\leqslant Ct^{-\beta/\alpha}.\tag{2.73}$$

Proof. We differentiate $S_{\beta,1}(t,y)$ defined in (1.6) with respect to y:

$$\begin{aligned}|\nabla S_{\beta,1}(t,y)| &= \left|\frac{1}{\beta(2\pi)^d}\nabla\int_{\mathbb{R}^d}\int_0^\infty e^{ipy}e^{-a|p|^\alpha t^\beta x}x^{-1-1/\beta}w(x^{-1/\beta},\beta,1)dxdp\right| \\ &= \frac{1}{\beta(2\pi)^d}\int_{\mathbb{R}^d}\int_0^\infty |ip|e^{ipy}e^{-a|p|^\alpha t^\beta x}x^{-1-1/\beta}w(x^{-1/\beta},\beta,1)dxdp \\ &= \frac{1}{\beta(2\pi)^d}\int_{\mathbb{R}^d}\int_0^\infty |p|e^{ipy}e^{-a|p|^\alpha t^\beta x}x^{-1-1/\beta}w(x^{-1/\beta},\beta,1)dxdp. \end{aligned} \tag{2.74}$$

Here we use the asymptotic expansions from Theorems 7.2.1 and 7.2.2 and Theorem 7.3.2, which are in the appendix as equations (5.2) and (5.6), and we use the inequality (7.40) in [2], which also appears in the appendix for reader's convenience, as (5.11) and (5.12). For I_A in the case $|y|>t^{\beta/\alpha}$ we use that for $\xi>1$, $\xi^{-1/\beta}<1$ and $w(\xi^{-1/\beta},\beta,1)<(\xi^{-1/\beta})^q$, for any $q>1$. Then

$$\begin{aligned}|\nabla I_A| &\leqslant C\int_{|y|^\alpha t^{-\beta}}^\infty \xi^{-1/\alpha-1-1/\beta-d/\alpha}w(\xi^{-1/\beta},\beta,1)t^{-\beta/\alpha-d\beta/\alpha}d\xi \\ &\leqslant Ct^{-\beta/\alpha-d\beta/\alpha}(|y|^\alpha t^{-\beta})^{-1/\alpha-d/\alpha-1/\beta-q/\beta} \\ &\leqslant Ct^{1+q}|y|^{-1-q\alpha/\beta-\alpha/\beta-d}. \end{aligned} \tag{2.75}$$

Integrating gives

$$\begin{aligned}\int_{|y|>t^{\beta/\alpha}}|\nabla I_A|\,dy &\leqslant C\int_{|y|>t^{\beta/\alpha}} t^{1+q}|y|^{-d+d-1-1-q\alpha/\beta-\alpha/\beta}dy \\ &= Ct^{1+q-\beta/\alpha-q-1} = Ct^{-\beta/\alpha}. \end{aligned} \tag{2.76}$$

Now, let us look at I_B in the case $|y|>t^{\beta/\alpha}$. Proposition 1 in the Appendix and the change of variables $\xi^{-1/\beta}=z$ yield

$$\begin{aligned}|\nabla I_B| &\leqslant C\int_0^{|y|^\alpha t^{-\beta}} \xi^{-1-1/\beta}w(\xi^{-1/\beta},\beta,1)t^{-\beta+\beta}|y|^{-\alpha-1-d-\alpha}t^\beta d\xi \\ &\leqslant C|y|^{-d-1}\int_{|y|^{-\alpha/\beta}t}^\infty w(z,\beta,1)dz \leqslant C|y|^{-d-1}. \end{aligned} \tag{2.77}$$

Integration gives

$$\int_{|y|>t^{\beta/\alpha}}|\nabla I_B|\,dy \leqslant C\int_{|y|>t^{\beta/\alpha}}|y|^{-d+d-1-1}dy \leqslant Ct^{-\beta/\alpha}. \tag{2.78}$$

Now, let us look at I_A in the case $|y|<t^{\beta/\alpha}$.

$$|\nabla I_A| \leqslant C\int_{|y|^\alpha t^{-\beta}}^\infty \xi^{-1-1/\beta-1/\alpha-d/\alpha}w(\xi^{-1/\beta},\beta,1)t^{-\beta/\alpha-\beta d/\alpha}d\xi. \tag{2.79}$$

We split this integral into cases $\xi \in (|y|^{\alpha}t^{-\beta},1)$ and $\xi \in [1,\infty)$. For $\xi \in (|y|^{\alpha}t^{-\beta},1)$, $\xi^{-1/\beta} > 1$ and we may use that $w(\xi^{-1/\beta},\beta,1) \sim (\xi^{-1/\beta})^{-1-\beta}$. So

$$\begin{aligned}
|\nabla I_A| &\leqslant C\int_{|y|^{\alpha}t^{-\beta}}^{1} \xi^{-1-1/\beta} w(\xi^{-1/\beta},\beta,1) t^{-\beta/\alpha-\beta d/\alpha}\xi^{-1/\alpha-d/\alpha}d\xi \\
&\leqslant C\int_{|y|^{\alpha}t^{-\beta}}^{\infty} t^{-\beta/\alpha-\beta d/\alpha}\xi^{-1/\alpha-d/\alpha}d\xi \\
&= Ct^{-\beta/\alpha-\beta d/\alpha} - Ct^{-\beta}|y|^{-1-d+\alpha}.
\end{aligned} \tag{2.80}$$

Integration yields

$$\begin{aligned}
\int_{|y|\leqslant t^{\beta/\alpha}} |\nabla I_A|\,dy \leqslant C\int_{|y|\leqslant t^{\beta/\alpha}} y^{d-1}t^{-\beta/\alpha-\beta d/\alpha}dy& \\
+C\int_{|y|<t^{\alpha/\beta}} t^{-\beta}|y|^{-1-\alpha-1-d+d}dy& \\
\leqslant Ct^{-\beta/\alpha} + Ct^{-\beta}(t^{\beta/\alpha})^{-1+\alpha} \leqslant Ct^{-\beta/\alpha}&.
\end{aligned} \tag{2.81}$$

As for $\xi \in [1,\infty)$, then $\xi^{-1/\beta} < 1$ and so $w(\xi^{-1/\beta},\beta,1) \sim C$ and

$$\begin{aligned}
\int_1^{\infty} \xi^{-2-1/\beta}C\xi^{-d/\alpha}t^{-\beta/\alpha-\beta d/\alpha}d\xi& \\
\leqslant t^{-\beta/\alpha-\beta d/\alpha}C\int_1^{\infty}\xi^{-2-1/\beta-d/\alpha}d\xi& \\
\leqslant t^{-\beta/\alpha-\beta d/\alpha}C\left(1-\lim_{K\to\infty}\frac{1}{K}\right)& \\
= Ct^{-\beta/\alpha-\beta d/\alpha}&.
\end{aligned} \tag{2.82}$$

Integration yields

$$\int_{|y|<t^{\beta/\alpha}} t^{-\beta/\alpha-\beta d/\alpha}|y|^{d-1}dy \leqslant Ct^{-\beta/\alpha-\beta d/\alpha}t^{\beta d/\alpha} = Ct^{-\beta/\alpha}. \tag{2.83}$$

Finally, I_B in the case $|y| \leqslant t^{\beta/\alpha}$

$$\begin{aligned}
|\nabla I_B| &\leqslant C\int_0^{|y|^{\alpha}t^{-\beta}} \xi^{-1-1/\beta}w(\xi^{-1/\beta},\beta,1)\xi^{-1}t^{-\beta}|y|^{\alpha-1}|y|^{-d-\alpha}t^{\beta}\xi\, d\xi \\
&\leqslant C\int_0^{|y|^{\alpha}t^{-\beta}} \xi^{-1-1/\beta}w(\xi^{-1/\beta},\beta,1)|y|^{-1-d}d\xi \\
&\leqslant C\int_0^{|y|^{\alpha}t^{-\beta}} \xi^{-1-1/\beta}(\xi^{-1/\beta})^{-1-\beta}|y|^{-1-d}d\xi \\
&\leqslant C|y|^{-1-d-\alpha}t^{-\beta}.
\end{aligned} \tag{2.84}$$

Integration yields

$$\begin{aligned}
\int_{|y|\leqslant t^{\beta/\alpha}} |\nabla I_B|\,dy \leqslant C\int_{|y|\leqslant t^{\beta/\alpha}} |y|^{\alpha-1-1-d+d}t^{-\beta}dy& \\
= Ct^{-\beta}(t^{\beta/\alpha})^{\alpha-1} = Ct^{-\beta/\alpha}&.
\end{aligned} \tag{2.85}$$

Hence (2.76), (2.78), (2.81), (2.83) and (2.85) together with the assumption $|f_0(y)| < C$ yield (2.73). □

THEOREM 6. *For $S_{\beta,1}$ defined in (1.6), in the case $\alpha = 2$ and assuming $|f_0(y)| < C$*

$$\int_{\mathbb{R}^d} S_{\beta,1}(t, y-x) f_0(x) dx \leqslant Ct^0. \tag{2.86}$$

Proof. Using (2.35)

$$\int_{\mathbb{R}^d} S_{\beta,1}(t,y)dy = \int_0^\infty \int_{\mathbb{R}^d} (xt^\beta)^{-d/2} e^{-y^2/(4axt^\beta)} x^{-1-1/\beta} w(x^{-1/\beta},\beta,1) dydx$$
$$= C\int_0^\infty x^{-1-1/\beta} w(x^{-1/\beta},\beta,1)dx. \tag{2.87}$$

We split this integral into two parts: $x \in [0,1]$ and $x \in (1,\infty)$. In the first case $x \leqslant 1$ and $x^{-1/\beta} > 1$ so we may use $w(x^{-1/\beta},\beta,1) \sim (x^{-1/\beta})^{-1-\beta}$. In the case $x > 1$ we may use that $w(x^{-1/\beta},\beta,1) \sim C$. So we obtain

$$\int_0^1 x^{-1-1/\beta} w(x^{-1/\beta},\beta,1)dx = \int_0^1 dx = 1, \tag{2.88}$$

and

$$\int_1^\infty x^{-1-1/\beta} w(x^{-1/\beta},\beta,1)dx = \int_1^\infty x^{-1-1/\beta} C dx$$
$$= C\left(\lim_{K\to\infty} K^{-1/\beta} - 1^{-1/\beta}\right) = C. \tag{2.89}$$

Together with the assumption $|f_0(y)| < C$, the result (2.86) follows. □

THEOREM 7. *For $S_{\beta,1}$ defined as in (1.6), in the case $\alpha = 2$, $\beta \in (0,1)$ and assuming $|f_0(y)| < C$*

$$\int_{\mathbb{R}^d} \nabla S_{\beta,1}(t,y) f_0(x-y) dy \leqslant Ct^{-\beta/2}. \tag{2.90}$$

Proof. We use the representation of $S_{\beta,1}(t,y)$ in (2.54) and write

$$\int_{\mathbb{R}^d} \nabla S_{\beta,1}(t,y)dy = \int_0^\infty x^{-3/2-1/\beta} t^{-\beta/2} w(x^{-1/\beta},\beta,1)dx. \tag{2.91}$$

We split the above integral into two: for $x \in [0,1]$ and for $x > 1$. In the case $x \in [0,1]$ we use that $w(x^{-1/\beta},\beta,1) \sim (x^{-1/\beta})^{-1-\beta}$. In the case $x > 1$ we use that $w(x^{-1/\beta},\beta,1) \sim C$. So we get

$$t^{-\beta/2} \int_0^1 x^{-3/2-1/\beta} w(x^{-1/\beta},\beta,1)dx$$
$$= Ct^{-\beta/2} \int_0^1 x^{-1/2} dx = Ct^{-\beta/2}/2 \tag{2.92}$$

and

$$t^{-\beta/2}\int_1^{\infty} x^{-3/2-1/\beta} w(x^{-1/\beta},\beta,1)dx = Ct^{-\beta/2}. \tag{2.93}$$

So from (2.92) and (2.93) and that $|f_0(y)| < C$ and we obtain (2.90). $\square$

3. Smoothing properties for the linear equation

Let us denote by $C^p(\mathbb{R}^d)$ the space of p times continuously differentiable functions. By C^1_∞ we shall denote functions f in $C^1(\mathbb{R}^d)$ such that f and ∇f are rapidly decreasing continuous functions on $\mathbb{R}^d$, with the sum of sup-norms of the function and all of its derivatives up to and including the order p as the corresponding norm. The sup-norm is $\|f\| = \sup_{t\in[0,T]} \|f(t)\|$. Let us denote by H^p_1 the Sobolev space of functions with generalised derivatives up to and including p, being in $L^1(\mathbb{R}^d)$. Here and in what follows we often identify the function $f_0(y)$ with the function $f(t,y) = f_0(y)$, $\forall t \geqslant 0$.

THEOREM 8. (Solution regularity) *For $\alpha \in (1,2]$ and $\beta \in (0,1)$ the resolving operator*

$$\Psi_t(f_0) = \int_{\mathbb{R}^d} S_{\beta,1}(t,y-x) f_0(x)dx + \int_0^t \int_{\mathbb{R}^d} G_\beta(t-s,y-x)h(s,x)dxds \tag{3.1}$$

satisfies the following properties

- $\Psi_t : C^p(\mathbb{R}^d) \mapsto C^p(\mathbb{R}^d)$, *and* $\|\Psi_t\|_{C^p} < C(t)$,
- $\Psi_t : H^p_1(\mathbb{R}^d) \mapsto H^p_1(\mathbb{R}^d)$ *and* $\|\Psi_t\|_{H^p_1} < C(t)$.

Proof. We look at the C^p norm of $\Psi_t f_0$ and use Theorem 1

$$\begin{aligned}\|\Psi_t(f_0)\|_{C^p(\mathbb{R}^d)} &\leqslant \int_{\mathbb{R}^d} S_{\beta,1}(t,y)|f_0^{(p)}|dy + Ct^\beta \sup_{s\in[0,t]} \|h(s,\cdot)\|_{C^p(\mathbb{R}^d)} \\ &\leqslant C\|f_0\|_{C^p(\mathbb{R}^d)} + Ct^\beta \sup_{s\in[0,t]} \|h(s,\cdot)\|_{C^p(\mathbb{R}^d)},\end{aligned} \tag{3.2}$$

for some constant $C > 0$. Analogously,

$$\begin{aligned}\|\Psi_t f_0\|_{H^p_1(\mathbb{R}^d)} &\leqslant \int_{\mathbb{R}^d} S_{\beta,1}(t,y)|f_0^{(p)}|dy + Ct^\beta \sup_{s\in[0,t]} \|h(s,\cdot)\|_{H^p_1(\mathbb{R}^d)} \\ &\leqslant C\|f_0\|_{H^p_1(\mathbb{R}^d)} + Ct^\beta \sup_{s\in[0,t]} \|h\|_{H^p_1(\mathbb{R}^d)}.\end{aligned} \tag{3.3}$$

$\square$

THEOREM 9. (Solution smoothing) *For $\alpha \in (1,2]$ and $\beta \in (0,1)$ the resolving operator (3.1) satisfies the following smoothing properties*

- *If* $f_0, h \in C^p(\mathbb{R}^d)$ *uniformly in time, then* $f \in C^{p+1}(\mathbb{R}^d)$ *and for any* $s \in (0,t]$

$$\|\Psi_t(f_0)\|_{C^{p+1}(\mathbb{R}^d)} \leqslant Ct^{-\beta/\alpha}\|f_0\|_{C^p(\mathbb{R}^d)} + Ct^{\beta-\beta/\alpha}\|h\|_{C^p(\mathbb{R}^d)} \tag{3.4}$$

- *If* $f_0, h \in H_1^p(\mathbb{R}^d)$ *uniformly in time, then* $f \in H_1^{p+1}(\mathbb{R}^d)$ *and for any* $s \in (0,t]$

$$\|\Psi_t(f_0)\|_{H_1^{p+1}(\mathbb{R}^d)} \leqslant Ct^{-\beta/\alpha}\|f_0\|_{H_1^p(\mathbb{R}^d)} + Ct^{\beta-\beta/\alpha}\|h\|_{H_1^p(\mathbb{R}^d)}. \tag{3.5}$$

In particular we may choose $p = 0$, *when* $H_1^0(\mathbb{R}^d) = L^1(\mathbb{R}^d)$.

Proof. We study the $C^{p+1}(\mathbb{R}^d)$ norm of $\Psi_t(f_0)$ and use theorems 1 and 2

$$\begin{aligned}
\|\Psi_t(f_0)\|_{C^{p+1}} &\leqslant \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\nabla_x S_{\beta,1}(t,x-y)f_0^{(p)}(y)\right|dy \\
&\quad + \sup_{x\in\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}\left|\nabla_x G_\beta(t-s,x-y)h_y^{(p)}(s,y)\right|dyds \\
&\leqslant Ct^{-\beta/\alpha}\sup_{x\in\mathbb{R}^d}|f_0^{(p)}(x)| + C\sup_{x\in\mathbb{R}^d}|h^{(p)}(s,x)|\int_0^t (t-s)^{\beta-\beta/\alpha-1}ds \\
&\leqslant Ct^{-\beta/\alpha}\|f_0\|_{C^p} + Ct^{\beta-\beta/\alpha}\|h\|_{C^p}.
\end{aligned} \tag{3.6}$$

The proof for (3.5) is analogous. □

Similar results apply for the non-linear equation (0.1).

4. Well-posedness

Now we study well-posedness of the full non-linear equation (0.1):

$$D_{0,t}^{*\beta} f(t,y) = -a(-\Delta)^{-\alpha/2} f(t,y) + H(t,y,\nabla f(t,y)), \tag{4.1}$$

with the initial condition $f(0,y) = f_0(y)$, and $a > 0$ is a constant. This FDE has the following mild form:

$$\begin{aligned}
f(t,y) &= \int_{\mathbb{R}^d} f_0(x) S_{\beta,1}(t,y-x)dx \\
&\quad + \int_0^t\int_{\mathbb{R}^d} G_\beta(t-s,y-x)H(s,x,\nabla f(s,x))dxds,
\end{aligned} \tag{4.2}$$

which follows from (1.3).

LEMMA 1. *Let us define by* $C([0,T],C_\infty^1(\mathbb{R}^d))$ *the space of functions* $f(t,y)$, *defined for* $t \in [0,T], y \in \mathbb{R}^d$, *such that* $f(t,y)$ *is continuous in* t *and* $f(t,\cdot) \in C_\infty^1(\mathbb{R}^d)$ *for all* $t \in [0,T]$. *Denote by* $B_{f_0}^T$ *the closed convex subset of* $C([0,T],C_\infty^1(\mathbb{R}^d))$ *consisting* *of functions with* $f(0,\cdot) = f_0(\cdot) = S_0(\cdot)$ *for some given function* S_0*. Let us define a non-linear mapping* $f \to \{\Psi_t(f)\}$ *defined for* $f \in B^T_{f_0}$*:*

$$\Psi_t(f)(y) = \int_{\mathbb{R}^d} f_0(x) S_{\beta,1}(t, y-x)dx \\ + \int_0^t \int_{\mathbb{R}^d} G_\beta(t-s, y-x) H(s,x,\nabla f(s,x))dxds. \tag{4.3}$$

Suppose $H(s,y,p)$ *is Lipschitz in* p *with the Lipschitz constant* L*. Let us take* $f_1, f_2 \in B^T_{f_0}$*. Then for any* $t \in [0,T]$*:*

$$\|\Psi^n_t(f_1) - \Psi^n_t(f_2)\|_{C^1} \leqslant \frac{(\beta - \beta/\alpha)L^n((\beta - \beta/\alpha)^{-1} t^{(\beta-\beta/\alpha)})^n}{n^{n\beta - n\beta/\alpha + 1}} \sup_{s\in[0,t]} \|f_1 - f_2\|_{C^1}. \tag{4.4}$$

Proof. Due to regularity estimates for $S_{\beta,1}$ and G_β:

$$\|\Psi_t(f_1) - \Psi_t(f_2)\|_{C^1} \leqslant CLt^{\beta-\beta/\alpha} \sup_{s\in[0,t]} \|f_1 - f_2\|_{C^1}. \tag{4.5}$$

and

$$\|\Psi^2_t(f_1) - \Psi^2_t(f_2)\|_{C^1} \leqslant C^2 L^2 \sup_{s\in[0,t]} \|f_1 - f_2\|_{C^1} \int_0^t (t-s)^{\beta-\beta/\alpha-1} s^{\beta-\beta/\alpha} ds. \tag{4.6}$$

We calculate the integral above using the change of variables $z = s/t$:

$$\int_0^t (t-s)^{\beta-\beta/\alpha-1} s^{\beta-\beta/\alpha} ds \\ = \int_0^1 t^{\beta-\beta/\alpha-1}(1-z)^{\beta-\beta/\alpha-1} z^{\beta-\beta/\alpha} t^{\beta-\beta/\alpha+1} dz \\ = t^{2\beta-2\beta/\alpha} B(\beta - \beta/\alpha + 1, \beta - \beta/\alpha). \tag{4.7}$$

Now, when we estimate $\|\Psi^3_t(f_1) - \Psi^3_t(f_2)\|_{C^1}$ we calculate

$$\int_0^t s^{2\beta-2\beta/\alpha}(t-s)^{\beta-\beta/\alpha-1} ds \\ = t^{\beta-\beta/\alpha-1} \int_0^1 t^{2\beta-2\beta/\alpha+1} z^{2\beta-2\beta/\alpha}(1-z)^{\beta-\beta/\alpha-1} dz \\ = t^{3(\beta-\beta/\alpha)} B(2\beta - 2\beta/\alpha + 1, \beta - \beta/\alpha). \tag{4.8}$$

This yields

$$\|\Psi^3_t(f_1) - \Psi^3_t(f_2)\|_{C^1} \\ \leqslant C^3 L^3 t^{3\beta-3\beta/\alpha} B(2\beta - 2\beta/\alpha + 1, \beta - \beta/\alpha) \sup_{s\in[0,t]} \|f_1 - f_2\|_{C^1}. \tag{4.9}$$

As the inductive step, assume that the following is true for some $n \in \mathbb{N}$:

$$\|\Psi_t^n(f_1) - \Psi_t^n(f_2)\|_{C^1} \leqslant R_{n,\alpha,\beta} \sup_{s\in[0,t]} \|f_1 - f_2\|_{C^1}. \tag{4.10}$$

where

$$R_{n,\alpha,\beta} = C^n L^n t^{n\beta - n\beta/\alpha} \frac{(\Gamma(\beta - \beta/\alpha))^{n-1}\Gamma(\beta - \beta/\alpha + 1)}{\Gamma(n\beta - n\beta/\alpha + 1)} \tag{4.11}$$

Let us check that then (4.10) holds for $k = n+1$.

$$\begin{aligned}
&\|\Psi_t^{n+1}(f_1) - \Psi_t^{n+1}(f_2)\|_{C^1} \\
&\quad = \left\| \int_0^t \int_{\mathbb{R}^d} G_\beta(t-s,x-y)\,(H(s,y,\nabla\Psi_t^n(f_1)) - H(s,y,\nabla\Psi_t^n(f_2))) \right\|_{C^1} \\
&\quad \leqslant CL \int_0^t (t-s)^{\beta-\beta/\alpha-1} ds \|\Psi_t^n(f_1) - \Psi_t^n(f_2)\|_{C^1} \\
&\quad \leqslant C^{n+1}L^{n+1}t^{n\beta-n\beta/\alpha} M_n \int_0^t (t-s)^{\beta-\beta/\alpha-1} s^{n\beta-n\beta/\alpha} ds \sup_{s\in[0,t]} \|f_1-f_2\|_{C^1} \\
&\quad \leqslant C^{n+1}L^{n+1}t^{n\beta-n\beta/\alpha} M_n B_n \sup_{s\in[0,t]} \|f_1-f_2\|_{C^1} \\
&\quad \leqslant C^{n+1}L^{n+1}t^{n\beta-n\beta/\alpha} M_{n+1} \sup_{s\in[0,t]} \|f_1-f_2\|_{C^1},
\end{aligned} \tag{4.12}$$

where

$$M_n = \frac{(\Gamma(\beta - \beta/\alpha))^{n-1}\Gamma(\beta - \beta/\alpha + 1)}{\Gamma(n\beta - n\beta/\alpha + 1)}, \tag{4.13}$$

M_{n+1} is as in (4.13) with n replaced by $n+1$, and B_n is the Beta function

$$B_n = B(n\beta - n\beta/\alpha + 1, \beta - \beta/\alpha). \tag{4.14}$$

The inequality (4.12) is (4.10) with $k = n$ replaced by $k = n+1$. We have shown (4.10) is true for $k = 1$ and $k = 2$. So by induction on k we obtain (4.10) for any $k \in \mathbb{N}$. Using that $g(x) = x^n$ is a convex function for $n \in \mathbb{N}$ it may be shown by Jensen's inequality that

$$(\Gamma(\beta - \beta/\alpha))^n \leqslant \frac{\Gamma(n\beta - n\beta/\alpha - n + 1)}{n^{n\beta - n\beta/\alpha - n + 1}}. \tag{4.15}$$

Using Stirling's formula we now obtain the quotient approximation

$$\frac{\Gamma(n(\beta - \beta/\alpha) + B)}{\Gamma(n(\beta - \beta/\alpha) + A)} \approx (n(\beta - \beta/\alpha))^{B-A}. \tag{4.16}$$

Let us substitute $A=1$ and $B=-n+1$. Then

$$\begin{aligned}
&\|\Psi_t^n(f_1)-\Psi_t^n(f_2)\|_{C^1}\\
&\qquad\leqslant \frac{\Gamma(1+\beta-\beta/\alpha)t^{n\beta-n\beta/\alpha}}{n^{n\beta-n\beta/\alpha+1}(\beta-\beta/\alpha)^n\Gamma(\beta-\beta/\alpha)}\sup_{s\in[0,t]}\|f_1-f_2\|_{C^1}\\
&\qquad\leqslant \frac{t^{n\beta-n\beta/\alpha}}{n^{n\beta-n\beta/\alpha+1}(\beta-\beta/\alpha)^{n-1}}\sup_{s\in[0,t]}\|f_1-f_2\|_{C^1},
\end{aligned}\tag{4.17}$$

so (4.4) holds. □

THEOREM 10. *Assume* $\alpha\in(1,2]$, $\beta\in(0,1)$ *and that*

- $H(s,y,p)$ *is Lipschitz in* p *with the Lipschitz constant* L *independent of* y.
- $|H(s,y,0)|\leqslant h$, *for a constant* h *independent of* y.
- $f_0(y)\in C^1_\infty(\mathbb{R}^d)$.

Then the equation (4.2) *has a unique solution* $S(t,y)\in C^1_\infty(\mathbb{R}^d)$.

Proof. Let us denote by $C([0,T],C^1_\infty(\mathbb{R}^d))$ and $B^T_{f_0}$ as in Lemma 1. Let $\Psi_t(f)$ be defined as in (4.3). Take $f_1(s,x),f_2(s,x)\in B^T_{f_0}$. Note that due to our choice of f_1,f_2,

$$\int_{\mathbb{R}^d} f_1(0,x)S_{\beta,1}(t,y-x)dx=\int_{\mathbb{R}^d} f_2(0,x)S_{\beta,1}(t,y-x)dx.\tag{4.18}$$

We would like to prove the existence and uniqueness result for all $t\leqslant T$ and any $T\geqslant 0$. For this we use (4.4) in Lemma 1. As $n\to\infty$, n^n grows faster than m^n for any fixed $m>0$. Hence for any $t\geqslant 0$

$$\|\Psi_t^n(f_1)-\Psi_t^n(f_2)\|_{C^1}\leqslant\frac{L^n(t^{\beta-\beta/\alpha}(\beta-\beta/\alpha)^{-1})^n(\beta-\beta/\alpha)}{n^{n\beta-n\beta/\alpha+1}}\sup_{s\in[0,t]}\|f_1-f_2\|_{C^1}.\tag{4.19}$$

The sum

$$\sum_{n=1}^{\infty}\frac{(t^{\beta-\beta/\alpha}(\beta-\beta/\alpha)^{-1})^n}{n^{n(\beta-\beta/\alpha)+1}}$$

is convergent by the ratio test. By Weissinger's fixed point theorem, see [1] Theorem D.7, Ψ_t has a unique fixed point f^* such that for any $f_1\in B^T_{f_0}$

$$\|\Psi_t^n(f_1)-f^*\|_{C^1}\leqslant\sum_{k=n}^{\infty}\frac{(t^{\beta-\beta/\alpha}(\beta-\beta/\alpha)^{-1})^n(\beta-\beta/\alpha)}{n^{n\beta-n\beta/\alpha+1}}\|\Psi_t(f_1)-f_1\|_{C^1}.\tag{4.20}$$

So $S(t,y)=f^*$ is the solution of (4.2) of class $C^1_\infty(\mathbb{R}^d)$. □

THEOREM 11. *Assume* $\alpha \in (1,2]$, $\beta \in (0,1)$ *and that*

- $H(s,y,p)$ *is Lipschitz in* p *with the Lipschitz constant* L_1 *independent of* y.
- H *is Lipschitz in* y *independently of* p, *with a Lipschitz constant* L_2

$$|H(s,y_1,p) - H(s,y_2,p)| \leqslant L_2|y_1 - y_2|(1+|p|) \tag{4.21}$$

- $|H(s,y,0)| \leqslant h$, *for a constant* h *independent of* y.
- $f_0(y) \in C^2_\infty(\mathbb{R}^d)$.

Then there exists a unique solution $f^*(t,y)$ *of the FDE equation (4.1) for* $\beta \in (0,1)$ *and* $\alpha \in (1,2]$, *and* f^* *satisfies*

$$\operatorname*{ess\,sup}_y |\nabla^2(f^*(t,y))| < C. \tag{4.22}$$

Proof. First, we work with the mild form of the equation (4.1). Let $B^{T,2}_{f_0}$ denote the subset of $B^T_{f_0}$ which is twice continuously differentiable in y and with $f(0,y) = S_0(y)$, for all $y \in \mathbb{R}^d$. Let the mapping Ψ_t on $B^{T,2}_{f_0}$ be defined as in (4.3). Take $f_0 \in B^{T,2}_{f_0}$, which continues $f_0(y) = S_0(y)$ to all $t \geqslant 0$. Then

$$\begin{aligned}
\|\Psi_t(f_0)\|_{C^2} \leqslant{} & Ct^{\beta-\beta/\alpha}\|H(s,x,\nabla f_0(x))\|_{C^1} \\
& + \|\int_{\mathbb{R}^d} S_{\beta,1}(t,y-x) f_0(x)dx\|_{C^2} \\
\leqslant{} & Ct^{\beta-\beta/\alpha}L_1\|f_0\|_{C^2} + Ct^{\beta-\beta/\alpha}L_2\|f_0\|_{C^1} \\
& + Ct^{\beta-\beta/\alpha}\|\nabla f_0(x)\|_{C^0} + C_3 \\
\leqslant{} & Lt^{\beta-\beta/\alpha}\|f_0\|_{C^2} + Ct^{\beta-\beta/\alpha}\|f_0(x)\|_{C^1} + C_3 \\
\leqslant{} & Ct^{\beta-\beta/\alpha}(\|f_0\|_{C^2}+1) + C_3.
\end{aligned} \tag{4.23}$$

Iterations and induction yield

$$\|\Psi^n_t(f_0)\|_{C^2} \leqslant C_3 \sum_{m=1}^n t^{m\beta - m\beta/\alpha} K_m + \sum_{m=1}^n t^{m(\beta-\beta/\alpha)} C_m (1 + \|f_0\|_{C^2}), \tag{4.24}$$

for constants $K_m = B_2 \times \cdots \times B_{m-1}$ and $C_m = B_2 \times \cdots \times B_m$, where

$$B_k = B(k\beta - k\beta/\alpha + 1, \quad \beta - \beta/\alpha),$$

for any $k \in \mathbb{N}$. We use that for x large and y fixed $B(x,y) \sim \Gamma(y)x^{-y}$ to obtain that $B_{m+1} < B_m$, for all $m \in \mathbb{N}$ which yields that the sums $\sum_{m=1}^n t^{m\beta-m\beta/\alpha}K_m$ and $\sum_{m=1}^n t^{m\beta-m\beta/\alpha}C_m$ are convergent as $n \to \infty$. So for some constants A_1, A_2, $C_{f_0} > 0$,

$$\|\Psi^n_t f_0\|_{C^2} < A_1 + A_2\|f_0\|_{C^2} < C_{f_0}. \tag{4.25}$$

Hence, $\forall n \in \mathbb{N}$

$$\|\nabla(\Psi_t^n f_0)\|_{Lip} < C_{f_0}. \tag{4.26}$$

Hence, we obtain

$$\|\lim_{n\to\infty} \nabla(\Psi_t^n f_0)\|_{Lip} < 2C_{f_0}. \tag{4.27}$$

By Rademacher's theorem it follows that $\lim_{n\to\infty}(\nabla^2(\Psi_t^n(f_0))$ exists almost everywhere. We invite the reader to see [26] for the Rademacher's theorem and its proof. From the previous theorem $\lim_{n\to\infty}\Psi_t^n(f_0) = f^*$. The limit is understood in the sense of convergence in $C_\infty^1(\mathbb{R}^d)$. Therefore f^* satisfies (4.22). ☐

THEOREM 12. *Assume $\alpha \in (1,2], \beta \in (0,1)$ and that*

- *$H(s,y,p)$ is Lipschitz in p with the Lipschitz constant L independent of y.*
- *H is Lipschitz in y independently of p, with a Lipschitz constant L_2*

$$|H(s,y_1,p) - H(s,y_2,p)| \leqslant L_2|y_1 - y_2|(1+|p|) \tag{4.28}$$

- *$|H(s,y,0)| \leqslant h$, for a constant h independent of y.*
- *$f_0(y) \in C_\infty^2(\mathbb{R}^d)$.*

Then a solution to the mild form

$$\begin{aligned} f(t,y) = &\int_{\mathbb{R}^d} S_{\beta,1}(t,x-y) f_0(y)dy \\ &+ \int_0^t \int_{\mathbb{R}^d} G_\beta(t-s,x-y) H(s,y,\nabla f(s,y)) ds dy \end{aligned} \tag{4.29}$$

which satisfies (4.22), is a classical solution to

$$D_{0,t}^{*\beta} f(t,y) = -(-\Delta)^{\alpha/2} f(t,y) + H(t,y,\nabla f(t,y)). \tag{4.30}$$

Proof. Let us define $\Psi_t(f)$ as in (4.3). By [1]

$$\begin{aligned} \hat{f}(t,p) = &\hat{f}_0(p) E_{\beta,1}(-a|p|^\alpha t^\beta) \\ &+ \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-a(t-s)^\beta |p|^\alpha)) \hat{H}(s,y,p) ds, \end{aligned} \tag{4.31}$$

is equivalent to

$$D_{0,t}^{*\beta} \hat{f}(t,p) = -a|p|^\alpha \hat{f}(t,p) + \hat{H}(t,y,\nabla f(t,y)), \tag{4.32}$$

which in turn is equivalent to (4.30) as its Fourier transform. Also, (4.29) is equivalent to (1.3) as its inverse Fourier transform. Therefore (4.29) is equivalent to (4.30).

We may carry out these equivalence procedures when $D^{*\beta}_{0,t}\Psi_t(f)$ and $-(-\Delta)^{\alpha/2}f$ are defined for f satisfying (4.22).

Due to theorem assumptions:

$$|H(s,y,\nabla f(s,\cdot))| \leqslant h + L|\nabla f(s,\cdot)| < \infty. \tag{4.33}$$

So

$$\begin{aligned} D^{*\beta}_{0,t} &\left(\int_0^t \int_{\mathbb{R}^d} G_\beta(t,y) H(s,y,\nabla f(t,y)) dy ds \right) \\ &\leqslant \frac{C}{\Gamma[1-\beta]} \int_0^t (t-s)^{-\beta} s^\beta ds \leqslant C_1 \int_0^1 (t-tz)^{-\beta} \beta (tz)^{\beta-1} t dz \\ &\leqslant C_1 \beta \int_0^1 (1-z)^{1-\beta-1} z^{\beta-1} dz \\ &\leqslant C_1 \beta B(1-\beta,\beta) < \infty. \end{aligned} \tag{4.34}$$

Similarly

$$D^{*\beta}_{0,t} \int_{\mathbb{R}^d} S_{\beta,1}(t,x-y) f_0(y) dy \tag{4.35}$$

exists when $f_0(y)$ gives dependence of $\int_{\mathbb{R}^d} S_{\beta,1}(t,x-y) f_0(y) dy$ on t such as t^k, where $k > -1$. This is because

$$\begin{aligned} \int_0^t (t-s)^{-\beta} \left(\frac{d}{ds} s^k \right) ds &= t^{k+1-\beta} \int_0^1 (1-z)^{-\beta} z^{k-1} dz \\ &= t^{k+1-\beta} B(1-\beta,k+1), \end{aligned} \tag{4.36}$$

where for any $\beta \in (0,1)$ the Beta function $B(1-\beta,k+1)$ is defined for $k+1>0$. Hence, due to (4.33), (4.34) and (4.35), $D^{*\beta}_{0,t}\Psi_t(f)$ is defined for the solution f for (4.30). For f satisfying (4.22), when $\alpha \in (1,2]$, $-(-\Delta)^{\alpha/2}f$ is defined. Now, let us study the solution $f^*(t,y)$

$$\begin{aligned} f^*(t,y) = &\int_0^t \int_{\mathbb{R}^d} G_\beta(t-s,y-x) H(s,x,\nabla f^*(t,x)) dx ds \\ &+ \int_{\mathbb{R}^d} S_{\beta,1}(t,y-x) f_0(x) dx. \end{aligned} \tag{4.37}$$

Differentiating twice w.r.t. y gives:

$$\begin{aligned} \nabla^2 &\int_0^t \int_{\mathbb{R}^d} G_\beta(t-s,y-x) H(s,x,\nabla f^*(t,x)) dx ds \\ &= \int_0^t \int_{\mathbb{R}^d} \nabla_y G_\beta(t-s,y-x) \nabla_x H(s,x,\nabla f^*(s,x)) dx ds. \end{aligned} \tag{4.38}$$

From the representations of $G_\beta(t,y)$ and $\nabla G_\beta(t,y)$ used in theorems (1), (2) it is clear that $\nabla G_\beta(t,y)$ exists and is continuous in t and in y. From theorem 10 we know ∇f^* exists and is Lipschitz continuous. Since we assumed H to be Lipschitz, it follows from Rademacher's theorem that $\nabla_x H(s,x,\nabla f^*(s,x))$ is almost everywhere defined and bounded. Hence (4.38) represents a continuous function in y and in t. Since $f_0 \in C^2_\infty(\mathbb{R}^d)$ and due to theorem (4)

$$\nabla^2 \int_{\mathbb{R}^d} S_{\beta,1}(t,y-x) f_0(x)dx = \int_{\mathbb{R}^d} S_{\beta,1}(t,x)\nabla^2 f_0(y-x)dx < \infty. \tag{4.39}$$

Thus, $\nabla^2 f^*(t,y)$ exists and so $f^*(t,y) \in C^2_\infty(\mathbb{R}^d)$. This completes the necessary requirements for the solution of the mild form (4.29) to be the solution of (4.30) of class $C^2_\infty(\mathbb{R}^d)$, i.e. a solution in the classical sense. □

5. Appendix

Let us recall the asymptotic properties of stable densities defined in (2.1)

$$g(y,\alpha,\sigma) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \exp\{-\sigma|p|^\alpha\} e^{-ipy} dp, \tag{5.1}$$

see [2] for details. For $|y|/\sigma^{1/\alpha} \to 0$ the following asymptotic expansion for g holds

$$g(y,\alpha,\sigma) \sim \frac{|S^{d-2}|}{(2\pi\sigma^{1/\alpha})^d} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} a_k \left(\frac{|y|}{\sigma^{1/\alpha}}\right)^{2k}, \tag{5.2}$$

where

$$a_k = \alpha^{-1}\Gamma\left(\frac{2k+d}{\alpha}\right) B\left(k+\frac{1}{2}, \frac{d-1}{2}\right), \tag{5.3}$$

where

$$B(q,p) = \int_0^1 x^{p-1}(1-x)^q dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{5.4}$$

is the Beta function, and

$$|S^{d-2}| = 2\frac{\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})} \tag{5.5}$$

and $|S^0| = 2$, see [2] for the proof.

For $|y|/\sigma^{1/\alpha} \to \infty$ the following asymptotic expansion holds

$$g(y;\alpha,\sigma) \sim (2\pi)^{-(d+1)/2} \frac{2}{|y|^d} \sum_{k=1}^{\infty} \frac{a_k}{k!} (\sigma|y|^{-\alpha})^k \tag{5.6}$$

where

$$a_k = (-1)^{k+1} \sin\left(\frac{k\pi\alpha}{2}\right) \int_0^\infty \xi^{\alpha k + (d-1)/2} W_{0,\frac{d}{2}-1}(2\xi) d\xi \tag{5.7}$$

and $W_{0,n}(z)$ is the Whittaker function

$$W_{0,n}(z) = \frac{e^{-z/2}}{\Gamma(n+1/2)} \int_0^\infty [t(1+t/z)]^{n-1/2} e^{-t} dt, \tag{5.8}$$

see [2] for the proof.

In the case $d=1$ the stable density function $w(x,\beta,1)$ defined in (2.2) is infinitely smooth for $x=0$ and $w(x,\beta,1)=0$ for $x<0$. Hence w grows at 0 slower than any power. This gives rise to the inequalities such as $w(x,\beta,1) < C_q x^{q-1}$ for any $q>1$, for $x<1$. The property $w(x) \sim x^{-1-\beta}$ for $x \gg 1$, may be found for example in [2]. This may be deduced from the asymptotic expansions in equations 7.7 and 7.9 in [2] with $\gamma = 1$.

The following result is part of the proposition 7.3.2 from [2]:

PROPOSITION 1. *Let*

$$\phi(y,\alpha,\beta,\sigma) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |p|^\beta exp\{-i(p,y) - \sigma|p|^{-\alpha}\} dp, \tag{5.9}$$

so that

$$\frac{\partial \phi}{\partial \beta}(y,\alpha,\beta,\sigma) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |p|^\beta \log|p| exp\{-i(p,y) - \sigma|p|^{-\alpha}\} dp. \tag{5.10}$$

Then if $\frac{|y|}{\sigma^{1/\alpha}} \leqslant K$

$$|\phi(y,\alpha,\beta,\sigma)| \leqslant c\sigma^{-\beta/\alpha} g(y,\alpha,\sigma) \tag{5.11}$$

and if $\frac{|y|}{\sigma^{1/\alpha}} > K$

$$|\phi(y,\alpha,\beta,\sigma)| \leqslant c\sigma^{-1} |y|^{\alpha-\beta} g(y,\alpha,\sigma), \tag{5.12}$$

where g *is as in (5.1) and (2.1).*

Acknowledgements. The authors would like to thank BCAM, G. Pagnini and E. Scalas for organizing the BCAM FCPNLO 2013 workshop where this work was discussed, and the referee for the useful comments. This work is supported by the IPI RAN grants RFBR 11-01-12026 and 12-07-00115, and by the grant 4402 of the Ministry of Education and Science of Russia, and by the EPSRC grant EP/HO23364/1 through MASDOC, University of Warwick, UK.

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(Received February 25, 2014)

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