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EXISTENCE OF MILD SOLUTIONS FOR IMPULSIVE FRACTIONAL FUNCTIONAL INTEGRO–DIFFERENTIAL EQUATIONS

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Abstract. In this investigation, our aim is to develop the definition of mild solutions for impulsive fractional differential equations of order $\alpha \in (1,2)$ and obtain some sufficient conditions for existence of mild solutions using the analytic operator functions and fixed point theorems. We also verify the existence result with an example involving partial derivative.

1. Introduction

The main objective of this work is to develop the definition of mild solutions and study the existence of mild solutions for following problem

$$
\begin{aligned}
{}^{C}D_t^{\alpha}u(t) &= Au(t) + f(t, u_{\rho(t,u_t)}) \\
&\qquad + \int_0^t p(t-s)h(s, u_{\rho(s,u_s)})ds,\ t \in (s_i, t_{i+1}] \subset J,\ i = 0,1,\dots,N, \qquad (1.1)\\
u(t) &= \phi(t),\ u'(t) = \varphi(t),\ t \in [-d,0], \qquad (1.2)\\
u(t) &= g_i(t,u(t)),\ u'(t) = q_i(t,u(t)),\ t \in (t_i, s_i],\ i = 1,2,\dots,N, \qquad (1.3)
\end{aligned}
$$

where ${}^{C}D_t^{\alpha}$ denotes the Caputo's fractional derivative of order $\alpha \in (1,2)$ for the state $u(t)$ belong to a Banach space $(X, \|\cdot\|_X)$ and $A : D(A) \subset X \to X$ is a sectorial operator defined on X. Functions $f;h : J \times PC_0 \to X$; $p : J \to X$; $\rho : J \times PC_0 \to [-d,T]$; $T < \infty$ are appropriate and $J = (0,T]$ is an operational interval with $0 = t_0 = s_0 < t_1 \leqslant s_1 \leqslant t_2 < \dots < t_N \leqslant s_N \leqslant t_{N+1} = T$ are pre-fixed numbers named as impulsive points. The history function $u_t : [-d,0] \to X$ is element of $PC_0 = C([-d,0],X)$ and defined as $u_t(\theta) = u(t+\theta)$, $\theta \in [-d,0]$. The maps $\phi(t), \varphi(t)$ belong to PC_0 and $u'(t)$ is the ordinary derivative of $u(t)$ with respect to t. The nonlinear functions g_i, q_i belong to $C((t_i,s_i] \times X; X)$ for all $i = 1,2,\dots,N$, respectively.

Dynamical systems, which have evolutionary processes characterized by abrupt changes in the state at certain moments known as an impulse system. These changes appear due to disturbances, changing operation conditions and component failures, of the state. For example, biological and mechanical model subject to shock. For further details of impulsive effects, see the papers [8, 9, 10, 11, 12].

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However, it seems that the dynamical systems with evolutionary processes cannot be characterized by instantaneous impulses in pharmacotherapy [20]. Such a situation should be characterized by a new type of impulsive action, which starts at an arbitrary fixed point and stays active on a finite time interval which is known as a non instantaneous impulse. For example, consider the hemodynamic equilibrium of a person; the injection of drugs in the bloodstream and the consequent absorption of the body are gradual and continuous processes. For more information about such an impulse see [3, 20, 22].

The subject of fractional calculus is an interesting and important field of research. Fractional differential equations play an important role in describing memory and hereditary properties of many materials and processes. Fractional calculus has extensive applications in the engineering and scientific disciplines such as physics, chemistry, biophysics, control theory, aerodynamics, nonlinear oscillation of earthquake, polymer rheology, regular variation in thermodynamics, economics, etc. Further details can be found in the papers [4, 21, 23, 24, 25, 26, 27] and the references therein.

Functional differential equations with non integer order are those in which the time evolution of the state variable can depend on the past in some arbitrary way. Specific type of functional differential equations, namely delay differential equations with state dependent delay often appear in most important areas such as classical electrodynamics, populations models, model of price fluctuations, model of blood cell productions etc. For further details and some recent important contribution one can see the papers [1, 2, 5, 6, 7, 13, 14, 15, 16, 17, 18, 19] and the references therein.

In this survey, mainly we describe that the several author's study the existence results of solutions for impulsive fractional differential equations. Shu et al. [27] gave the definition of mild solution for fractional differential equations of order $\alpha \in (1,2)$ of the following form

$$\begin{aligned} D_t^{\alpha} u(t) &= Au(t) + f(t,u(t)) + \int_0^t q(t-s)g(s,u(s))ds, \ t \in [0,T], \\ u(0)+m(u) &= u_0 \in X, \ u'(0)+n(u) = u_1 \in X, \end{aligned}$$

and established the existence of mild solutions using the Krasnoselskii theorem and analytic operator theory.

Hernandez et al. [20] used the first time not instantaneous impulsive condition for the following abstract problem

$$\begin{aligned} u'(t) &= Au(t) + f(t,u(t)), \ t \in (s_i,t_{i+1}], \ i = 0,1,\dots,N, \\ u(t) &= g_i(t,u(t)), \ t \in (t_i,s_i], \ i = 1,2,\dots,N, \ u(0) = x_0, \end{aligned}$$

where $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup of bounded operators $(T(t))_{t \geqslant 0}$ defined on a Banach space X. In [20], authors introduced the concepts of mild and classical solutions and established the existence results for these types of problems by using fixed point theorems.

Recently, Kumar et al. [22] studied the following fractional order problem with not instantaneous impulse

$$ {}^{C}D_t^{\beta} u(t) + Au(t) = f(t,u(t),g(u(t))), \ t \in (s_i,t_{i+1}], \ i = 0,1,\cdots,N, \tag{1.4}$$

$$u(t) = g_i(t,u(t)),\ t \in (t_i,s_i],\ i = 1,2,\ldots,N,\ u(0) = u_0 \in H, \tag{1.5}$$

and by using the Banach fixed point theorem and condensing map they established the existence and uniqueness results.

Motivated by the above literature [20, 22, 27], we develop the definition of mild solution for the impulsive fractional differential equation of order $\alpha \in (1,2)$ and obtain the existence results for the problem (1.1)–(1.3).

Rest of this paper is organized in four sections. Second section provides some basic definitions, notations and propositions. We obtain the existence of the mild solutions of problem (1.1)–(1.3) in the third section and fourth section is concerned with an example.

2. Preliminaries

Let $(X, \|\cdot\|_X)$ denote a complex Banach space of functions with the norm

$$\|u\|_X = \sup_{t\in J}\{|u(t)| : u \in X\}.$$

Let $L(X)$ the space of bounded linear operators from X into X endowed with the norm of operators denoted by $\|\cdot\|_{L(X)}$.

As usual, $PC_0 = C([-d,0],X)$ (with $[-d,0] \subset \mathbb{R}$) is the space formed by all the continuous functions defined from $[-d,0]$ into X, endowed with the norm

$$\|u(t)\|_{PC_0} = \sup_{t\in[-d,0]} \|u(t)\|_X.$$

In case of impulse conditions, let

$$PC_t = PC([-d,t];X),\ 0 \leqslant t \leqslant T,$$

be a Banach space of all functions $u : [-d,t] \to X$, which are continuous every where except for a finite number of points $t_i \in (0,T)$, $i = 1,2,\ldots,N$, at which $u(t_i^+)$ and $u(t_i^-) = u(t_i)$ exist and endowed with the norm

$$\|u\|_{PC_t} = \sup_{t\in[-d,T]} \{\|u(t)\|_X, u \in PC_t\}.$$

For a function $u \in PC_t$ and $i \in \{0,1,\ldots,N\}$, we introduce the function $\bar{u}_i \in C([t_i,t_{i+1}];X)$ as

$$\bar{u}_i(t) = \begin{cases} u(t), \text{ for } t \in (t_i,t_{i+1}], \\ u(t_i^+), \text{ for } t = t_i. \end{cases}$$

Further, let

$$PC_t^1 = PC([-d,t];X),\ 0 \leqslant t \leqslant T,$$

be a Banach space of all functions $u:[-d,T]\to X$, which are continuously differentiable every where except for a finite number of points $t_i\in(0,T)$, $i=1,2,\dots,N$, at which $u'(t_i^+)$ and $u'(t_i^-)=u'(t_i)$ exist and endowed with the norm

$$\|u\|_{PC_t^1}=\sup_{t\in[-d,T]}\{\|u(t)\|_X,\|u'(t)\|_X,u\in PC_t^1\}.$$

For a function $u\in PC_t^1$ and $i\in\{0,1,\dots,N\}$, we introduce the function $\bar{u}_i\in C^1([t_i,t_{i+1}];X)$ as

$$\bar{u}_i(t)=\begin{cases}u'(t), \text{ for } t\in(t_i,t_{i+1}],\\ u'(t_i^+), \text{ for } t=t_i.\end{cases}$$

DEFINITION 2.1. Caputo's derivative of order $\alpha>0$ with lower limit a, for a function $f:[a,\infty)\to\mathbb{R}$ such that $f\in C^n([a,\infty),X)$ is defined as

$${}_aD_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t(t-s)^{n-\alpha-1}f^{(n)}(s)ds={}_aJ_t^{n-\alpha}f^{(n)}(t),$$

where $n-1<\alpha<n$, $a\geqslant 0$, $n\in N$.

DEFINITION 2.2. The Riemann-Liouville fractional integral operator of order $\alpha>0$ with lower limit a, for a function $f\in L^1_{loc}([a,\infty),X)$ is defined by

$${}_aJ_t^0f(t)=f(t),\ {}_aJ_t^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}f(s)ds,\quad t>0,$$

where $a\geqslant 0$, $n\in N$ and $\Gamma(\cdot)$ is the Euler gamma function.

DEFINITION 2.3. A two parameter function of the Mittag-Lefller type is defined by the series expansion

$$E_{\alpha,\beta}(y)=\sum_{k=0}^{\infty}\frac{y^k}{\Gamma(\alpha k+\beta)}=\frac{1}{2\pi\iota}\int_c\frac{\mu^{\alpha-\beta}e^\mu}{\mu^\alpha-y}d\mu,\ \alpha,\ \beta>0,\ y\in\mathbb{C},$$

where c is a contour which starts and ends at $-\infty$ and encircles the disc $|\mu|\leqslant|y|^{\frac{1}{\alpha}}$ counter clockwise. The Laplace integral of this function given by

$$\int_0^\infty e^{-\lambda t}t^{\beta-1}E_{\alpha,\beta}(\omega t^\alpha)dt=\frac{\lambda^{\alpha-\beta}}{\lambda^\alpha-\omega},\ Re\lambda>\omega^{\frac{1}{\alpha}},\ \omega>0.$$

DEFINITION 2.4. Let $A:D(A)\subset X\to X$ be a closed linear operator and $\alpha,\beta>0$. We say that A is the generator of (α,β) operator function if there exist $\omega\geqslant 0$ and a strongly continuous function $W_{\alpha,\beta}:\mathbb{R}^+\to L(X)$ such that $\{\lambda^\alpha: Re\lambda>\omega\}\subset\rho(A)$ and

$$\lambda^{\alpha-\beta}(\lambda^\alpha I-A)^{-1}u=\int_0^\infty e^{-\lambda t}W_{\alpha,\beta}(t)udt,\ Re\lambda>\omega,\ u\in X.$$

Here $W_{\alpha,\beta}(t)$ is called the operator function generated by A.

PROPOSITION. *Let $W_{\alpha,\beta}(t)$ be a operator function on X with generator A. Then, we have*

(a) $W_{\alpha,\beta}(t)D(A) \subset D(A)$ and $AW_{\alpha,\beta}(t)u = W_{\alpha,\beta}(t)Au$ for all $u \in D(A), t \geqslant 0$.

(b) Let $u \in D(A), t \geqslant 0$. Then $W_{\alpha,\beta}(t)u = u\frac{t^{\beta-1}}{\Gamma(\beta)} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} AW_{\alpha,\beta}(s)uds$.

(c) Let $u \in X, t \geqslant 0$. Then $\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} AW_{\alpha,\beta}(s)uds \in D(A)$ and

$$W_{\alpha,\beta}(t)u = u\frac{t^{\beta-1}}{\Gamma(\beta)} + A\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} W_{\alpha,\beta}(s)uds.$$

REMARK 2.5. The operator function $W_{\alpha,\beta}(t)$ is general case of α-resolvent family and solution operator. In case $\beta = 1$, operator function correspond to solution operator $S_\alpha(t)$ (definition 2.1 in [2]), whereas in the case $\beta = \alpha$, operator function correspond to α-resolvent family $T_\alpha(t)$ (definition (2.3) in [4]) and operator function correspond to $K_\alpha(t)$ in case $\beta = 2$ (see [27]).

DEFINITION 2.6. ([27]) Let $A : D(A) \subset X \to X$ be a closed linear operator. A is said to be sectorial of the type (M, θ, α, μ) if there exist $\mu \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$, $M > 0$, such that such that the α-resolvent of A exists outside the sector and following two conditions are satisfied:

(1) $\mu + S_\theta = \{\mu + \lambda^\alpha : \lambda \in \mathbb{C},\ |Arg(-\lambda^\alpha)| < \theta\}$,

(2) $\|(\lambda^\alpha I - A)^{-1}\|_{L(X)} \leqslant \frac{M}{|\lambda^\alpha - \mu|}$, $\lambda \notin \mu + S_\theta$,

where X is the complex Banach space with norm denoted by $\|\cdot\|_X$.

LEMMA 2.7. *Let f satisfies the uniform Holder condition with exponent $\beta \in (0,1]$ and A is a sectorial operator. Then Cauchy problem of order $\alpha \in (1,2)$*

$$ {}_a^C D_t^\alpha u(t) = Au(t) + f(t),\ t \in J = (a,T],\ a \geqslant 0, \tag{2.1}$$

$$ u(a) = u_0,\ u'(a) = u_1, \tag{2.2}$$

has a solution $u(t) \in C([a,T],\mathbb{R})$ if it satisfies the following integral equation

$$u(t) = S_\alpha(t-a)u_0 + K_\alpha(t-a)u_1 + \int_a^t T_\alpha(t-s)f(s)ds,$$

where $S_\alpha(t), K_\alpha(t), T_\alpha(t)$ generated by A and defined as

$$S_\alpha(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}d\lambda;\ K_\alpha(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-2}(\lambda^\alpha I - A)^{-1}d\lambda,$$

$$T_\alpha(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda^\alpha I - A)^{-1}d\lambda,$$

and Γ is a suitable path such that $\lambda^\alpha \notin \mu + S_\theta$ for $\lambda \in \Gamma$.

Proof. Let $t = w + a$, then the problem (2.1)–(2.2) translated into the form

$$ {}^{C}D_{w}^{\alpha}\tilde{u}(w) = A\tilde{u}(w) + \tilde{f}(w), \tag{2.3}$$

$$ \tilde{u}(0) = u_0,\ \tilde{u}'(0) = u_1. \tag{2.4}$$

Taking the Riemann-Liouville fractional integral operator on Eq. (2.3), we get

$$\tilde{u}(w) = -c_0 - c_1 w + \int_0^w \frac{(w-\xi)^{\alpha-1}}{\Gamma(\alpha)} A\tilde{u}(\xi)d\xi + \int_0^w \frac{(w-\xi)^{\alpha-1}}{\Gamma(\alpha)} \tilde{f}(\xi)d\xi,$$

using the initial condition given in Eq. (2.4), we get

$$\tilde{u}(w) = u_0 + u_1 w + \int_0^w \frac{(w-\xi)^{\alpha-1}}{\Gamma(\alpha)} A\tilde{u}(\xi)d\xi + \int_0^w \frac{(w-\xi)^{\alpha-1}}{\Gamma(\alpha)} \tilde{f}(\xi)d\xi. \tag{2.5}$$

Applying the Laplace transformation on Eq. (2.5), we have

$$L\{\tilde{u}(w)\} = \frac{u_0}{\lambda} + \frac{u_1}{\lambda^2} + \frac{AL\{\tilde{u}(w)\}}{\lambda^{\alpha}} + \frac{L\{\tilde{f}(w)\}}{\lambda^{\alpha}}, \tag{2.6}$$

by algebraic estimate, Eq. (2.6) become

$$L\{\tilde{u}(w)\} = \frac{\lambda^{\alpha-1}u_0}{\lambda^{\alpha}-A} + \frac{\lambda^{\alpha-2}u_1}{\lambda^{\alpha}-A} + \frac{L\{\tilde{f}(w)\}}{\lambda^{\alpha}-A}. \tag{2.7}$$

Now, taking the laplace inverse, on Eq. (2.7), we obtain

$$\tilde{u}(w) = S_{\alpha}(w)u_0 + K_{\alpha}(w)u_1 + \int_0^w T_{\alpha}(w-s)\tilde{f}(s)ds \tag{2.8}$$

By replacing $w = t - a$ in Eq. (2.8) then we get

$$u(t) = S_{\alpha}(t-a)u_0 + K_{\alpha}(t-a)u_1 + \int_a^t T_{\alpha}(t-s)f(s)ds.$$

Which is required result. □

LEMMA 2.8. *Let f satisfies the uniform Holder condition with exponent $\beta \in (0,1]$ and A is a sectorial operator. Then the following integral equation*

$$u(t) = \begin{cases} S_{\alpha}(t)(\phi(0)) + K_{\alpha}(t)(\varphi(0)) + \int_0^t T_{\alpha}(t-s)f(s,u_{\rho(s,u_s)})ds \\ + \int_0^t T_{\alpha}(t-s)\int_0^s q(s-\xi)h(\xi,u_{\rho(\xi,u_{\xi})})d\xi ds, & t \in (0,t_1], \\ S_{\alpha}(t-s_i)g_i(s_i,u(s_i)) + K_{\alpha}(t-s_i)q_i(s_i,u(s_i)) \\ + \int_{s_i}^t T_{\alpha}(t-s)f(s,u_{\rho(s,u_s)})ds \\ + \int_{s_i}^t T_{\alpha}(t-s)\int_0^s q(s-\xi)h(\xi,u_{\rho(\xi,u_{\xi})})d\xi ds, & t \in (s_i,t_{i+1}], \end{cases}$$

is the solution of the problem (1.1)–(1.3).

Proof. Let $t \in (0,t_1]$, we have following problem

$$ {}^C D_t^\alpha u(t) = Au(t) + f(t,u_{\rho(t,u_t)}) + \int_0^t p(t-s)h(s,u_{\rho(s,u_s)})ds, \tag{2.9}$$

$$ u(t) = \phi(t),\ u'(t) = \varphi(t),\ t \in [-d,0]. \tag{2.10}$$

By using the lemma 2.7 and the initial conditions $u(0) = \phi(0)$, $u'(0) = \varphi(0)$, we have the solution of system (2.9)–(2.10) in following integral equation

$$ u(t) = S_\alpha(t)(\phi(0)) + K_\alpha(t)(\varphi(0)) + \int_0^t T_\alpha(t-s)f(s,u_{\rho(s,u_s)})ds + \int_0^t T_\alpha(t-s)\int_0^s q(s-\xi)h(\xi,u_{\rho(\xi,u_\xi)})d\xi ds.$$

Let $t \in (s_i,t_{i+1}]$, we have following problem

$$ {}^C D_t^\alpha u(t) = Au(t) + f(t,u_{\rho(t,u_t)}) + \int_0^t p(t-s)h(s,u_{\rho(s,u_s)})ds, \tag{2.11}$$

$$ u(t) = g_i(t,u(t)),\ u'(t) = q_i(t,u(t)),\ t \in (t_i,s_i], \tag{2.12}$$

Again by using the lemma 2.7 and the impulsive conditions $u(s_i) = g_i(s_i,u(s_i))$, $u'(s_i) = q_i(s_i,u(s_i))$, we have the solution of system (2.11)–(2.12) in following integral equation

$$ u(t) = S_\alpha(t-s_i)g_i(s_i,u(s_i)) + K_\alpha(t-s_i)q_i(s_i,u(s_i)) + \int_{s_i}^t T_\alpha(t-s)f(s,u_{\rho(s,u_s)})ds + \int_{s_i}^t T_\alpha(t-s)\int_0^s q(s-\xi)h(\xi,u_{\rho(\xi,u_\xi)})d\xi ds,$$

This completes the proof. □

Now, we state the definition of mild solution of the problem (1.1)–(1.3).

DEFINITION 2.9. A function $u : [-d,T] \to X$ s.t. $u \in PC_t^1$ is called a mild solution of the problem (1.1)–(1.3) if $u(0) = \phi(0)$, $u'(0) = \varphi(0)$, $u(t) = g_j(t,u(t))$, $u'(t) = q_j(t,u(t))$ for $t \in (t_j,s_j]$ and each $j = 1,2,\ldots,N$, satisfies the following integral equation

$$ u(t) = \begin{cases} S_\alpha(t)(\phi(0)) + K_\alpha(t)(\varphi(0)) + \int_0^t T_\alpha(t-s)f(s,u_{\rho(s,u_s)})ds \\ + \int_0^t T_\alpha(t-s)\int_0^s q(s-\xi)h(\xi,u_{\rho(\xi,u_\xi)})d\xi ds, & t \in (0,t_1], \\ S_\alpha(t-s_i)g_i(s_i,u(s_i)) + K_\alpha(t-s_i)q_i(s_i,u(s_i)) \\ + \int_{s_i}^t T_\alpha(t-s)f(s,u_{\rho(s,u_s)})ds \\ + \int_{s_i}^t T_\alpha(t-s)\int_0^s q(s-\xi)h(\xi,u_{\rho(\xi,u_\xi)})d\xi ds, & t \in (s_i,t_{i+1}], \end{cases}$$

for every $i = 1,2,\ldots,N$.

THEOREM 2.10. (Schaefer's fixed point theorem[12]) *Let X be a Banach spaces and $P : X \to X$ be a completely continuous operator. If the set*

$$ E(F) = \{u \in X : u = \lambda Pu \text{ for some } \lambda \in [0,1]\}$$

is bounded, then P has at least a fixed point.

3. Existence results

We impose the following restrictions on problem (1.1)–(1.3) to prove the existence results. If $A : D(A) \subset X \to X$ is sectorial operator then strongly continuous functions $S_\alpha(t); K_\alpha(t)$ and $T_\alpha(t)$ generated by A are bounded as $\|S_\alpha(t)\| \leqslant M$; $\|K_\alpha(t)\| \leqslant M$; $\|T_\alpha(t)\| \leqslant M$. To obtain first result via Banach fixed point theorem, we need the following assumptions.

(H_1) There exist positive constants L_f, L_h such that

$$\|f(t,\varphi) - f(t,\xi)\|_X \leqslant L_f \|\varphi - \xi\|_{PC_0};\ \|h(t,\varphi) - h(t,\xi)\|_X \leqslant L_h \|\varphi - \xi\|_{PC_0}$$

for all $t \in J$, $\varphi, \xi \in PC_0$.

(H_2) There exist positive constants L_{g_i}, L_{q_i} such that

$$\|g_i(t,y) - g_i(t,z)\|_X \leqslant L_{g_i}\|y - z\|_X;\ \|q_i(t,y) - q_i(t,z)\|_X \leqslant L_{q_i}\|y - z\|_X,$$

for all $t \in J; y, z \in X$ and $i = 1, 2, \ldots N$.

THEOREM 3.1. *Let the assumptions $(H_1) - (H_2)$ hold and there exists a constant*

$$\Delta = C + MTL_f + MTL_h q^* < 1.$$

Then there exists a unique mild solution $u(t)$ on J of the problem (1.1)–(1.3).

Proof. We transform problem (1.1)–(1.3) into a fixed point problem. Consider the space $PC_T = \{u \in PC_t^1; u(0) = \phi(0), u'(0) = \varphi(0)\}$. Define an operator $P : PC_T \to PC_T$ by $Pu(t) = g_i(t, u(t)) : Pu'(t) = q_i(t, u(t)$ for $t \in (t_i, s_i]$, where

$$Pu(t) = \begin{cases} S_\alpha(t)(\phi(0)) + K_\alpha(t)(\varphi(0)) + \int_0^t T_\alpha(t-s) f(s, u_{\rho(s,u_s)}) ds \\ + \int_0^t T_\alpha(t-s) \int_0^s q(s-\xi) h(\xi, u_{\rho(\xi,u_\xi)}) d\xi ds, \quad \text{for all } t \in (0, t_1], \\ S_\alpha(t-s_i) g_i(s_i, u(s_i)) + K_\alpha(t-s_i) q_i(s_i, u(s_i)) + \int_{s_i}^t T_\alpha(t-s) f(s, u_{\rho(s,u_s)}) ds \\ + \int_{s_i}^t T_\alpha(t-s) \int_0^s q(s-\xi) h(\xi, u_{\rho(\xi,u_\xi)}) d\xi ds, \quad \text{for all } t \in (s_i, t_{i+1}]. \end{cases}$$

Let $u(t), u(t)^* \in PC_T$ for $t \in (0, t_1]$. Then we have

$$\begin{aligned} &\|Pu(t) - Pu(t)^*\|_X \\ \leqslant &\int_0^t \|T_\alpha(t-s)\|_{L(X)} \|f(s, u_{\rho(s,u_s)}) - f(s, u^*_{\rho(s,u^*_s)})\|_X ds \\ &+ \int_0^t \|T_\alpha(t-s)\| \int_0^s q(s-\xi) \|h(\xi, u_{\rho(\xi,u_\xi)}) - h(\xi, u^*_{\rho(\xi,u^*_\xi)})\|_X d\xi ds \\ \leqslant &(MTL_f + MTL_h q^*) \|u(t) - u(t)^*\|_{PC_T}. \end{aligned}$$

For $t \in (s_i, t_{i+1}]$, we have

$$\begin{aligned} &\|Pu(t) - Pu(t)^*\|_X \\ \leqslant &\|S_\alpha(t-s_i)\|_{L(X)} \|g_i(s_i, u(s_i)) - g_i(s_i, u^*(s_i))\|_X \\ &+ \|K_\alpha(t-s_i)\|_{L(X} \|q_i(s_i, u(s_i)) - q_i(s_i, u^*(s_i))\|_X \end{aligned}$$

$$
\begin{aligned}
&+\int_{s_i}^{t}\|T_\alpha(t-s)\|_{L(X}\|f(s,u_{\rho(s,u_s)})-f(s,u^*_{\rho(s,u^*_s)})\|_X ds\\
&+\int_{s_i}^{t}\|T_\alpha(t-s)\|_{L(X)}\int_0^s q(s-\xi)\|h(\xi,u_{\rho(\xi,u_\xi)})-h(\xi,u^*_{\rho(\xi,u^*_\xi)})\|_X d\xi ds\\
\leqslant\ &(MLg_i+MLq_i+MTL_f+MTL_hq^*)\|u(t)-u(t)^*\|_{PC_T}.
\end{aligned}
$$

For $t\in(t_i,s_i]$, we have

$$\|Pu(t)-Pu(t)^*\|_X\leqslant Lg_i\|u(t)-u(t)^*\|_{PC_T};\|Pu'(t)-Pu'(t)^*\|_X=Lq_i\|u(t)-u(t)^*\|_{PC_T}.$$

Let $C=\max\{MLg_i+MLq_i,Lq_i,Lg_i,\ \forall\, i=1,2,\ldots,N\}$. Then, for each $t\in[0,T]$, we have

$$
\begin{aligned}
\|Pu(t)-Pu(t)^*\|_X&\leqslant(C+MTL_f+MTL_hq^*)\|u(t)-u(t)^*\|_{PC_T}\\
&\leqslant\Delta\|u(t)-u(t)^*\|_{PC_T}.
\end{aligned}
$$

Since $\Delta<1$, therefore the operator P is a contraction and hence there exists a unique fixed point which is the mild solution of problem (1.1)–(1.3). This completes the proof. □

To obtain an existence result via Schaefer's fixed point theorem, we need the following assumptions.

(H_3) f,h are continuous functions and there exist functions $m_f(t),m_h(t)\in L^1(J,R^+)$ such that

$$\|f(t,\varphi)\|_X\leqslant m_f(t)\|\varphi\|_{PC_0};\ \|h(t,\varphi)\|_X\leqslant m_h(t)\|\varphi\|_{PC_0}$$

for all $t\in J$, $\varphi\in PC_0$.

(H_4) g,q are continuous functions and there exist positive constants M_g,M_q such that

$$\|g_i(t,y)\|_X\leqslant M_g;\ \|q_i(t,y)\|_X\leqslant M_q,$$

for all $t\in J$ and $y\in X$.

THEOREM 3.2. *Let the assumptions $(H_3)-(H_4)$ hold and*

$$M\|m_f\|_{L^1(J,R^+)}+Mq^*\|m_h(s)\|_{L^1(J,R^+)}<1.$$

Then problem (1.1)–(1.3) has at least one mild solution $u(t)$ on J.

Proof. Consider the operator P defined in theorem 3.1. With out loss of generality we show our result for $t\in(s_i,t_{i+1}]$. We show P has atleast one fixed point. Consider a sequence $u^n\to u$ in PC_T, so that

$$
\begin{aligned}
&\|Pu^n(t)-Pu(t)\|_X\\
\leqslant\ &\|S_\alpha(t-s_i)\|_{L(X}\|g_i(s_i,u^n(s_i))-g_i(s_i,u(s_i))\|_X\\
&+\|K_\alpha(t-s_i)\|_{L(X}\|q_i(s_i,u^n(s_i))-q_i(s_i,u(s_i))\|_X
\end{aligned}
$$

$$+\int_{s_i}^{t} \|T_\alpha(t-s)\|_{L(X}\|f(s,u^n_{\rho(s,u^n_s)})-f(s,u_{\rho(s,u_s)})\|_X ds$$
$$+\int_{s_i}^{t} \|T_\alpha(t-s)\|_{L(X}\int_0^s q(s-\xi)\|h(\xi,u_{\rho(\xi,u_\xi)})-h(\xi,u^n_{\rho(\xi,u^n_\xi)})\|_X d\xi ds.$$

Since function f,h,g,q are continuous, so $\|Pu^n(t)-Pu(t)\|_X \to 0$ as $n\to\infty$. This shows that P is continuous. Consider space $\mathscr{B}_r=\{u\in PC_T:\|u\|\leqslant r\}$, it is clear that $\mathscr{B}_r$ is a bounded, closed and convex subset in PC_T. If $u\in\mathscr{B}_r$, then we have

$$\begin{aligned}\|Pu(t)\|_X &\leqslant \|S_\alpha(t-s_i)\|_{L(X)}\|g_i(s_i,u(s_i))\|_X+\|K_\alpha(t-s_i)\|_{L(X)}\|q_i(s_i,u(s_i))\|_X\\
&\quad+\int_{s_i}^{t}\|T_\alpha(t-s)\|_{L(X}\|f(s,u_{\rho(s,u_s)})\|_X ds\\
&\quad+\int_{s_i}^{t}\|T_\alpha(t-s)\|_{L(X)}\int_0^s q(s-\xi)\|h(\xi,u_{\rho(\xi,u_\xi)})\|_X d\xi ds\\
&\leqslant MM_g+MM_q+Mr\|m_f\|_{L^1(J,R^+)}+Mrq^*\|m_h(s)\|_{L^1(J,R^+)}\leqslant C^*.\end{aligned}$$

This implies that P maps bounded set into bounded set in $\mathscr{B}_r$. Next, we show P is a family of equi-continuous functions in $\mathscr{B}_r$. Let $l_1,l_2\in(s_i,t_{i+1}]$ be such that $l_1<l_2$. Then

$$\begin{aligned}\|Pu(l_2)-Pu(l_1)\|_X &\leqslant \|S_\alpha(l_2-s_i)-S_\alpha(l_1-s_i)\|_{L(X)}\|g_i(s_i,u(s_i))\|_X\\
&\quad+\|K_\alpha(l_2-s_i)-K_\alpha(l_1-s_i)\|_{L(X)}\|q_i(s_i,u(s_i))\|_X\\
&\quad+\int_{s_i}^{l_1}\|T_\alpha(l_2-s)-T_\alpha(l_1-s)\|_{L(X)}\|f(s,u_{\rho(s,u_s)})\|_X ds\\
&\quad+\int_{l_1}^{l_2}\|T_\alpha(l_2-s)\|_{L(X)}\|\int_0^s q(s-\xi)\|h(\xi,u_{\rho(\xi,u_\xi)})d\xi\|_X ds\\
&\leqslant M_g\|S_\alpha(l_2-s_i)-S_\alpha(l_1-s_i)\|_{L(X)}\\
&\quad+M_q\|K_\alpha(l_2-s_i)-K_\alpha(l_1-s_i)\|_{L(X)}\\
&\quad+r\|m_f\|_{L^1(J,R^+)}\int_{s_i}^{l_1}\|T_\alpha(l_2-s)-T_\alpha(l_1-s)\|_{L(X)}ds\\
&\quad+rq^*\|m_h\|_{L^1(J,R^+)}(l_2-l_1)M.\end{aligned}$$

Since $S_\alpha(t),T_\alpha(t),K_\alpha(t)$ are strongly continuous, so $\lim_{l_2\to l_1}\|S_\alpha(l_2-s)-S_\alpha(l_1-s)\|_{L(X)}$; $\lim_{l_2\to l_1}\|K_\alpha(l_2-s)-K_\alpha(l_1-s)\|_{L(X)}$; $\lim_{l_2\to l_1}\|T_\alpha(l_2-s)-T_\alpha(l_1-s)\|_{L(X)}$ equal to 0, which implies that $\|P(u)(l_2)-P(u)(l_1)\|_X\to 0$ as $l_2\to l_1$. This proves that P is a family of equi-continuous functions. Hence by Arzela-Ascoli's theorem P is a compact operator. So, we conclude that the operator P is a completely continuous. Finally, we prove that P has a priori bounds in PC_T. Consider the set $\mathbb{E}=\{u\in PC_T$ such that $u\in\nu Pu$ for some $0<\nu<1\}$, we show that $\mathbb{E}$ is bounded. Let $u\in\mathbb{E}$ then $u=\nu Pu$ for some $0<\nu<1$. Then for every $t\in(s_i,t_{i+1}]$, we have

$$\begin{aligned}\|u(t)\|_X &\leqslant \|S_\alpha(t-s_i)\|_{L(X)}\|g_i(s_i,u(s_i))\|_X+\|K_\alpha(t-s_i)\|_{L(X)}\|q_i(s_i,u(s_i))\|_X\\
&\quad+\int_{s_i}^{t}\|T_\alpha(t-s)\|_{L(X}\|f(s,u_{\rho(s,u_s)})\|_X ds\end{aligned}$$

$$+\int_{s_i}^{t}\|T_\alpha(t-s)\|_{L(X)}\int_0^s q(s-\xi)\|h(\xi,u_{\rho(\xi,u_\xi)})\|_X d\xi ds$$

$$\leqslant MM_g+MM_q+M\|u\|_X\|m_f\|_{L^1(J,R^+)}+\|u\|_X Mg^*\|m_h(s)\|_{L^1(J,R^+)}$$

$$\|u(t)\|_X \leqslant \frac{MM_g+MM_q}{1-(M\|m_f\|_{L^1(J,R^+)}+Mg^*\|m_h(s)\|_{L^1(J,R^+)})}\leqslant \mathscr{C}^*.$$

Therefore by the theorem 2.10 there exist atleast one fixed point in PC_T. Hence, we conclude that problem (1.1)–(1.3) has a mild solution $u(t)$ on J. This is completes the proof. □

4. Application

We consider the following partial integro-differential equation with fractional derivative

$$\frac{\partial^\alpha}{\partial t^\alpha}u(t,x)=\frac{\partial^2}{\partial y^2}u(t,x)+\frac{u(t-\rho_1(t)\rho_2(\|u\|),x)}{49}$$

$$+\int_0^t \cos(t-\gamma)\frac{u(t-\rho_1(t)\rho_2(\|u\|),x)}{25}d\gamma,\ (t,x)\in\cup_{i=1}^N(s_i,t_{i+1}]\times(0,\pi], \quad (4.1)$$

$$u(t,x)=\phi(t,x);u'(t,x)=\varphi(t,x),\ t\in[-d,0],\ x\in(0,\pi], \quad (4.2)$$

$$u(t,x)=G_i(t,x);\ u'(t,x)=H_i(t,x),\ t\in(t_i,s_i]. \quad (4.3)$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is Caputo's fractional derivative of order $\alpha\in(1,2)$ and $\phi,\varphi\in PC_0$, Let $X=L^2(0,\pi]$ and define the operator $A:D(A)\subset X\to X$ by $Aw=w''$ with the domain $D(A):=\{w\in X:\ w,w'$ are absolutely continuous, $w''\in X, w(0)=0=w(\pi)\}$. Then

$$Aw=\sum_{n=1}^{\infty}n^2(w,w_n)w_n,\ w\in D(A),$$

where $w_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx)$, $n\in N$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geqslant 0}$ in X and is given by

$$T(t)\omega=\sum_{n=1}^{\infty}e^{-n^2t}(\omega,\omega_n)\omega_n,\ \text{for all }\omega\in X,\ \text{and every } t>0.$$

The subordination principle of solution operator implies that A is the infinitesimal generator of $K_\alpha(t),S_\alpha(t)$. Since $K_\alpha(t),S_\alpha(t)$ are strongly continuous on $[0,\infty)$ by uniformly bounded theorem, there exists a constant $M>0$ such that $\|S_\alpha(t)\|\leqslant M, \|K_\alpha(t)\|\leqslant M$ for $t\in(0,\pi]$. We assume that $\rho_i:[0,\infty)\to[0,\infty)$, $i=1,2$, are continuous functions.

Setting $u(t)(x)=u(t,x)$, and $\rho(t,\phi)=\rho_1(t)\rho_2(\|\phi(0)\|)$ we have

$$f(t,\phi)=\frac{\phi}{49};\ \ h(t,\phi)=\frac{\phi}{25};\ \ y(t)=G_i(t,y);\ \ y'(t)=H_i(t,y),$$

and the problem (4.1)–(4.2) can be written in the abstract form of problem (1.1)–(1.3). It is obvious that map f,h,g,q satisfy the assumptions H_1,H_2. Then there exists a unique mild solution of problem (4.1)–(4.3).

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