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ON THE CAUCHY PROBLEM OF A DELAY STOCHASTIC DIFFERENTIAL EQUATION OF ARBITRARY (FRACTIONAL) ORDERS

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Abstract. In this work, we are concerned with the Cauchy problem of a delay stochastic differential equation of arbitrary (fractional) orders. The existence (local) of a unique mean square continuous solution is proved. The continuous dependence of the solution on the initial random variable is studied.

1. Introduction

Let $I=[a,b]$. Let (Ω,F,P) be a fixed probability space, where Ω is a sample space, F is a σ-algebra and P is a probability measure.

We denote by $L_2(\Omega)$ the Banach space of random variables $X:\Omega\to R$ such that $\int_\Omega X^2\,dP<\infty$.

Let $X(t;\omega)=\{X(t),\ t\in I,\ \omega\in\Omega\}$ be a second order stochastic process, i.e., $E(X^2(t))<\infty$, $t\in I$.

Let $\mathfrak{R}=\mathfrak{R}(I,L_2(\Omega))$ be the class of all second order stochastic processes which are mean square (m.s.) Riemann integrable on I i.e.,

$$\int_a^b E(X^2(t))\,dt<\infty$$

The norm of $X\in\mathfrak{R}(I,L_2(\Omega))$ is given by

$$||X||_{\mathfrak{R}}^2=\int_a^b E(X^2(t))\,dt$$

DEFINITION 1. Let $X\in\mathfrak{R}(I,L_2(\Omega))$ and $\beta\in(0,1]$. The stochastic fractional order integral $I_a^\beta X$ is defined by

$$I_a^\beta X(t)=\int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}X(s)\,ds. \tag{1}$$

For the existence and properties of the integral $I_a^\beta X$ we have the following theorem ([4]–[7]).

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THEOREM 1. *Let $\alpha,\beta \in (0,1)$. If $X \in \Re(I,L_2(\Omega))$, then $I_a^\beta X$ exists in m.s. sense as a second order Riemann integrable stochastic process $I_a^\beta X \in \Re(I,L_2(\Omega))$. Further, the the following properties hold.*

(1) $I_a^\beta : \Re(I,L_2(\Omega)) \longrightarrow \Re(I,L_2(\Omega))$

(2) $I_a^\alpha I_a^\beta X(t) = I_a^\beta I_a^\alpha X(t) = I_a^{\alpha+\beta} X(t)$

(3) $L.i.m_{\beta\to 1} I_a^\beta X(t) = I_a X(t) = \int_a^t X(s)ds$

where L.i.m denote the mean square limit.

Let $C = C(I,L_2(\Omega))$ be the space of all second order stochastic processes which are mean square (m.s.) continuous on I. This space is a Banach space endowed with the norm

$$||X||_C = \sup_t \|X(t)\|_2, \quad \text{where} \quad \|X(t)\|_2 = (E(X^2(t)))^{\frac{1}{2}}.$$

DEFINITION 2. Let $X \in C^1(I,L_2(\Omega))$ (be a second order stochastic process which is m.s. differentiable with m.s. continuous derivative). The Caputo fractional derivative of order $\alpha \in (0,1]$ of the process X, denoted by $D_a^\alpha X(t)$ is defined by ([4]–[7])

$$D_a^\alpha X(t) = I_a^{1-\alpha} \frac{d}{dt} X(t) \in C(I,L_2(\Omega)). \tag{2}$$

For the properties of the fractional order stochastic derivative, we have the following theorem ([4]).

THEOREM 2. *Let $X \in C^1(I,L_2(\Omega))$, and $\alpha \in (0,1]$, then*

(1) $L.i.m_{\alpha\to 1} D_a^\alpha X(t) = \frac{d}{dt} X(t)$

(2) $L.i.m_{\alpha\to 0} D_a^\alpha X(t) = X(t) - X(a)$

(3) $I_a^\alpha D_a^\alpha X(t) = X(t) - X(a)$

(4) $D_a^\alpha I_a^\alpha X(t) = X(t)$.

For other properties of the fractional order derivative see ([4]).

When $a = 0$ we denote by D^α and I^β the operators D_a^α and I_a^β respectively.

Let $\alpha \in (0,1]$, ϕ_1, $\phi_2 : [0,T] \longrightarrow [0,T]$. Suppose that $\phi_1(t)$, $\phi_2(t) \leqslant t$ and X_0 be a random variable with $E(X_0)^2 < \infty$. Consider the Cauchy problem of the delay stochastic differential equation of arbitrary (fractional) orders

$$\frac{d}{dt} X(t) = f(t, X(\phi_1(t)), D^\alpha X(\phi_2(t))), \qquad t \in (0,T] \tag{3}$$

$$X(0) = X_0. \tag{4}$$

The delay fractional order differential equations have studied by some authors (see for example ([1]), ([3]) and ([9])).

Here we are concerned with the Cauchy problem (3)–(4). The existence of a unique solution $X \in C([0,T],L_2(\Omega))$ of the problem (3)–(4) is proved. The continuous dependence, of this solution, on the initial random variable X_o is studied.

2. Existence of solution

Consider the Cauchy the problem (3)–(4) under the following assumptions

(i) $f : [0,T] \times L_2(\Omega) \times L_2(\Omega) \longrightarrow L_2(\Omega)$ is m.s. continuous and satisfies Lipschitz condition

$$\|f(t,X_1,Y_1) - f(t,X_2,Y_2)\|_2 \leqslant k[\|X_1 - X_2\|_2 + \|Y_1 - Y_2\|_2],$$

where k is constant

(ii) $\phi_1,\ \phi_2 : [0,T] \longrightarrow [0,T]$ are continuous real valued functions, $\phi_1(t),\ \phi_2(t) \leqslant t$, and ϕ_2 has a bounded derivative such that $\|\phi_2'\|_2 \leqslant M$.

Now, let $\frac{d}{dt}X(t) = Y(t) \in C([0,T],L_2(\Omega))$, then we obtain (see [8])

$$X(t) = X_0 + \int_0^t Y(s)\, ds \in C^1([0,T],L_2(\Omega)) \tag{5}$$

and

$$X(\phi_1(t)) = X_0 + \int_0^{\phi_1(t)} Y(s)\, ds$$

where Y is the solution of the stochastic functional integral equation

$$Y(t) = f(t,X_0 + \int_0^{\phi_1(t)} Y(s)\, ds, I^{1-\alpha}\phi_2'Y(\phi_2(t))). \tag{6}$$

Hence we have proved the following lemma.

LEMMA 1. *Let the assumptions (i)–(ii) be satisfied. The solution of the problem (3)–(4) can be represented by equation (5) where y is the solution of the stochastic functional integral equation (6).*

THEOREM 3. *Let the assumptions (i)–(ii) be satisfied. If*

$$K = \left(kT + \frac{kMT^{1-\alpha}}{\Gamma(2-\alpha)}\right) < 1,$$

then the stochastic functional integral equation (6) has (locally) a unique solution $Y \in C([0,T],L_2(\Omega))$.

Proof. Define the operator F as the following

$$FY(t) = f(t,X_0 + \int_0^{\phi_1(t)} Y(s)\, ds, I^{1-\alpha}\phi_2'Y(\phi_2(t))).$$

Now we prove that

$$F : C([0,T],L_2(\Omega)) \longrightarrow C([0,T],L_2(\Omega)).$$

For this, let $t_1,t_2 \in [0,T]$ such that $|t_2-t_1|<\delta$ and $Y \in C([0,T],L_2(\Omega))$, then we have

$$\begin{aligned}
FY(t_2)-FY(t_1) &= f(t_2,X_0+\int_0^{\phi_1(t_2)} Y(s)\,ds,\ I^{1-\alpha}\phi_2' Y(\phi_2(t))|_{t=t_2}) \\
&\quad -f(t_1,X_0+\int_0^{\phi_1(t_1)} Y(s)\,ds,\ I^{1-\alpha}\phi_2' Y(\phi_2(t))|_{t=t_1}) \\
&= f(t_2,X_0+\int_0^{\phi_1(t_2)} Y(s)\,ds, I^{1-\alpha}\phi_2' Y(\phi_2(t))|_{t=t_2}) \\
&\quad -f(t_2,X_0+\int_0^{\phi_1(t_2)} Y(s)\,ds,\ I^{1-\alpha}\phi_2' Y(\phi_2(t))|_{t=t_1}) \\
&\quad +f(t_2,X_0+\int_0^{\phi_1(t_2)} Y(s)\,ds, I^{1-\alpha}\phi_2' Y(\phi_2(t))|_{t=t_1}) \\
&\quad -f(t_1,X_0+\int_0^{\phi_1(t_1)} Y(s)\,ds,\ I^{1-\alpha}\phi_2' Y(\phi_2(t))|_{t=t_1}) \\
&= (f(t_2,X_0+\int_0^{\phi_1(t_2)} Y(s)\,ds, I^{1-\alpha}\phi_2' Y(\phi_2(t))|_{t=t_2}) \\
&\quad -f(t_2,X_0+\int_0^{\phi_1(t_2)} Y(s)\,ds,\ I^{1-\alpha}\phi_2' Y(\phi_2(t))|_{t=t_1})) \\
&\quad +(f(t_2,X_0+\int_0^{\phi_1(t_2)} Y(s)\,ds, I^{1-\alpha}\phi_2' Y(\phi_2(t))|_{t=t_1}) \\
&\quad -f(t_2,X_0+\int_0^{\phi_1(t_1)} Y(s)\,ds,\ I^{1-\alpha}\phi_2' Y(\phi_2(t))|_{t=t_1}) \\
&\quad +f(t_2,X_0+\int_0^{\phi_1(t_1)} Y(s)\,ds, I^{1-\alpha}\phi_2' Y(\phi_2(t))|_{t=t_1}) \\
&\quad -f(t_1,X_0+\int_0^{\phi_1(t_1)} Y(s)\,ds,\ I^{1-\alpha}\phi_2' Y(\phi_2(t))|_{t=t_1})).
\end{aligned}$$

Then

$$\begin{aligned}
\|FY(t_2)-FY(t_1)\|_2 &\leqslant \|f(t_2,X_0+\int_0^{\phi_1(t_2)} Y(s)\,ds,\ I^{1-\alpha}\phi_2' Y(\phi_2(t))|_{t=t_2}) \\
&\quad -f(t_2,X_0+\int_0^{\phi_1(t_2)} Y(s)\,ds,\ I^{1-\alpha}\phi_2' Y(\phi_2(t))|_{t=t_1}))\|_2 \\
&\quad +\|f(t_2,X_0+\int_0^{\phi_1(t_2)} Y(s)\,ds,\ I^{1-\alpha}\phi_2' Y(\phi_2(t))|_{t=t_1}) \\
&\quad -f(t_2,X_0+\int_0^{\phi_1(t_1)} Y(s)\,ds, I^{1-\alpha}\phi_2' Y(\phi_2(t))|_{t=t_1})\|_2 \\
&\quad +\|f(t_2,X_0+\int_0^{\phi_1(t_1)} Y(s)\,ds,\ I^{1-\alpha}\phi_2' Y(\phi_2(t))|_{t=t_1}) \\
&\quad -f(t_1,X_0+\int_0^{\phi_1(t_1)} Y(s)\,ds, I^{1-\alpha}\phi_2' Y(\phi_2(t))|_{t=t_1})\|_2
\end{aligned}$$

$$\begin{aligned}
&\leqslant k\|I^{1-\alpha}\phi_2'Y(\phi_2(t))|_{t=t_2} - I^{1-\alpha}\phi_2'Y(\phi_2(t))|_{t=t_1}\|_2 \\
&\quad + k\|X_0 + \int_0^{\phi_1(t_2)} Y(s)\,ds - X_0 - \int_0^{\phi_1(t_1)} Y(s)\,ds\|_2 \\
&\quad + \|f(t_2, X_0 + \int_0^{\phi_1(t_1)} Y(s)\,ds, I^{1-\alpha}\phi_2'Y(\phi_2(t))|_{t=t_1}) \\
&\quad - f(t_1, X_0 + \int_0^{\phi_1(t_1)} Y(s)\,ds,\ I^{1-\alpha}\phi_2'Y(\phi_2(t))|_{t=t_1})\|_2 \\
&\leqslant kM\|Y\|_C \left| \int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)}\,ds - \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)}\,ds \right| \\
&\quad + k\| \int_0^{\phi_1(t_1)} Y(s)\,ds + \int_{\phi_1(t_1)}^{\phi_1(t_2)} Y(s)\,ds - \int_0^{\phi_1(t_1)} Y(s)\,ds\|_2 \\
&\quad + \|f(t_2, X_0 + \int_0^{\phi_1(t_1)} Y(s)\,ds, I^{1-\alpha}\phi_2'Y(\phi_2(t))|_{t=t_1}) \\
&\quad - f(t_1, X_0 + \int_0^{\phi_1(t_1)} Y(s)\,ds,\ I^{1-\alpha}\phi_2'Y(\phi_2(t))|_{t=t_1})\|_2 \\
&\leqslant kM\|Y\|_C \left(\frac{t_2^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{t_1^{1-\alpha}}{\Gamma(2-\alpha)} \right) + kM(\phi_1(t_2) - \phi_1(t_1)) \\
&\quad + \|f(t_2, X_0 + \int_0^{\phi_1(t_1)} Y(s)\,ds, I^{1-\alpha}\phi_2'Y(\phi_2(t))|_{t=t_1}) \\
&\quad - f(t_1, X_0 + \int_0^{\phi_1(t_1)} Y(s)\,ds,\ I^{1-\alpha}\phi_2'Y(\phi_2(t))|_{t=t_1})\|_2.
\end{aligned}$$

This proves that $F : C([0,T], L_2(\Omega)) \longrightarrow C([0,T], L_2(\Omega))$.

Secondly, we prove that F is contraction.

For this, let $X,\ Y \in C([0,T], L_2(\Omega))$, then we have

$$\begin{aligned}
&\|FY(t) - FX(t)\|_2 \\
&\leqslant \|f(t, X_0 + \int_0^{\phi_1(t)} Y(s)\,ds,\ I^{1-\alpha}\phi_2'(t)Y(\phi_2(t))) \\
&\quad - f(t, X_0 + \int_0^{\phi_1(t)} X(s)\,ds, I^{1-\alpha}\phi_2'(t)X(\phi_2(t)))\|_2 \\
&\leqslant k\|X_0 + \int_0^{\phi_1(t)} Y(s)\,ds - X_0 - \int_0^{\phi_1(t)} X(s)\,ds\|_2 \\
&\quad + k\|I^{1-\alpha}\phi_2'(t)Y(\phi_2(t)) - I^{1-\alpha}\phi_2'(t)X(\phi_2(t))\|_2 \\
&\leqslant k\| \int_0^{\phi_1(t)} Y(s)\,ds - \int_0^{\phi_1(t)} X(s)\,ds\|_2 + \|I^{1-\alpha}\phi_2'(t)(Y(\phi_2(t)) - X(\phi_2(t)))\|_2 \\
&\leqslant k\| \int_0^{\phi_1(t)} (Y(s) - X(s))\,ds\|_2 + k\mid I^{1-\alpha}\phi_2'(t)\mid \|(Y(\phi_2(t)) - X(\phi_2(t)))\|_C \\
&\leqslant k\phi_1(t)\|Y - X\|_C + k\left|\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right| M\|Y - X\|_C
\end{aligned}$$

$$\leqslant kT\|Y-X\|_C+\frac{kMT^{1-\alpha}}{\Gamma(2-\alpha)}\|Y-X\|_C$$
$$\leqslant \left(kT+\frac{kMT^{1-\alpha}}{\Gamma(2-\alpha)}\right)\|Y-X\|_C$$
$$\leqslant K\|Y-X\|_C.$$

Hence

$$\|FY-FX\|_C\leqslant K\|Y-X\|_C.$$

Since $K=(kT+\frac{kMT^{1-\alpha}}{\Gamma(2-\alpha)})<1$ (by assumption), F is contraction operator. By the Banach fixed point theorem [2], there exists a unique solution $Y\in C([0,T],L_2(\Omega))$ of the integral equation (6). □

Now from Lemma 1 and Theorem 3 we can prove the following corollary.

COROLLARY 1. *Let the assumptions of Theorem 3 be satisfied. Then the problem (3)–(4) has (locally) a unique solution* $X\in C([0,T],L_2(\Omega))$.

3. Continuous dependence

DEFINITION 3. The solution of the problem (3)–(4) is said to depend continuously on the initial random variable X_0 if $\forall\varepsilon>0$, $\exists\delta\ (\varepsilon)>0$ s.t. $\|X_0-\widetilde{X}_0\|_2\leqslant\delta$ implies that $\|X-\widetilde{X}\|_C\leqslant\varepsilon$.

THEOREM 4. *Let the assumptions of Theorem 3 be satisfied, then the solution of the initial value problem (3)–(4) depends continuously on the initial random variable.*

Proof. Let $\varepsilon>0$ be given. Choose $\delta=\delta\ (\varepsilon)$ such that

$$\|X_0-\widetilde{X}_0\|_2\leqslant\delta,$$

then we have

$$\|Y(t)-\widetilde{Y}(t)\|_2$$
$$\leqslant \|f(t,X_0+\int_0^{\phi_1(t)}Y(s)ds,I^{1-\alpha}\phi_2'Y(\phi_2(t)))-f(t,\widetilde{X}_0+\int_0^{\phi_1(t)}\widetilde{Y}(s)ds,I^{1-\alpha}\phi_2'\widetilde{Y}(\phi_2(t)))\|_2$$
$$\leqslant k[\|X_0+\int_0^{\phi_1(t)}Y(s)\ ds-\widetilde{X}_0-\int_0^{\phi_1(t)}\widetilde{Y}(s)\ ds\|_2+\|I^{1-\alpha}\phi_2'Y(\phi_2(t))-I^{1-\alpha}\phi_2'\widetilde{Y}(\phi_2(t))\|_2]$$
$$\leqslant k[\|X_0-\widetilde{X}_0\|_2+\int_0^{\phi_1(t)}\|Y(s)-\widetilde{Y}(s)\|_2\ ds+|\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}|\ \|\phi_2'(t)\|_2\|Y(\phi_2(t))-\widetilde{Y}(\phi_2(t))\|_2]$$
$$\leqslant k[\|X_0-\widetilde{X}_0\|_2+\phi_1(t)\|Y-\widetilde{Y}\|_C+\frac{MT^{1-\alpha}}{\Gamma(2-\alpha)}\|Y-\widetilde{Y}\|_C]$$
$$\leqslant k[\|X_0-\widetilde{X}_0\|_2+T\|Y-\widetilde{Y}\|_C+\frac{MT^{1-\alpha}}{\Gamma(2-\alpha)}\|Y-\widetilde{Y}\|_C],$$

then

$$(1-K)\|Y-\widetilde{Y}\|_C \leqslant k\|X_0-\widetilde{X}_0\|_2$$

and

$$\|Y-\widetilde{Y}\|_C \leqslant \frac{k}{1-K}\|X_0-\widetilde{X}_0\|_2.$$

Now

$$\begin{aligned}\|X(t)-\widetilde{X}(t)\|_2 &\leqslant \|X_0-\widetilde{X}_0\|_2 + t\|Y-\widetilde{Y}\|_C \\ &\leqslant \|X_0-\widetilde{X}_0\|_2 + T\|Y-\widetilde{Y}\|_C \\ &\leqslant \|X_0-\widetilde{X}_0\|_2 + \frac{Tk}{1-K}\|X_0-\widetilde{X}_0\|_2 \\ &\leqslant \left(1+\frac{Tk}{1-K}\right)\|X_0-\widetilde{X}_0\|_2 \\ &\leqslant \delta\left(1+\frac{Tk}{1-K}\right) \leqslant \varepsilon\end{aligned}$$

such that $\varepsilon = \delta(1+\frac{Tk}{1-K})$.

Hence the solution of the problem (3)–(4) depends continuously on the initial random variable X_0 for $t \in [0,T]$. □

4. Examples

Here, as an application of our results, we give the following two examples.

EXAMPLE 1. Let $\beta_1,\ \beta_2 \in (0,1]$. As ϕ_1 and ϕ_2, one can tack, for example $\phi_1(t) = \beta_1 t$ and $\phi_2(t) = \beta_2 t$.

Let the assumptions of Theorem 3 be satisfied. Then the problem

$$\frac{d}{dt}X(t) = f(t, X(\beta_1 t), D^{\alpha}X(\beta_2 t)), \qquad t \in (0,T] \tag{7}$$

$$X(0) = X_0. \tag{8}$$

has (locally) a unique solution $X \in C([0,T], L_2(\Omega))$. This solution depends continuously on the initial random variable X_0 for $t \in [0,T]$.

EXAMPLE 2. Let $\gamma_1,\ \gamma_2 \geqslant 1$. As ϕ_1 and ϕ_2, one can tack, for example $\phi_1(t) = t^{\gamma_1}$ and $\phi_2(t) = t^{\gamma_2}$.

Let the assumptions of Theorem 3 be satisfied. Then the problem

$$\frac{d}{dt}X(t) = f(t, X(t^{\gamma_1}), D^{\alpha}X(t^{\gamma_2})), \qquad t \in (0,1] \tag{9}$$

$$X(0) = X_0. \tag{10}$$

has (locally) a unique solution $X \in C([0,1], L_2(\Omega))$. This solution depends continuously on the initial random variable X_0 for $t \in [0,1]$.

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