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POSITIVE SOLUTIONS TO ITERATIVE SYSTEMS OF FRACTIONAL ORDER THREE–POINT BOUNDARY VALUE PROBLEMS WITH RIEMANN–LIOUVILLE DERIVATIVE

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Abstract. In this paper, we determine the eigenvalues λ_i, $1 \leqslant i \leqslant n$, for which there exist positive solutions for the iterative system of fractional order three-point boundary value problems by applying fixed point theorem.

1. Introduction

The theory of fractional order differential equations is important due to its demonstrated applications in various fields of science and engineering such as physics, chemistry, control systems, flow in porous media, electromagnetics, mechanics, and so forth [13, 14, 3, 11, 2, 10]. The positive solutions of boundary value problems associated with ordinary differential equations were studied by many authors [4, 8, 7]. Establishing positive solutions to fractional order boundary value problems are gained momentum, for some recent developments on the topic, see [1, 5, 9, 15, 16] and the references therein.

This paper is concerned with determining the eigenvalues λ_i, $1 \leqslant i \leqslant n$, for which there exist positive solutions to the iterative system of fractional order three-point boundary value problems

$$\left.\begin{aligned} D_{0^+}^{q_1} u_i(t) + \lambda_i p_i(t) f_i\Big(u_{i+1}(t)\Big) = 0,\ 1 \leqslant i \leqslant n,\ 0 < t < 1,\\ u_{n+1}(t) = u_1(t), \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} u_i^{(j)}(0) = 0,\ j = 0,1,2,\cdots,n-2,\\ \beta D_{0^+}^{q_2} u_i(1) = \alpha D_{0^+}^{q_2} u_i(\xi), \end{aligned}\right\} \tag{2}$$

where $q_1 \in (n-1, n]$, $n \geqslant 2$, $\xi \in (0,1)$, $q_2 \in (1, q_1)$, α, β are positive real numbers and $D_{0^+}^{q_1}, D_{0^+}^{q_2}$ are the standard Riemann–Liouville fractional order derivatives.

We assume that the following conditions hold throughout the paper:

(A1) $f_i : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, for $1 \leqslant i \leqslant n$,

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(A2) $p_i : [0,1] \to \mathbb{R}^+$ is continuous and p_i does not vanish identically on any closed subinterval of $[0,1]$, for $1 \leqslant i \leqslant n$,

(A3) $\mathscr{M} = \beta - \alpha\xi^{q_1-q_2-1} > 0$,

(A4) each of
$$f_{i0} = \lim_{x\to 0^+} \frac{f_i(x)}{x} \text{ and } f_{i\infty} = \lim_{x\to\infty} \frac{f_i(x)}{x},$$
for $1 \leqslant i \leqslant n$, exists as positive real numbers.

By a positive solution of the fractional order boundary value problem (1)–(2), we mean $\Big(u_1(t), u_2(t), \cdots, u_n(t)\Big) \in \Big(C^{[q_1]+1}[0,1]\Big)^n$ satisfying (1)–(2) with $u_i(t) \geqslant 0, i = 1,2,3,\cdots,n,$ for all $t \in [0,1]$ and $(u_1, u_2, \cdots, u_n) \neq (0,0,\cdots,0)$.

The rest of the paper is organized as follows. In Section 2, we construct the Green's function for the fractional order boundary value problem and estimate the bounds for the Green's function. Later, we express the solution of the boundary value problem (1)–(2) into an equivalent integral equation. In Section 3, we establish criteria to determine the eigenvalues for which the fractional order boundary value problem (1)–(2) has at least one positive solution in a cone by using Guo–Krasnosel'skii fixed point theorem. In Section 4, as an application, we illustrate our results with an example.

2. Green's function and bounds

In this section, we constructed an equivalent integral equation for the fractional order boundary value problem (1)–(2) and the kernel involved in the integral equation is named as the Green's function and bounds for the Green's function are estimated.

LEMMA 1. *Let $\Delta = \Gamma(q_1)\mathscr{M} \neq 0$. If $h(t) \in C[0,1]$, then the fractional order differential equations,*
$$D_{0^+}^{q_1} u_1(t) + h(t) = 0,\ t \in (0,1), \tag{3}$$
satisfying the boundary conditions
$$\left.\begin{aligned} u_1^{(j)}(0) = 0,\ j = 0,1,2,\cdots,n-2,\\ \beta D_{0^+}^{q_2} u_1(1) = \alpha D_{0^+}^{q_2} u_1(\xi), \end{aligned}\right\} \tag{4}$$
has a unique solution
$$u_1(t) = \int_0^1 G(t,s)h(s)ds,$$
where $G(t,s)$ is the Green's function for the problem (3), (4) and is given by
$$G(t,s) = \begin{cases} \underset{t\in[0,\xi]}{G(t,s)} = \begin{cases} G_{11}(t,s),\ 0 \leqslant t \leqslant s \leqslant \xi < 1, \\ G_{12}(t,s),\ 0 \leqslant s \leqslant \min\{t,\xi\} < 1, \end{cases} \\ \\ \underset{t\in[\xi,1]}{G(t,s)} = \begin{cases} G_{21}(t,s),\ 0 \leqslant \max\{t,\xi\} \leqslant s \leqslant 1, \\ G_{22}(t,s),\ 0 < \xi \leqslant s \leqslant t \leqslant 1, \end{cases} \end{cases} \tag{5}$$

$$G_{11}(t,s) = \frac{1}{\Delta}\Big[\beta t^{q_1-1}(1-s)^{q_1-q_2-1} - \alpha t^{q_1-1}(\xi-s)^{q_1-q_2-1}\Big],$$
$$G_{12}(t,s) = \frac{1}{\Delta}\Big[\beta t^{q_1-1}(1-s)^{q_1-q_2-1} - \mathscr{M}(t-s)^{q_1-1} - \alpha t^{q_1-1}(\xi-s)^{q_1-q_2-1}\Big],$$
$$G_{21}(t,s) = \frac{1}{\Delta}\Big[\beta t^{q_1-1}(1-s)^{q_1-q_2-1}\Big],$$
$$G_{22}(t,s) = \frac{1}{\Delta}\Big[\beta t^{q_1-1}(1-s)^{q_1-q_2-1} - \mathscr{M}(t-s)^{q_1-1}\Big].$$

Proof. Let $u_1(t) \in C^{[q_1]+1}[0,1]$ be the solution of fractional order boundary value problem given by (3) and (4). An equivalent integral equation for (3) is given by

$$u_1(t) = \frac{-1}{\Gamma(q_1)}\int_0^t (t-s)^{q_1-1}h(s)ds + c_1t^{q_1-1} + c_2t^{q_1-2} + \cdots + c_nt^{q_1-n}.$$

Using the conditions (4), we can determine $c_n = c_{n-1} = \cdots = c_2 = 0$ and

$$c_1 = \frac{1}{\Delta}\left[\beta\int_0^1 (1-s)^{q_1-q_2-1}h(s)ds - \alpha\int_0^\xi (\xi-s)^{q_1-q_2-1}h(s)ds\right].$$

Thus, the unique solution of (3), (4) is

$$\begin{aligned} u_1(t) &= \frac{t^{q_1-1}}{\Delta}\Big[\beta\int_0^1 (1-s)^{q_1-q_2-1}h(s)ds - \alpha\int_0^\xi (\xi-s)^{q_1-q_2-1}h(s)ds\Big] \\ &\quad - \frac{\mathscr{M}}{\Delta}\int_0^t (t-s)^{q_1-q_2-1}h(s)ds \\ &= \int_0^1 G(t,s)h(s)ds. \quad \square \end{aligned}$$

LEMMA 2. *Assume that the condition (A3) is satisfied. Then the Green's function $G(t,s)$ given in (5) satisfies the following inequalities*

$$(i)\ G(t,s) \geqslant 0,\ for\ all\ (t,s) \in [0,1]\times[0,1],$$
$$(ii)\ G(t,s) \leqslant G(1,s),\ for\ all\ (t,s) \in [0,1]\times[0,1],$$
$$(iii)\ G(t,s) \geqslant \tau^{q_1-1}G(1,s),\ for\ all\ (t,s) \in [\tau,1]\times[0,1],$$

where $\tau \in (0,1)$.

Proof. The Green's function $G(t,s)$ is given in (5).
Let $0 \leqslant t \leqslant s \leqslant \xi \leqslant 1$. Then, we have

$$\begin{aligned} G_{11}(t,s) &= \frac{1}{\Delta}\Big[\beta t^{q_1-1}(1-s)^{q_1-q_2-1} - \alpha t^{q_1-1}(\xi-s)^{q_1-q_2-1}\Big] \\ &\geqslant \frac{1}{\Delta}\Big[\beta t^{q_1-1}(1-s)^{q_1-q_2-1} - \alpha t^{q_1-1}(\xi-\xi s)^{q_1-q_2-1}\Big] \\ &= \frac{t^{q_1-1}}{\Delta}\Big[\mathscr{M}(1-s)^{-q_2}\Big](1-s)^{q_1-1} \\ &= \frac{t^{q_1-1}}{\Delta}\Big[\mathscr{M}\Big(1+q_2s+O(s^2)\Big)\Big](1-s)^{q_1-1} \geqslant 0. \end{aligned}$$

Let $0 \leqslant s \leqslant \min\{t,\xi\} \leqslant 1$. Then, we have

$$
\begin{aligned}
G_{12}(t,s) &= \frac{1}{\Delta}\Big[\beta t^{q_1-1}(1-s)^{q_1-q_2-1} - \mathscr{M}(t-s)^{q_1-1} - \alpha t^{q_1-1}(\xi-s)^{q_1-q_2-1}\Big] \\
&\geqslant \frac{1}{\Delta}\Big[\beta t^{q_1-1}(1-s)^{q_1-q_2-1} - \mathscr{M}(t-ts)^{q_1-1} - \alpha t^{q_1-1}(\xi-\xi s)^{q_1-q_2-1}\Big] \\
&= \frac{t^{q_1-1}}{\Delta}\Big[\mathscr{M}((1-s)^{-q_2}-1)\Big](1-s)^{q_1-1} \\
&= \frac{t^{q_1-1}}{\Delta}\Big[q_2 s\mathscr{M} + O(s^2)\Big](1-s)^{q_1-1} \geqslant 0.
\end{aligned}
$$

Let $0 \leqslant \max\{t,\xi\} \leqslant s \leqslant 1$. Then, we have

$$
G_{21}(t,s) = \frac{1}{\Delta}\Big[\beta t^{q_1-1}(1-s)^{q_1-q_2-1}\Big] \geqslant 0.
$$

Let $0 \leqslant \xi \leqslant s \leqslant t \leqslant 1$. Then, we have

$$
\begin{aligned}
G_{22}(t,s) &= \frac{1}{\Delta}\Big[\beta t^{q_1-1}(1-s)^{q_1-q_2-1} - \mathscr{M}(t-s)^{q_1-1}\Big] \\
&\geqslant \frac{1}{\Delta}\Big[\beta t^{q_1-1}(1-s)^{q_1-q_2-1} - \mathscr{M}(t-ts)^{q_1-1}\Big] \\
&= \frac{t^{q_1-1}}{\Delta}\Big[\beta(1-s)^{-q_2} - \mathscr{M}\Big](1-s)^{q_1-1} \\
&= \frac{t^{q_1-1}}{\Delta}\Big[\beta q_2 s + \alpha\xi^{q_1-q_2-1} + O(s^2)\Big](1-s)^{q_1-1} \geqslant 0.
\end{aligned}
$$

Now we prove the inequality (ii).

Let $0 \leqslant t \leqslant s \leqslant \xi \leqslant 1$. Then, we have

$$
\begin{aligned}
\frac{\partial G_{11}(t,s)}{\partial t} &= \frac{(q_1-1)}{\Delta}\Big[\beta t^{q_1-2}(1-s)^{q_1-q_2-1} - \alpha t^{q_1-2}(\xi-s)^{q_1-q_2-1}\Big] \\
&\geqslant \frac{(q_1-1)}{\Delta}\Big[\beta t^{q_1-2}(1-s)^{q_1-q_2-1} - \alpha t^{q_1-2}(\xi-\xi s)^{q_1-q_2-1}\Big] \\
&= \frac{(q_1-1)t^{q_1-2}}{\Delta}\Big[\mathscr{M}(1-s)^{-q_2}\Big](1-s)^{q_1-1} \\
&= \frac{(q_1-1)t^{q_1-1}}{\Delta}\Big[\mathscr{M}\Big(1+q_2 s+O(s^2)\Big)\Big](1-s)^{q_1-1} \geqslant 0.
\end{aligned}
$$

Therefore, $G_{11}(t,s)$ is increasing in t, which implies $G_{11}(t,s) \leqslant G_{11}(1,s)$.

Let $0 \leqslant s \leqslant \min\{t, \xi\} \leqslant 1$. Then, we have

$$\begin{aligned}
&\frac{\partial G_{12}(t,s)}{\partial t} \\
&= \frac{(q_1-1)}{\Delta}\Big[\beta t^{q_1-2}(1-s)^{q_1-q_2-1} - \mathscr{M}(t-s)^{q_1-2} - \alpha t^{q_1-2}(\xi-s)^{q_1-q_2-1}\Big] \\
&\geqslant \frac{(q_1-1)}{\Delta}\Big[\beta t^{q_1-2}(1-s)^{q_1-q_2-1} - \mathscr{M}(t-ts)^{q_1-2} - \alpha t^{q_1-2}(\xi-\xi s)^{q_1-q_2-1}\Big] \\
&= \frac{(q_1-1)t^{q_1-2}}{\Delta}\Big[\mathscr{M}\Big((1-s)^{-(q_2-1)}-1\Big)\Big](1-s)^{q_1-2} \\
&= \frac{(q_1-1)t^{q_1-2}}{\Delta}\Big[(q_2-1)s\mathscr{M} + O(s^2)\Big](1-s)^{q_1-2} \\
&\geqslant 0.
\end{aligned}$$

Therefore, $G_{12}(t,s)$ is increasing in t, which implies $G_{12}(t,s) \leqslant G_{12}(1,s)$.

Let $0 \leqslant \max\{t, \xi\} \leqslant s \leqslant 1$. Then, we have

$$\frac{\partial G_{21}(t,s)}{\partial t} = \frac{(q_1-1)}{\Delta}\Big[\beta t^{q_1-2}(1-s)^{q_1-q_2-1}\Big] \geqslant 0.$$

Therefore, $G_{21}(t,s)$ is increasing in t, which implies $G_{21}(t,s) \leqslant G_{21}(1,s)$.

Let $0 \leqslant \xi \leqslant s \leqslant t \leqslant 1$. Then, we have

$$\begin{aligned}
\frac{\partial G_{22}(t,s)}{\partial t} &= \frac{(q_1-1)}{\Delta}\Big[\beta t^{q_1-2}(1-s)^{q_1-q_2-1} - \mathscr{M}(t-s)^{q_1-2}\Big] \\
&\geqslant \frac{(q_1-1)}{\Delta}\Big[\beta t^{q_1-2}(1-s)^{q_1-q_2-1} - \mathscr{M}(t-ts)^{q_1-2}\Big] \\
&= \frac{(q_1-1)t^{q_1-2}}{\Delta}\Big[\beta(1-s)^{-(q_2-1)} - \mathscr{M}\Big](1-s)^{q_1-2} \\
&= \frac{(q_1-1)t^{q_1-2}}{\Delta}\Big[\beta(q_2-1)s + O(s^2) + \alpha\xi^{q_1-q_2-1}\Big](1-s)^{q_1-2} \\
&\geqslant 0.
\end{aligned}$$

Therefore, $G_{22}(t,s)$ is increasing in t, which implies $G_{22}(t,s) \leqslant G_{22}(1,s)$.

Finally we can establish the inequality (iii).

Let $0 \leqslant t \leqslant s \leqslant \xi \leqslant 1$ and $t \in [\tau, 1]$. Then

$$\begin{aligned}
G_{11}(t,s) &= \frac{1}{\Delta}\Big[\beta t^{q_1-1}(1-s)^{q_1-q_2-1} - \alpha t^{q_1-1}(\xi-s)^{q_1-q_2-1}\Big] \\
&= \frac{t^{q_1-1}}{\Delta}\Big[\beta(1-s)^{q_1-q_2-1} - \alpha(\xi-s)^{q_1-q_2-1}\Big] \\
&= t^{q_1-1}G_{11}(1,s) \\
&\geqslant \tau^{q_1-1}G_{11}(1,s).
\end{aligned}$$

Let $0 \leqslant s \leqslant \min\{t,\xi\} \leqslant 1$ and $t \in [\tau,1]$. Then

$$\begin{aligned}
G_{12}(t,s) &= \frac{1}{\Delta}\Big[\beta t^{q_1-1}(1-s)^{q_1-q_2-1} - \mathscr{M}(t-s)^{q_1-1} - \alpha t^{q_1-1}(\xi-s)^{q_1-q_2-1}\Big] \\
&\geqslant \frac{1}{\Delta}\Big[\beta t^{q_1-1}(1-s)^{q_1-q_2-1} - \mathscr{M}(t-ts)^{q_1-1} - \alpha t^{q_1-1}(\xi-s)^{q_1-q_2-1}\Big] \\
&= \frac{t^{q_1-1}}{\Delta}\Big[\beta(1-s)^{q_1-q_2-1} - \mathscr{M}(1-s)^{q_1-1} - \alpha(\xi-s)^{q_1-q_2-1}\Big] \\
&= t^{q_1-1}G_{12}(1,s) \geqslant \tau^{q_1-1}G_{12}(1,s).
\end{aligned}$$

Let $0 \leqslant \max\{t,\xi\} \leqslant s \leqslant 1$ and $t \in [\tau,1]$. Then

$$\begin{aligned}
G_{21}(t,s) &= \frac{1}{\Delta}\Big[\beta t^{q_1-1}(1-s)^{q_1-q_2-1}\Big] \\
&= t^{q_1-1}G_{21}(1,s) \\
&\geqslant \tau^{q_1-1}G_{21}(1,s).
\end{aligned}$$

Let $0 \leqslant \xi \leqslant s \leqslant t \leqslant 1$ and $t \in [\tau,1]$. Then

$$\begin{aligned}
G_{22}(t,s) &= \frac{1}{\Delta}\Big[\beta t^{q_1-1}(1-s)^{q_1-q_2-1} - \mathscr{M}(t-s)^{q_1-1}\Big] \\
&\geqslant \frac{t^{q_1-1}}{\Delta}\Big[\beta(1-s)^{q_1-q_2-1} - \mathscr{M}(1-s)^{q_1-1}\Big] \\
&= t^{q_1-1}G_{22}(1,s) \geqslant \tau^{q_1-1}G_{22}(1,s).
\end{aligned}$$

where $\tau \in (0,1)$ satisfies $\int_\tau^1 G(1,s)p_i(s)ds > 0, 1 \leqslant i \leqslant n.$ □

An n-tuple $\Big(u_1(t),u_2(t),\cdots,u_n(t)\Big)$ is a solution of the three-point boundary value problem (1)–(2) if and only if $u_i(t) \in C^{[q_1]+1}[0,1]$ satisfies

$$\begin{aligned}
u_1(t) =&\lambda_1\int_0^1 G(t,s_1)p_1(s_1)f_1\Bigg(\lambda_2\int_0^1 G(s_1,s_2)p_2(s_2)\cdots \\
&f_{n-1}\Bigg(\lambda_n\int_0^1 G(s_{n-1},s_n)p_n(s_n)f_n\Big(u_1(s_n)\Big)ds_n\Bigg)\cdots ds_2\Bigg)ds_1,
\end{aligned}$$

and

$$u_i(t) = \lambda_i\int_0^1 G(t,s)p_i(s)f_i\Big(u_{i+1}(s)\Big)ds,\ 0<t<1,\ 2 \leqslant i \leqslant n,$$

where

$$u_{n+1}(t) = u_1(t), \quad 0<t<1.$$

In establishing our main result, we will employ the following fixed point theorem due to Guo–Krasnosel'skii [6, 12].

THEOREM 1. [6, 12] *Let X be a Banach Space, $P \subseteq X$ be a cone and suppose that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is completely continuous operator such that either*

(i) $\| Tu \| \leqslant \| u \|$, $u \in P \cap \partial\Omega_1$ *and* $\| Tu \| \geqslant \| u \|$, $u \in P \cap \partial\Omega_2$, *or*

(ii) $\| Tu \| \geqslant \| u \|$, $u \in P \cap \partial\Omega_1$ *and* $\| Tu \| \leqslant \| u \|$, $u \in P \cap \partial\Omega_2$ *holds.*

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Positive solutions in a cone

In this section, we establish criteria to determine the eigenvalues for which the fractional order boundary value problem (1)–(2) has at least one positive solution.

Let $X = \Big\{ x : x \in C[0,1] \Big\}$ be the Banach space equipped with the norm

$$\|x\| = \max_{t \in [0,1]} |x(t)|.$$

Define a cone $P \subset X$ by

$$P = \Big\{ x \in X \mid x(t) \geqslant 0 \text{ on } [0,1] \text{ and } \min_{t \in [\tau,1]} x(t) \geqslant \tau^{q_1 - 1} \|x\| \Big\}.$$

Now, we define an integral operator $T : P \to X$, for $u_1 \in P$, by

$$\left.\begin{aligned} Tu_1(t) = \lambda_1 \int_0^1 G(t,s_1) p_1(s_1) f_1 \Bigg(\lambda_2 \int_0^1 G(s_1,s_2) p_2(s_2) \cdots \\ f_{n-1} \Bigg(\lambda_n \int_0^1 G(s_{n-1},s_n) p_n(s_n) f_n \Big(u_1(s_n) \Big) ds_n \Bigg) \cdots ds_2 \Bigg) ds_1. \end{aligned}\right\} \tag{6}$$

Notice from $(A1)$, $(A2)$ and Lemma 2 that, for $u_1 \in P$, $Tu_1(t) \geqslant 0$ on $[0,1]$. Also, we have

$$\begin{aligned} Tu_1(t) \leqslant \lambda_1 \int_0^1 G(1,s_1) p_1(s_1) f_1 \Bigg(\lambda_2 \int_0^1 G(s_1,s_2) p_2(s_2) \cdots \\ f_{n-1} \Bigg(\lambda_n \int_0^1 G(s_{n-1},s_n) p_n(s_n) f_n \Big(u_1(s_n) \Big) ds_n \Bigg) \cdots ds_2 \Bigg) ds_1 \end{aligned}$$

so that

$$\left.\begin{aligned} \|Tu_1\| \leqslant \lambda_1 \int_0^1 G(1,s_1) p_1(s_1) f_1 \Bigg(\lambda_2 \int_0^1 G(s_1,s_2) p_2(s_2) \cdots \\ f_{n-1} \Bigg(\lambda_n \int_0^1 G(s_{n-1},s_n) p_n(s_n) f_n \Big(u_1(s_n) \Big) ds_n \Bigg) \cdots ds_2 \Bigg) ds_1. \end{aligned}\right\} \tag{7}$$

Next, if $u_1 \in P$, we have from Lemma 2 and (7) that

$$\begin{aligned}\min_{t\in[\tau,1]} Tu_1(t) &= \min_{t\in[\tau,1]} \lambda_1 \int_0^1 G(t,s_1)p_1(s_1)f_1\Bigg(\lambda_2 \int_0^1 G(s_1,s_2)p_2(s_2)\cdots \\ &\qquad f_{n-1}\Bigg(\lambda_n \int_0^1 G(s_{n-1},s_n)p_n(s_n)f_n\Big(u_1(s_n)\Big)ds_n\Bigg)\cdots ds_2\Bigg)ds_1 \\ &\geqslant \lambda_1 \tau^{q_1-1} \int_0^1 G(1,s_1)p_1(s_1)f_1\Bigg(\lambda_2 \int_0^1 G(s_1,s_2)p_2(s_2)\cdots \\ &\qquad f_{n-1}\Bigg(\lambda_n \int_0^1 G(s_{n-1},s_n)p_n(s_n)f_n\Big(u_1(s_n)\Big)ds_n\Bigg)\cdots ds_2\Bigg)ds_1 \\ &\geqslant \tau^{q_1-1}\|Tu_1\|.\end{aligned}$$

Therefore, $\min\limits_{t\in[\tau,1]} Tu_1(t) \geqslant \tau^{q_1-1}\|Tu_1\|$. Hence, $Tu_1 \in P$ and so $T: P \to P$. Further, the operator T is completely continuous operator by an application of the Arzela–Ascoli Theorem.

Now, we seek suitable fixed point of T belonging to the cone P. For our first result, define positive numbers $\mathscr{N}_1$ and $\mathscr{N}_2$, by

$$\mathscr{N}_1 = \max_{1\leqslant i\leqslant n}\left\{\left[\tau^{q_1-1}\int_\tau^1 G(1,s)p_i(s)ds f_{i\infty}\right]^{-1}\right\}$$

and

$$\mathscr{N}_2 = \min_{1\leqslant i\leqslant n}\left\{\left[\int_0^1 G(1,s)p_i(s)ds f_{i0}\right]^{-1}\right\}.$$

THEOREM 2. *Assume that the conditions* $(A1)$–$(A4)$ *are satisfied. Then, for each* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *satisfying*

$$\mathscr{N}_1 < \lambda_i < \mathscr{N}_2, \quad 1 \leqslant i \leqslant n, \tag{8}$$

there exists an n*-tuple* $(u_1, u_2, \cdots, u_n)$ *satisfying* (1)–(2) *such that* $u_i(t) > 0$, $1 \leqslant i \leqslant n$ *on* $(0,1)$.

Proof. Let λ_i, $1 \leqslant i \leqslant n$, be given as in (8). Now, let $\varepsilon > 0$ be chosen such that

$$\max_{1\leqslant i\leqslant n}\left\{\left[\tau^{q_1-1}\int_\tau^1 G(1,s)p_i(s)ds(f_{i\infty}-\varepsilon)\right]^{-1}\right\} \leqslant \min_{1\leqslant i\leqslant n}\lambda_i$$

and

$$\max_{1\leqslant i\leqslant n}\lambda_i \leqslant \min_{1\leqslant i\leqslant n}\left\{\left[\int_0^1 G(1,s)p_i(s)ds(f_{i0}+\varepsilon)\right]^{-1}\right\}.$$

We seek fixed point of the completely continuous operator $T : P \to P$ defined by (6). Now, from the definitions of f_{i0}, $1 \leqslant i \leqslant n$, there exists an $H_1 > 0$ such that, for each $1 \leqslant i \leqslant n$,

$$f_i(x) \leqslant (f_{i0} + \varepsilon)x, \ 0 < x \leqslant H_1.$$

Let $u_1 \in P$ with $\|u_1\| = H_1$. We first have from Lemma 2 and the choice of ε, for $0 \leqslant s_{n-1} \leqslant 1$,

$$\begin{aligned}
&\lambda_n \int_0^1 G(s_{n-1}, s_n) p_n(s_n) f_n\Big(u_1(s_n)\Big) ds_n \\
&\quad \leqslant \lambda_n \int_0^1 G(1, s_n) p_n(s_n)(f_{n0} + \varepsilon) u_1(s_n) ds_n \\
&\quad \leqslant \lambda_n \int_0^1 G(1, s_n) p_n(s_n) ds_n (f_{n0} + \varepsilon)\|u_1\| \\
&\quad \leqslant \|u_1\| = H_1.
\end{aligned}$$

It follows in a similar manner from Lemma 2 and the choice of ε that, for $0 \leqslant s_{n-2} \leqslant 1$,

$$\begin{aligned}
&\lambda_{n-1} \int_0^1 G(s_{n-2}, s_{n-1}) p_{n-1}(s_{n-1}) \\
&\qquad\qquad \times f_{n-1}\left(\lambda_n \int_0^1 G(s_{n-1}, s_n) p_n(s_n) f_n\Big(u_1(s_n)\Big) ds_n\right) ds_{n-1} \\
&\quad \leqslant \lambda_{n-1} \int_0^1 G(1, s_{n-1}) p_{n-1}(s_{n-1}) ds_{n-1} (f_{n-1,0} + \varepsilon)\|u_1\| \\
&\quad \leqslant \|u_1\| = H_1.
\end{aligned}$$

Continuing with this bootstrapping argument, we have

$$\begin{aligned}
&\lambda_1 \int_0^1 G(t, s_1) p_1(s_1) f_1\Bigg(\lambda_2 \int_0^1 G(s_1, s_2) p_2(s_2) \cdots \\
&\qquad\qquad f_n\Big(u_1(s_n)\Big) ds_n\Big) \cdots ds_2\Bigg) ds_1 \leqslant H_1, \text{ for } 0 \leqslant t \leqslant 1,
\end{aligned}$$

so that, $Tu_1(t) \leqslant H_1$, for $0 \leqslant t \leqslant 1$. If we set $\Omega_1 = \Big\{x \in X \mid \|x\| < H_1\Big\}$, then

$$\|Tu_1\| \leqslant \|u_1\|, \text{ for } u_1 \in P \cap \partial\Omega_1. \tag{9}$$

Next, from the definitions of $f_{i\infty}$, $1 \leqslant i \leqslant n$, there exists $\overline{H}_2 > 0$ such that, for each $1 \leqslant i \leqslant n$,

$$f_i(x) \geqslant (f_{i\infty} - \varepsilon)x, \ x \geqslant \overline{H}_2.$$

Let $H_2 = \max\left\{2H_1, \dfrac{\overline{H}_2}{\tau^{q_1-1}}\right\}$. Let $u_1 \in P$ and $\|u_1\| = H_2$. Then,

$$\min_{t \in [\tau,1]} u_1(t) \geqslant \tau^{q_1-1}\|u_1\| \geqslant \overline{H}_2.$$

Then, from Lemma 2 and choice of ε, for $0 \leqslant s_{n-1} \leqslant 1$, we have that

$$\begin{aligned}
\lambda_n & \int_0^1 G(s_{n-1}, s_n) p_n(s_n) f_n\Big(u_1(s_n)\Big) ds_n \\
& \geqslant \tau^{q_1-1} \lambda_n \int_\tau^1 G(1, s_n) p_n(s_n)(f_{n\infty} - \varepsilon) u_1(s_n) ds_n \\
& \geqslant \tau^{q_1-1} \lambda_n \int_\tau^1 G(1, s_n) p_n(s_n) ds_n (f_{n\infty} - \varepsilon) \|u_1\| \\
& \geqslant \|u_1\| = H_2.
\end{aligned}$$

It follows in a similar manner from Lemma 2 and choice of ε, for $0 \leqslant s_{n-2} \leqslant 1$,

$$\begin{aligned}
\lambda_{n-1} & \int_0^1 G(s_{n-2}, s_{n-1}) p_{n-1}(s_{n-1}) \\
& \quad \times f_{n-1}\left(\lambda_n \int_0^1 G(s_{n-1}, s_n) p_n(s_n) f_n\Big(u_1(s_n)\Big) ds_n\right) ds_{n-1} \\
& \geqslant \tau^{q_1-1} \lambda_{n-1} \int_\tau^1 G(1, s_{n-1}) p_{n-1}(s_{n-1}) ds_{n-1} (f_{n-1,\infty} - \varepsilon) \|u_1\| \\
& \geqslant \|u_1\| = H_2.
\end{aligned}$$

Again, using a bootstrapping argument, we have

$$\begin{aligned}
\lambda_1 \int_0^1 G(t, s_1) p_1(s_1) f_1\Bigg(\lambda_2 \int_0^1 G(s_1, s_2) p_2(s_2) \cdots \\
f_n\Big(u_1(s_n)\Big) ds_n\Big) \cdots ds_2\Bigg) ds_1 \geqslant H_2,
\end{aligned}$$

so that $Tu_1(t) \geqslant H_2 = \|u_1\|$. Hence, $\|Tu_1\| \geqslant \|u_1\|$. So if we set

$$\Omega_2 = \Big\{x \in X \mid \|x\| < H_2\Big\},$$

then

$$\|Tu_1\| \geqslant \|u_1\|, \text{ for } u_1 \in P \cap \partial\Omega_2. \tag{10}$$

Applying Theorem 1 to (9) and (10), we obtain that T has a fixed point $u_1 \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$. As such, setting $u_1 = u_{n+1}$, we obtain a positive solution $(u_1, u_2, \cdots, u_n)$ of (1)–(2) given iteratively by

$$u_i(t) = \lambda_i \int_0^1 G(t, s) p_i(s) f_i\Big(u_{i+1}(s)\Big) ds, \; i = n, n-1, \cdots, 1.$$

The proof is completed. □

Prior to our next result, we define the positive numbers $\mathcal{N}_3$ and $\mathcal{N}_4$ by

$$\mathcal{N}_3 = \max_{1 \leqslant i \leqslant n} \left\{ \left[\tau^{q_1-1} \int_\tau^1 G(1, s) p_i(s) ds f_{i0} \right]^{-1} \right\}$$

and

$$\mathcal{N}_4 = \min_{1\leqslant i\leqslant n}\left\{\left[\int_0^1 G(1,s)p_i(s)ds f_{i\infty}\right]^{-1}\right\}.$$

THEOREM 3. *Assume that the conditions* $(A1)-(A4)$ *are satisfied. Then, for each* $\lambda_1,\lambda_2,\cdots,\lambda_n$ *satisfying*

$$\mathcal{N}_3 < \lambda_i < \mathcal{N}_4,\ \ 1\leqslant i\leqslant n, \tag{11}$$

there exists an n*-tuple* $(u_1,u_2,\cdots,u_n)$ *satisfying* (1)–(2) *such that* $u_i(t)>0,\ 1\leqslant i\leqslant n$ *on* $(0,1)$.

Proof. Let λ_i, $1\leqslant i\leqslant n$ be given as in (11). Now, let $\varepsilon>0$ be chosen such that

$$\max_{1\leqslant i\leqslant n}\left\{\left[\tau^{q_1-1}\int_\tau^1 G(1,s)p_i(s)ds(f_{i0}-\varepsilon)\right]^{-1}\right\}\leqslant \min_{1\leqslant i\leqslant n}\lambda_i$$

and

$$\max_{1\leqslant i\leqslant n}\lambda_i \leqslant \min_{1\leqslant i\leqslant n}\left\{\left[\int_0^1 G(1,s)p_i(s)ds(f_{i\infty}+\varepsilon)\right]^{-1}\right\}.$$

Let T be the cone preserving, completely continuous operator that was defined by (6). From the definition of f_{i0}, $1\leqslant i\leqslant n$ there exists $\overline{H}_3>0$ such that, for each $1\leqslant i\leqslant n$,

$$f_i(x)\geqslant (f_{i0}-\varepsilon)x,\quad 0<x\leqslant \overline{H}_3.$$

Also, from the definitions of f_{i0}, it follows that $f_{i0}(0)=0,\ \ 1\leqslant i\leqslant n$, and so there exist $0<K_n<K_{n-1}<\cdots<K_2<\overline{H}_3$ such that

$$\lambda_i f_i(t)\leqslant \frac{K_{i-1}}{\int_0^1 G(1,s)p_i(s)ds},\quad t\in[0,K_i],\ \ 3\leqslant i\leqslant n,\ \text{and}$$

$$\lambda_2 f_2(t)\leqslant \frac{\overline{H}_3}{\int_0^1 G(1,s)p_2(s)ds},\quad t\in[0,K_2].$$

Choose $u_1\in P$ with $\|u_1\|=K_n$. Then, we have

$$\begin{aligned}\lambda_n\int_0^1 G(s_{n-1},s_n)p_n(s_n)f_n\Big(u_1(s_n)\Big)ds_n &\leqslant \lambda_n\int_0^1 G(1,s_n)p_n(s_n)f_n\Big(u_1(s_n)\Big)ds_n\\ &\leqslant \frac{\int_0^1 G(1,s_n)p_n(s_n)K_{n-1}ds_n}{\int_0^1 G(1,s_n)p_n(s_n)ds_n}=K_{n-1}.\end{aligned}$$

Continuing with this bootstrapping argument, it follows that

$$\lambda_2\int_0^1 G(1,s_2)p_2(s_2)f_2\Bigg(\lambda_3\int_0^1 G(s_2,s_3)p_3(s_3)\cdots f_n\Big(u_1(s_n)\Big)ds_n\Bigg)\cdots ds_3\Bigg)ds_2\leqslant \overline{H}_3.$$

Then,

$$Tu_1(t) = \lambda_1 \int_0^1 G(t,s_1)p_1(s_1)f_1\Bigg(\lambda_2 \int_0^1 G(s_1,s_2)p_2(s_2)\cdots$$

$$f_n\Big(u_1(s_n)\Big)ds_n\Big)\cdots ds_2\Bigg)ds_1$$

$$\geqslant \tau^{q_1-1}\lambda_1 \int_\tau^1 G(1,s_1)p_1(s_1)(f_{10}-\varepsilon)\|u_1\|ds_1 = \|u_1\|.$$

So, $\|Tu_1\| \geqslant \|u_1\|$. If we put $\Omega_1 = \Big\{x \in X \mid \|x\| < K_n\Big\}$, then

$$\|Tu_1\| \geqslant \|u_1\|, \text{ for } u_1 \in P \cap \partial\Omega_1. \tag{12}$$

Since each $f_{i\infty}$ is assumed to be a positive real number, it follows that f_i, $1 \leqslant i \leqslant n$, is unbounded at ∞.

For each $1 \leqslant i \leqslant n$, set $f_i^*(x) = \sup\limits_{0 \leqslant s \leqslant x} f_i(s)$. Then, it is straightforward that, for each $1 \leqslant i \leqslant n$, f_i^* is a nondecreasing real-valued function, $f_i \leqslant f_i^*$ and

$$\lim_{x\to\infty} \frac{f_i^*(x)}{x} = f_{i\infty}.$$

Next, by definition of $f_{i\infty}$, $1 \leqslant i \leqslant n$, there exists $\overline{H}_4$ such that, for each $1 \leqslant i \leqslant n$,

$$f_i^*(x) \leqslant (f_{i\infty}+\varepsilon)x, \quad x \geqslant \overline{H}_4.$$

It follows that there exists $H_4 > \max\Big\{2\overline{H}_3, \overline{H}_4\Big\}$ such that, for each $1 \leqslant i \leqslant n$,

$$f_i^*(x) \leqslant f_i^*(H_4), \ \ 0 < x \leqslant H_4.$$

Choose $u_1 \in P$ with $\|u_1\| = H_4$. Then, using the usual bootstrapping argument, we have

$$\begin{aligned}
Tu_1(t) &= \lambda_1 \int_0^1 G(t,s_1)p_1(s_1)f_1(\lambda_2\cdots)ds_1 \\
&\leqslant \lambda_1 \int_0^1 G(t,s_1)p_1(s_1)f_1^*(\lambda_2\cdots)ds_1 \\
&\leqslant \lambda_1 \int_0^1 G(1,s_1)p_1(s_1)f_1^*(H_4)ds_1 \\
&\leqslant \lambda_1 \int_0^1 G(1,s_1)p_1(s_1)ds_1(f_{1\infty}+\varepsilon)H_4 \\
&\leqslant H_4 = \|u_1\|,
\end{aligned}$$

and so $\|Tu_1\| \leqslant \|u_1\|$. So, if we let $\Omega_2 = \Big\{x \in X \mid \|x\| < H_4\Big\}$, then

$$\|Tu_1\| \leqslant \|u_1\|, \text{ for } u_1 \in P \cap \partial\Omega_2. \tag{13}$$

Applying Theorem 1 to (12)–(13), we obtain that T has a fixed point $u_1 \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$, which in turn with $u_1 = u_{n+1}$, yields an n-tuple $(u_1, u_2, \cdots, u_n)$ satisfying (1)–(2) for the chosen values of λ_i, $1 \leqslant i \leqslant n$. The proof is completed. $\square$

4. Example

In this section, as an application, we demonstrate our results with an example. Consider the system of fractional order boundary value problem,

$$\left.\begin{aligned} D_{0^+}^{2.5}u_1(t)+\lambda_1 p_1(t)f_1(u_2)=0, \quad t\in(0,1),\\ D_{0^+}^{2.5}u_2(t)+\lambda_2 p_2(t)f_2(u_3)=0, \quad t\in(0,1),\\ D_{0^+}^{2.5}u_3(t)+\lambda_3 p_3(t)f_3(u_1)=0, \quad t\in(0,1), \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} &u_i(0)=0, \quad u_i'(0)=0,\\ &5D_{0^+}^{1.5}u_i(1)=2D_{0^+}^{1.5}u_i\left(\frac{1}{2}\right), \quad i=1,2,3, \end{aligned}\right\} \tag{15}$$

where

$$f_1(u_2)=\frac{u_2\left[1050-999.35e^{-3u_2}\right]\left[870-798.5e^{-5u_2}\right]}{1+99\sqrt{\pi}},$$

$$f_2(u_3)=\frac{u_3\left[3000-2984.65e^{-7u_3}\right]\left[567-489.5e^{-u_3}\right]}{3+87\sqrt{\pi}},$$

$$f_3(u_1)=\frac{u_1\left[1200-986.25e^{-2u_1}\right]\left[1050-999.2e^{-3u_1}\right]}{2+78\sqrt{\pi}},$$

and $p_1(t)=p_2(t)=p_3(t)=1$. Then the Green's function $G(t,s)$ for the boundary value problem,

$$D_{0^+}^{2.5}u_1(t)+h(t)=0, \quad t\in(0,1),$$

$$u_1(0)=0, \quad u_1'(0)=0, \quad 5D_{0^+}^{1.5}u_1(1)=2D_{0^+}^{1.5}u_1\left(\frac{1}{2}\right),$$

and is given by

$$G(t,s)=\begin{cases} \underset{t\in[0,\frac{1}{2}]}{G_1(t,s)} = \begin{cases} G_{11}(t,s), \ 0\leqslant t\leqslant s\leqslant \frac{1}{2}<1,\\ G_{12}(t,s), \ 0\leqslant s\leqslant \min\{t,\frac{1}{2}\}<1, \end{cases}\\[2ex] \underset{t\in[\frac{1}{2},1]}{G(t,s)} = \begin{cases} G_{21}(t,s), \ 0\leqslant \max\{t,\frac{1}{2}\}\leqslant s\leqslant 1,\\ G_{22}(t,s), \ 0<\frac{1}{2}\leqslant s\leqslant t\leqslant 1, \end{cases} \end{cases}$$

$$\begin{aligned} G_{11}(t,s) &= \tfrac{25}{112}\left[5t^{1.5}(1-s)^{0.3}-2t^{1.5}(\tfrac{1}{2}-s)^{1.5}\right],\\ G_{12}(t,s) &= \tfrac{25}{112}\left[5t^{1.5}(1-s)^{0.3}-3.38(t-s)^{1.5}-2t^{1.5}(\tfrac{1}{2}-s)^{1.5}\right],\\ G_{21}(t,s) &= \tfrac{25}{112}\left[5t^{1.5}(1-s)^{0.3}\right],\\ G_{22}(t,s) &= \tfrac{25}{112}\left[5t^{1.5}(1-s)^{0.3}-3.38(t-s)^{1.5}\right]. \end{aligned}$$

By direct calculations, we determine $\tau=0.65, \tau^{1.5}=0.52405$, $f_{10}=20.5214$, $f_{20}=50.5308$, $f_{30}=77.4216$, $f_{1\infty}=5176.4312$, $f_{2\infty}=10820.3708$ and $f_{3\infty}=8983.8675$. Employing Theorem 2, we get an eigenvalue interval $0.00033829<\lambda_i<0.01504484$, for $i=1,2,3$ in which the fractional order boundary value problem (14)–(15) has at least one positive solution.

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