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LIE SYMMETRY ANALYSIS TO GENERAL TIME–FRACTIONAL KORTEWEG–DE VRIES EQUATIONS

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Abstract. In present paper, two class of the general time-fractional Korteweg-de Vries equations (KdVs) are considered, a systematic investigation to derive Lie point symmetries of the equations are presented and compared. Each of them has been transformed into a nonlinear ordinary differential equation with a new independent variable are investigated. The derivative corresponding to time-fractional in the reduced formula is known as the Erdélyi-Kober fractional derivative.

1. Introduction

In recent years, physics and mathematics fields have devoted considerable effort to the study of solutions to partial differential equations (PDEs). Among PDEs, the KdV equation is an important mathematical model with wide applications in solid state physics, plasma physics, fluid physics and quantum field theory [11, 12]. More specifically, KdV equation generically describes the dynamics near long-wave-length primary instabilities in the presence of appropriate symmetries [18], it has attracted a great deal of interest as a model for complex temporal-spatio physics in spatially extended systems [5], as a model pattern formations on unstable flame fronts and thin hydrodynamic films [29], and as a paradigm for finite-dimensional physics [20], such as the model of Kaup-Kupershmidt equation arises in the study of the capillary-gravity waves, see [1, 19]. In many methods for solving the equation, Lie symmetry analysis method [3, 13, 14, 21, 31] can provide an effective procedure for conservations laws, explicit and numerical solutions of a wide and general class of differential systems representing real physical problems.

Although Lie symmetry analysis plays a significant role in the analysis of PDEs, it has not been widely applied for studying the invariance properties of fractional partial differential equation (FPDE). During the past three decades or so, fractional calculus has obtained considerable popularity and importance as generalizations of integer-order evolution equations, and used to model problems in neurons, hydrology, viscoelasticity and rheology, image processing, mechanics, mechatronics, physics, finance and control theory, see [2, 10, 16, 17, 23, 24, 27, 28]. Gazizov et al [8] adapted the methods of Lie continuous groups for symmetry analysis, the equations with the derivative of the order α $(0 < \alpha < 1)$ have finite-dimensional groups of admissible transformations, examples of constructing symmetries of FPDE and using these symmetries for constructing

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exact solutions of equations under consideration are also presented. Liu [15] has made complete group classifications on the fractional fifth-order KdV type of equation and investigated the symmetry reductions and exact solutions by the Lie symmetry analysis method. It is worth to mention that in [4] a symmetry group of scaling transformations is determined for a PDE of fractional order α, containing among particular cases the diffusion equation, the wave equation, and the fractional diffusion-wave equation. For its group-invariant solutions, an ordinary differential equation of fractional order with the new independent variable $z = xt^{\alpha/2}$ is derived. Sahadevan and Bakkyaraj [25] have derived Lie point symmetries to time-fractional generalized Burgers and KdVs, and have shown that each of equation has been transformed into a nonlinear ordinary differential equation of fractional order with a new independent variable by using of the obtained Lie point symmetries. Other results we can refer to the literature [7, 9, 30]. The purpose of this paper is to investigate Lie symmetry analysis is useful in the analysis of general time-fractional KdVs. Taking the advantage of Riemann-Liouville's approach that the initial conditions for fractional differential equation take on the traditional form as for integer-order differential equation, the time-fractional KdVs are considered and extent Lie symmetry analysis to derive their infinitesimals.

We provide some background material of the fractional calculus used throughout the remaining sections of the present paper. The books [6, 22, 26] develop fractional calculus and various definitions of fractional integration and differentiation.

DEFINITION 1. The Riemann-Liouville fractional derivative $D_t^{\alpha}u(x,t)$ with respect to t is defined as

$$D_t^{\alpha}u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} D_t^m \int_{t_1}^{t} (t-\tau)^{m-\alpha-1} u(\tau,x)\, d\tau, & m-1<\alpha<m, \\ D_t^m u(x,t), & \alpha = m \in \mathbb{N}, \end{cases} \qquad t_1 < t < t_2,$$

where $D_q^p(\cdot) = \frac{\partial^p}{\partial q^p}(\cdot)$, $p \in \mathbb{N}^+$.

Note that

i) Leibnitz' formula for the Riemann-Liouville fractional derivative takes the form

$$D_t^{\alpha}\Big(\mu(x,t)\nu(x,t)\Big) = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n}\mu(x,t) D_t^n \nu(x,t), \quad \alpha > 0,$$

where $\binom{\alpha}{n} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}$.

ii) Faà di Bruno's formula for the integer-order derivative is

$$D_t^n\left(\psi(\phi(t))\right) = \sum_{m=0}^{n}\sum_{l=0}^{m} \binom{m}{l} \frac{1}{m!} (-\phi(t))^l D_t^n (\phi(t))^{m-l} D_\phi^m \psi(\phi).$$

2. Lie symmetry analysis to FPDE

We present brief details of Lie symmetry analysis for FPDE with two independent variables. Consider a scalar TFPDE having the form

$$D_t^{\alpha}u + F(x,t,u,u_x,u_{2x},u_{3x},u_{4x},u_{5x},\ldots) = 0, \quad t_1 < t < t_2, \tag{1}$$

with $0 < \alpha < 1$, where $u_{kx} = \partial^k/\partial x^k$.

Let us assume that the above TFPDE, (1) is invariant under a one parameter ε continuous transformations

$$\begin{aligned}
&\bar{t} = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2), \quad \bar{x} = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2), \quad \bar{u} = u + \varepsilon\eta(x,t,u) + O(\varepsilon^2),\\
&D^{\alpha}_{\bar{t}}\bar{u} = D^{\alpha}_t u + \varepsilon\zeta^0_{\alpha} + O(\varepsilon^2), \quad D_{\bar{x}}\bar{u} = D_x u + \varepsilon\zeta^1_1 + O(\varepsilon^2), \quad D^2_{\bar{x}}\bar{u} = D^2_x u + \varepsilon\zeta^1_2 + O(\varepsilon^2),\\
&D^3_{\bar{x}}\bar{u} = D^3_x u + \varepsilon\zeta^1_3 + O(\varepsilon^2), \quad D^4_{\bar{x}}\bar{u} = D^4_x u + \varepsilon\zeta^1_4 + O(\varepsilon^2), \quad D^5_{\bar{x}}\bar{u} = D^5_x u + \varepsilon\zeta^1_5 + O(\varepsilon^2),\\
&\ldots,
\end{aligned} \tag{2}$$

where τ, ξ and η are infinitesimals and ζ^1_1, ζ^1_2, $\ldots$, ζ^1_5 and ζ^0_{α} are extended infinitesimals of orders $1,2,\ldots,5$ and α, respectively. The explicit expression for ζ^1_1, ζ^1_2, $\ldots$, ζ^1_5 are

$$\begin{aligned}
\zeta^1_1 &= \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t,\\
\zeta^1_2 &= \eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\eta_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_x u_t - \xi_{uu}u_x^3 - \tau_{uu}u_x^2 u_t - 2\tau_x u_{xt}\\
&\quad + (\eta_u - 2\xi_x)u_{xx} - \tau_u u_{xx}u_t - 2\tau_u u_{xt}u_x - 3\xi_u u_x u_{xx},\\
\zeta^1_3 &= \eta_{xxx} + (3\eta_{xxu} - \xi_{xxx})u_x - \tau_{xxx}u_t + (3\eta_{xuu} - 3\xi_{xxu})u_x^2 - 3\tau_{xxu}u_x u_t + (\eta_{uuu} - 3\xi_{xuu})u_x^3\\
&\quad - \tau_u u_t u_{xxx} + (3\eta_{xu} - 3\xi_{xx})u_{xx} - 3\tau_{xx}u_{xt} - 3\tau_{xuu}u_t u_x^2 + (3\eta_{uu} - 9\xi_{xu})u_x u_{xx} - 3\tau_{xu}u_t u_{xx}\\
&\quad - 6\tau_{xu}u_x u_{xt} - 3\tau_x u_{xxt} + (\eta_u - 3\xi_x)u_{xxx} - \xi_{uuu}u_x^4 - 6\xi_{uu}u_x^2 u_{xx} - 3\tau_{uu}u_x^2 u_{xt} - \tau_{uuu}u_t u_x^3\\
&\quad - 3\xi_u u_{xx}^2 - 3\tau_u u_x u_{xxt} - 3\tau_u u_{xt}u_{xx} - 3\tau_{uu}u_x u_t u_{xx} - 4\xi_u u_x u_{xxx},\\
\zeta^1_4 &= \eta_{xxxx} + (4\eta_{xxxu} - \xi_{xxxx})u_x + (6\eta_{xxu} - 4\xi_{xxx} - 3\tau_{xxu})u_{xx} + (4\eta_{xu} - 6\xi_{xx})u_{xxx} - 4\tau_{xxx}u_{xt}\\
&\quad - \tau_{xxxx}u_t - 6\tau_{xx}u_{xxt} - 4\tau_x u_{xxxt} + (\eta_u - 4\xi_x)u_{xxxx} + (6\eta_{xxuu} - 4\xi_{xxxu})u_x^2 - 4\tau_{xxxu}u_x u_t\\
&\quad + (\eta_{uuuu} - 4\xi_{xuuu})u_x^4 + (12\eta_{xuu} - 12\xi_{xxu})u_x u_{xx} - 6\tau_{xxuu}u_x^2 u_t - 12\tau_{xxu}u_x u_{tx} - 4\tau_{xu}u_t u_{xxx}\\
&\quad + (6\eta_{uuu} - 24\xi_{xuu})u_x^2 u_{xx} - 4\tau_{uu}u_t u_x u_{xxx} - 4\tau_u u_{tx}u_{xxx} - \tau_u u_t u_{xxxx} - 4\tau_{xuuu}u_t u_x^3\\
&\quad - 12\tau_{xuu}u_{tx}u_x^2 - 12\tau_{xuu}u_t u_x u_{xx} + (3\eta_{uu} - 12\xi_{xu})u_{xx}^2 + (4\eta_{uu} - 16\xi_{xu})u_x u_{xxx} - 3\tau_{xxu}u_t u_{xx}\\
&\quad - 12\tau_{xu}u_{tx}u_{xx} - 12\tau_{xu}u_x u_{xxt} - \xi_{uuuu}u_x^5 - 10\xi_{uuu}u_x^3 u_{xx} - 15\xi_{uu}u_x u_{xx}^2 - 10\xi_{uu}u_x^2 u_{xxx}\\
&\quad - 4\tau_u u_x u_{xxxt} - 12\tau_{uu}u_x u_{xt}u_{xx} - 6\tau_{uu}u_x^2 u_{xxt} - \tau_{uuuu}u_x^4 u_t - 6\tau_{uuu}u_t u_x^2 u_{xx} - 10\xi_u u_{xx}u_{xxx}\\
&\quad - 6\tau_u u_{xx}u_{xxt} - 3\tau_{uu}u_t u_{xx}^2 - 4\tau_{uuu}u_x^3 u_{xt} + (4\eta_{xuuu} - 6\xi_{xxuu})u_x^3 - 5\xi_u u_x u_{xxxx},\\
\zeta^1_5 &= \eta_{xxxxx} + (5\eta_{xxxxu} - \xi_{xxxxx})u_x + (10\eta_{xxxuu} - 5\xi_{xxxxu})u_x^2 - 3\tau_{xxu}u_{xxx} - 5\tau_u u_x u_{xxxxt}\\
&\quad + (30\eta_{xxuu} - 24\xi_{xxxu} - 3\tau_{xxuu})u_x u_{xx} + (10\eta_{xxxu} - 5\xi_{xxxx} - 3\tau_{xxxu})u_{xx} - 10\tau_{xx}u_{xxxt}\\
&\quad + (5\eta_{xu} - 10\xi_{xx})u_{xxxx} - 5\tau_{xxxx}u_{xt} - 10\tau_{xxx}u_{xxt} - \tau_{xxxxx}u_t - 5\tau_{xxxxu}u_x u_t - 30\tau_{xxu}u_x u_{xxt}\\
&\quad - 5\tau_x u_{xxxxt} + (\eta_u - 5\xi_x)u_{xxxxx} + (10\eta_{xxuuu} - 10\xi_{xxxuu})u_x^3 + (30\eta_{xuuu} - 54\xi_{xxuu})u_x^2 u_{xx}\\
&\quad + (5\eta_{xuuuu} - 10\xi_{xxuuu})u_x^4 + (\eta_{uuuuu} - 5\xi_{xuuuu})u_x^5 + (10\eta_{uuuu} - 50\xi_{xuuu})u_x^3 u_{xx}\\
&\quad - 20\tau_{xxxu}u_x u_{xt} + (15\eta_{xuu} - 24\xi_{xxu})u_{xx}^2 - 10\tau_{xxuuu}u_x^3 u_t - 27\tau_{xxuu}u_x u_t u_{xx} - 30\tau_{xxuu}u_x^2 u_{xt}\\
&\quad - 27\tau_{xxu}u_{xx}u_{xt} + (15\eta_{uuu} - 75\xi_{xuu})u_x u_{xx}^2 + (10\eta_{uuu} - 50\xi_{xuu})u_x^2 u_{xxx} - 7\tau_{xxu}u_t u_{xxx}\\
&\quad - 20\tau_{xuu}u_x u_t u_{xxx} - 20\tau_{xu}u_{xt}u_{xxx} - 5\tau_{xu}u_t u_{xxxx} - 10\tau_{uuu}u_x^2 u_t u_{xxx} - 20\tau_{uu}u_x u_{xt}u_{xxx}
\end{aligned}$$

$$-10\tau_{uu}u_t u_{xx}u_{xxx}-5\tau_{uu}u_x u_t u_{xxxx}-10\tau_u u_{xxt}u_{xxx}-5\tau_u u_{xt}u_{xxxx}-\tau_u u_t u_{xxxxx}$$
$$-20\tau_{xuuu}u_{xt}u_x^3-30\tau_{xuu}u_{xxt}u_x^2-60\tau_{xuu}u_{xt}u_x u_{xx}-15\tau_{xuu}u_t u_{xx}^2+(10\eta_{uu}-50\xi_{xu})u_{xx}u_{xxx}$$
$$-7\tau_{xxxu}u_t u_{xx}-30\tau_{xu}u_{xxt}u_{xx}-20\tau_{xu}u_x u_{xxxt}-\xi_{uuuuu}u_x^6-15\xi_{uuuu}u_x^4 u_{xx}-20\xi_{uuu}u_x^3 u_{xxx}$$
$$-45\xi_{uuu}u_x^2 u_{xx}^2-15\xi_{uu}u_{xx}^3-50\xi_{uu}u_x u_{xx}u_{xxx}-15\xi_{uu}u_x^2 u_{xxxx}-5\tau_{xuuu}u_x^4 u_{xt}$$
$$-30\tau_{uuu}u_x^2 u_{xx}u_{xt}-10\tau_{uuu}u_x^3 u_{xxt}-15\tau_{uu}u_{xt}u_{xx}^2-30\tau_{uu}u_x u_{xx}u_{xxt}-10\tau_{uu}u_x^2 u_{xxxt}$$
$$-10\tau_{uuuu}u_t u_x^3 u_{xx}-15\tau_{uuu}u_x u_t u_{xx}^2-10\xi_u u_{xxx}^2-15\xi_u u_{xx}u_{xxxx}-10\tau_u u_{xx}u_{xxxt}$$
$$+(5\eta_{uu}-25\xi_{xu})u_x u_{xxxx}+(20\eta_{xuu}-34\xi_{xxu})u_x u_{xxx}-5\tau_{xuuuu}u_t u_x^4-\tau_{uuuuu}u_x^5 u_t$$
$$+(10\eta_{xxu}-10\xi_{xxx})u_{xxx}-10\tau_{xxxuu}u_x^2 u_t-30\tau_{xuuu}u_t u_x^2 u_{xx}-6\xi_u u_x u_{xxxxx},$$
$$\ldots,$$

with infinitesimal generator

$$G=\tau(x,t,u)D_t+\xi(x,t,u)D_x+\eta(x,t,u)D_u. \tag{3}$$

If the vector field (3) generates a symmetry of (1), then G must satisfy Lie's symmetry condition

$$Pr^{(n)}G(\Delta)|_{\Delta=0}=0,$$

where $\Delta=D_t^\alpha u+F(x,t,u,u_x,u_{2x},u_{3x},u_{4x},u_{5x},\ldots)$.

Since the lower terminal of the integral in Riemann-Liouville time-fractional derivative is fixed and, therefore it should be invariant with respect to the transformations (2). Such invariance condition yields

$$\tau(x,t,u)|_{t=0}=0. \tag{4}$$

DEFINITION 2. A solution $u=\upsilon(x,t)$ is said to be an invariant solution of TFPDE (1) if and only if

(i) $u=\upsilon(x,t)$ is an invariant surface, i.e.

$$G\upsilon(x,t)=0\Longleftrightarrow(\tau(x,t,u)D_t+\xi(x,t,u)D_x+\eta(x,t,u)D_u)\upsilon=0,$$

(ii) $u=\upsilon(x,t)$ satisfies TFPDE (1).

The α-th extended infinitesimal related to the Riemann-Liouville time-fractional derivative with (4) reads

$$\zeta_\alpha^0=D_t^\alpha(\eta)+\xi D_t^\alpha u_x-D_t^\alpha(\xi u_x)+D_t^\alpha\big(D_t(\tau)u\big)-D_t^{\alpha+1}(\tau u)+\tau D_t^{\alpha+1}u. \tag{5}$$

Using the Leibnitz' formula for the Riemann-Liouville fractional derivative, the form (5) can be presented as

$$\zeta_\alpha^0=D_t^\alpha(\eta)-\alpha D_t\tau D_t^\alpha u-\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n\xi D_t^{\alpha-n}u_x-\sum_{n=1}^{\infty}\binom{\alpha}{n+1}D_t^{n+1}\tau D_t^{\alpha-n}u. \tag{6}$$

Further, using Faà di Bruno's formula along with the Leibnitz' formula for the Riemann-Liouville fractional derivative with $f(x,t)=1$, one can written the first term $D_t^\alpha(\eta)$ in (6) as

$$D_t^\alpha(\eta)=D_t^\alpha\eta+\eta_u D_t^\alpha u-uD_t^\alpha\eta_u+\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n\eta_u D_t^{\alpha-n}u+\omega,$$

where

$$\omega=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}(-u)^r D_t^m(u^{k-r})D^{n-m+k}\eta,$$

and $D^{n-m+k}\eta=\frac{\partial^{n-m+k}}{\partial t^{n-m}\partial u^k}\eta$. As a consequence the α-th extended infinitesimal presented in (6) becomes

$$\zeta_\alpha^0=D_t^\alpha\eta+(\eta_u-\alpha D_t\tau)D_t^\alpha u-uD_t^\alpha\eta_u+\omega+\sum_{n=1}^{\infty}\left(\binom{\alpha}{n}D_t^n\eta_u-\binom{\alpha}{n+1}D_t^{n+1}\tau\right)D_t^{\alpha-n}u$$
$$-\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}u_x.$$

For the invariance of TFPDE (1) under transformations (2), we obtain

$$D_{\bar t}^\alpha\bar u+F(\bar x,\bar t,\bar u,\bar u_{\bar x},\bar u_{2\bar x},\bar u_{3\bar x},\bar u_{4\bar x},\bar u_{5\bar x},\ldots)=0, \tag{7}$$

for any solution $u=u(x,t)$ of TFPDE (1). Taking into account the higher order of the nonlinear TFPDEs, expanding (7) about $\varepsilon=0$ and making use of infinitesimals and their extensions (2) and equating the coefficients of ε, and neglecting the terms of higher powers of ε, we give the revised invariant equation (see [25]) of TFPDE

$$\zeta_\alpha^0+\xi\frac{\partial F}{\partial x}+\tau\frac{\partial F}{\partial t}+\eta\frac{\partial F}{\partial u}+\zeta_1^1\frac{\partial F}{\partial u_x}+\zeta_2^1\frac{\partial F}{\partial u_{2x}}+\zeta_3^1\frac{\partial F}{\partial u_{3x}}+\zeta_4^1\frac{\partial F}{\partial u_{4x}}+\zeta_5^1\frac{\partial F}{\partial u_{5x}}+\cdots=0, \tag{8}$$

which is known as the invariant equation of TFPDE (1). Now solving the invariant equation (8) with (1), we can determine τ,ξ,η explicitly. Notice that the expression for ω given in (7) vanishes when the infinitesimal η is linear in u.

3. Time-fractional KdV equation (I)

The general forth-order time-fractional KdV equation

$$D_t^\alpha u+au^5+bu^3u_x+cuu_x^2+du^2u_{2x}+eu_xu_{2x}+fuu_{3x}+u_{4x}=0, \tag{9}$$

with $0<\alpha<1$ and six constant parameters a,b,c,d,e,f are invariant under a one parameter transformations (2), and so the transformed equation is read as

$$D_{\bar t}^\alpha\bar u+a\bar u^5+b\bar u^3\bar u_{\bar x}+c\bar u\bar u_{\bar x}^2+d\bar u^2\bar u_{2\bar x}+e\bar u_{\bar x}\bar u_{2\bar x}+f\bar u\bar u_{3\bar x}+\bar u_{4\bar x}=0. \tag{10}$$

Making use of transformations (2) in (10), we obtain the invariant equation of (9)

$$\begin{aligned}\zeta_\alpha^0 &+ (bu^3+2cuu_x+eu_{2x})\zeta_1^1+(du^2+eu_x)\zeta_2^1+fu\zeta_3^1+\zeta_4^1\\ &+\eta(5au^4+3bu^2u_x+cu_x^2+2duu_{2x}+fu_{3x})=0.\end{aligned} \tag{11}$$

Such a structure of (11) allows one to reduce it to a system of infinitely many linear TFPDEs. Substituting the expressions for ζ_k^1 $(k=1,2,\dots,4)$ and ζ_α^0 into (11) and equating various powers of derivatives of u to zero, we obtain an over determined system of linear equations

$$\begin{cases}\xi_u=\xi_t=\tau_u=\tau_x=\eta_{uu}=0,\\ 4\xi_x-\alpha\tau_t=0,\\ \binom{\alpha}{n}D_t^n\eta_u-\binom{\alpha}{n+1}D_t^{n+1}\tau=0 \quad \text{for} \quad n=1,2,\dots,\\ D_t^\alpha\eta-uD_t^\alpha\eta_u+bu^3\eta_x+du^2\eta_{xx}+fu\eta_{xxx}+\eta_{xxxx}+5a\eta u^4=0,\\ bu^3(\alpha\tau_t-\xi_x)+2cu\eta_x+du^2(2\eta_{xu}-\xi_{xx})+e\eta_{xx}+fu(3\eta_{xxu}-\xi_{xxx})\\ \quad +(4\eta_{xxxu}-\xi_{xxxx})+3b\eta u^2=0.\end{cases} \tag{12}$$

Solving system (12) consistently, we obtain the explicit forms of infinitesimals

$$\xi=a_1x+b_1, \quad \tau=\frac{4a_1}{\alpha}t, \quad \eta=-a_1u,$$

where $a_1\neq 0$, b_1 are constants. Hence the infinitesimal operator becomes

$$G=(a_1x+b_1)D_x+\frac{4a_1}{\alpha}tD_t-a_1uD_u,$$

and so the underlying Lie algebra of the time-fractional KdV equation is two dimensional with basis

$$G_1=D_x, \quad G_2=xD_x+\frac{4}{\alpha}tD_t-uD_u. \tag{13}$$

It is easy to check that the symmetry generators found in (13) form a closed Lie algebra

$$[G_1,G_1]=[G_2,G_2]=0, \quad [G_1,G_2]=[G_2,G_1]=-G_1.$$

The similarity variable and similarity transformation corresponding to the infinitesimal generator G_2 can be obtained by solving the associated characteristic equation given by

$$\frac{dx}{x}=\frac{\alpha dt}{4t}=-\frac{du}{u},$$

which take the forms

$$u=t^{-\frac{\alpha}{4}}\varphi(z), \quad z=xt^{-\frac{\alpha}{4}}. \tag{14}$$

THEOREM 1. *The similarity transformation* $u = t^{-\frac{\alpha}{4}}\varphi(z)$ *along with the similarity variable* $z = xt^{-\frac{\alpha}{4}}$ *reduces the time-fractional KdV equation (9) to the nonlinear constant coefficient ordinary differential equation with variable* z

$$\begin{aligned}\left(P_{\frac{4}{\alpha}}^{1-\frac{5\alpha}{4},\alpha}\varphi\right)(z) + a\varphi(z)^5 + b\varphi(z)^3\varphi'(z) + c\varphi(z)\varphi'(z)^2 + d\varphi(z)^2\varphi''(z)\\ + e\varphi'(z)\varphi''(z) + f\varphi(z)\varphi'''(z) + \varphi''''(z) = 0,\end{aligned} \tag{15}$$

where $' = \frac{d}{dz}$, *with the Erdélyi-Kober fractional differential operator*

$$\left(P_{\delta}^{\tau,\alpha}\chi\right)(z) := \prod_{j=0}^{m-1}\left(\tau + j - \frac{1}{\delta}z\frac{d}{dz}\right)\left(K_{\delta}^{\tau+\alpha,m-\alpha}\chi\right)(z), \quad z>0,\ \alpha>0,\ \delta>0, \tag{16}$$

$$m = \begin{cases} [\alpha]+1, & \alpha \notin \mathbb{N},\\ \alpha, & \alpha \in \mathbb{N},\end{cases}$$

and

$$\left(K_{\delta}^{\tau,\alpha}\chi\right)(z) := \begin{cases} \frac{1}{\Gamma(\alpha)}\int_1^{\infty}(v-1)^{\alpha-1}v^{-\tau-\alpha}\chi(zv^{\frac{1}{\delta}})dv, & \alpha>0,\\ \chi(z), & \alpha=0,\end{cases}$$

is the Erdélyi-Kober fractional integral operator.

Proof. Let $n-1 \leqslant \alpha \leqslant n$, $n = 1,2,\ldots$. Thus the Riemann-Liouville time-fractional derivative for the similarity transformation (14) becomes

$$D_t^{\alpha}u = D_t^n\left(\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}s^{-\frac{\alpha}{4}}\varphi(xs^{-\frac{\alpha}{4}})ds\right).$$

Set $v = \frac{t}{s}$. Then the above equation can be written as

$$D_t^{\alpha}u = D_t^n\left(t^{n-\frac{5\alpha}{4}}\frac{1}{\Gamma(n-\alpha)}\int_1^{\infty}(v-1)^{n-\alpha-1}v^{-(n+1-\frac{5\alpha}{4})}\varphi(zv^{\frac{\alpha}{4}})dv\right).$$

Following the definition of the Erdélyi-Kober fractional integral operator given in (16), we have

$$D_t^{\alpha}u = D_t^n\left(t^{n-\frac{5\alpha}{4}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}\varphi\right)(z)\right). \tag{17}$$

In order to simplify (17), we consider the relation $(z = xt^{-\frac{\alpha}{4}},\ \phi \in C^1(0,\infty))$,

$$tD_t\phi(z) = tx\left(-\frac{\alpha}{4}\right)t^{-\frac{\alpha}{4}-1}D_z\phi(z) = -\frac{\alpha}{4}zD_z\phi(z),$$

and so, we have

$$\begin{aligned}D_t^n\left(t^{n-\frac{5\alpha}{4}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}\right)\right) &= D_t^{n-1}\left(D_t\left(t^{n-\frac{5\alpha}{4}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}\varphi\right)(z)\right)\right)\\ &= D_t^{n-1}\left(t^{n-1-\frac{5\alpha}{4}}\left(n-\frac{5\alpha}{4}-\frac{\alpha}{4}zD_z\right)\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}\varphi\right)(z)\right).\end{aligned}$$

Repeating the similar procedure for $n-1$ times, we get

$$D_t^n\left(t^{n-\frac{5\alpha}{4}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}\right)\right)=t^{-\frac{5\alpha}{4}}\prod_{j=0}^{n-1}\left(1+j-\frac{5\alpha}{4}-\frac{\alpha}{4}zD_z\right)\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}\varphi\right)(z).$$

Then the above equation can be written as

$$D_t^n\left(t^{n-\frac{5\alpha}{4}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}\right)\right)=t^{-\frac{5\alpha}{4}}\left(P_{\frac{4}{\alpha}}^{1-\frac{5\alpha}{4},\alpha}\varphi\right)(z),$$

and so we obtain an expression for the time-fractional derivative

$$D_t^\alpha u=t^{-\frac{5\alpha}{4}}\left(P_{\frac{4}{\alpha}}^{1-\frac{5\alpha}{4},\alpha}\varphi\right)(z).$$

Continuing further we find that the time-fractional KdV equation (9) reduces into an ordinary differential equation of fractional order

$$\begin{aligned}\left(P_{\frac{4}{\alpha}}^{1-\frac{5\alpha}{4},\alpha}f\right)(z)=&-a\varphi(z)^5-b\varphi(z)^3\varphi'(z)-c\varphi(z)\varphi'(z)^2-d\varphi(z)^2\varphi''(z)-e\varphi'(z)\varphi''(z)\\&-f\varphi(z)\varphi'''(z)-\varphi''''(z).\quad\square\end{aligned}$$

4. Time-fractional KdV equation (II)

The general fifth-order time-fractional KdV equation

$$D_t^\alpha u+au^6+bu^4u_x+cu^2u_x^2+du^3u_{2x}+eu_{2x}^2+fu^2u_{3x}+gu_xu_{3x}+huu_{4x}+u_{5x}=0, \tag{18}$$

with $0<\alpha<1$ and eight constant parameters a,b,c,d,e,f,g,h are invariant under a one parameter transformations (2), and so the transformed equation is read as

$$D_{\bar t}^\alpha \bar u+a\bar u^6+b\bar u^4\bar u_{\bar x}+c\bar u^2\bar u_{\bar x}^2+d\bar u^3\bar u_{2\bar x}+e\bar u_{2\bar x}^2+f\bar u^2\bar u_{3\bar x}+g\bar u_{\bar x}\bar u_{3\bar x}+h\bar u\bar u_{4\bar x}+\bar u_{5\bar x}=0. \tag{19}$$

Making use of transformations (2) in (19), we obtain the invariant equation of (18)

$$\begin{aligned}&\zeta_\alpha^0+(bu^4+2cu^2u_x+gu_{3x})\zeta_1^1+(du^3+2eu_{2x})\zeta_2^1+(fu^2+gu_x)\zeta_3^1+hu\zeta_4^1+\zeta_5^1\\&\quad+\eta(6au^5+4bu^3u_x+2cuu_x^2+3du^2u_{2x}+2fuu_{3x}+gu_{4x})=0.\end{aligned} \tag{20}$$

Substituting the expressions for ζ_k^1 $(k=1,2,\ldots,5)$ and ζ_α^0 into (20) and equating various powers of derivatives of u to zero, we have

$$\begin{cases}\xi_u=\xi_t=\tau_u=\tau_x=\eta_{uu}=0,\\ 5\xi_x-\alpha\tau_t=0,\\ \binom{\alpha}{n}D_t^n\eta_u-\binom{\alpha}{n+1}D_t^{n+1}\tau=0 \quad \text{for} \quad n=1,2,\ldots,\\ D_t^\alpha\eta-uD_t^\alpha\eta_u+bu^4\eta_x+du^3\eta_{xx}+fu^2\eta_{xxx}+hu\eta_{xxxx}+\eta_{xxxxx}+6a\eta u^5=0,\\ bu^4(\alpha\tau_t-\xi_x)+2cu^2\eta_x+du^3(2\eta_{xu}-\xi_{xx})+g\eta_{xxx}+fu^2(3\eta_{xxu}-\xi_{xxx})\\ \quad +hu(4\eta_{xxxu}-\xi_{xxxx})+5\eta_{xxxxu}-\xi_{xxxxx}+4b\eta u^3=0.\end{cases} \tag{21}$$

Solving linear system (21) consistently, it yields the explicit forms of infinitesimals

$$\xi = a_1 x + b_1, \quad \tau = \frac{5a_1 t}{\alpha}, \quad \eta = -a_1 u,$$

where $a_1 \neq 0, b_1$ are constants. Hence the infinitesimal operator is

$$G = (a_1 x + b_1) D_x + \frac{5a_1 t}{\alpha} D_t - a_1 u D_u,$$

and so the underlying Lie algebra of the time-fractional KdV equation is two dimensional with basis

$$G_1 = D_x, \quad G_2 = x D_x + \frac{5t}{\alpha} D_t - u D_u. \tag{22}$$

It is easy to check that the symmetry generators found in (22) form a closed Lie algebra

$$[G_1, G_1] = [G_2, G_2] = 0, \quad [G_1, G_2] = [G_2, G_1] = -G_1.$$

The similarity variable and similarity transformation corresponding to the infinitesimal generator G_2 can be obtained by solving the associated characteristic equation given by

$$\frac{dx}{x} = \frac{\alpha dt}{5t} = -\frac{du}{u},$$

which take the forms

$$u = t^{-\frac{\alpha}{5}} \varphi(z), \quad z = x t^{-\frac{\alpha}{5}}.$$

THEOREM 2. *The similarity transformation $u = t^{-\frac{\alpha}{5}} \varphi(z)$ along with the similarity variable $z = x t^{-\frac{\alpha}{5}}$ reduces the time-fractional KdV equation (18) to the nonlinear constant coefficient ordinary differential equation with variable z*

$$\begin{aligned} &\left(P_{\frac{5}{\alpha}}^{1-\frac{6\alpha}{5},\alpha} \varphi\right)(z) + a\varphi(z)^6 + b\varphi(z)^4 \varphi'(z) + c\varphi(z)^2 \varphi'(z)^2 + d\varphi(z)^3 \varphi''(z) \\ &\quad + e\varphi''(z)^2 + f\varphi^2 \varphi'''(z) + g\varphi'(z)\varphi''''(z) + h\varphi(z)\varphi''''(z) + \varphi'''''(z) = 0. \end{aligned} \tag{23}$$

Proof. The proof is similar to Theorem 1. □

REMARK 1. We reduce the time-fractional KdV equation (9) and (18) to the nonlinear constant coefficient ordinary differential equation (15) and (23), respectively. Unfortunately, the above nonlinear ordinary differential equations with fractional order α $(0 < \alpha < 1)$ is not solvable generally. However, for some special cases, such as the linear equations and the initial value problems, the exact analytic solutions to the nonlinear equations can be solved by the power series method with Mittag-Leffler function. The details are omitted in this paper.

5. Discussion

Using Lie point symmetries, the presented analysis we illustrate the application of Lie symmetry approach to study two class of the general time-fractional KdVs, and derived their Lie point symmetries. All of the geometric vector fields of the equations are obtained, which including the two class of nonlinear evolution equations as its special cases. In view of the angle of geometry, the vector fields of such equations are all symmetries (point symmetries or point transformations). Then the symmetry reductions are considered. The reduction of dimension in the symmetry algebra is due to the fact that each of the time FPDEs is not invariant under time translation symmetry, and can be transformed into a nonlinear ordinary differential equation of fractional order. How to get the exact solutions to the nonlinear equation is a difficult problem. In particular, for the equations (15) and (23) with arbitrary fractional order α $(0 < \alpha < 1)$, there is no general method for dealing with exact explicit solutions to the equations as far as we know. We hope to investigate it in the future.

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