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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR FRACTIONAL ORDER m–POINT BOUNDARY VALUE PROBLEMS

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Abstract. This paper deals with the existence of solutions for a class of fractional order differential equations having m-points boundary conditions involving the Caputo fractional derivative. Moreover the nonlinearity also depend on the Caputo fractional derivative. We obtain sufficient conditions for the existence and uniqueness of solutions via Schauder's fixed-point theorem and Banach contraction principle. We provide an example to illustrate the applicability of our results.

1. Introduction

Fractional differential equations have extensive applications in real life problems. These applications can be found in various scientific and engineering disciplines such as physics, chemistry, biology, viscoelasticity, control theory, signal processing etc. [1, 2, 3, 4, 5]. Moreover, most of the authors also considered the fractional differential equations as an object of mathematical investigations, we refer the readers to [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] and the references therein for recent development of the theory.

It is worthwhile to mention that Caputo fractional derivatives play important role in applied problems as it provides known physical interpretation for initial and boundary conditions. On the other hand, the standard Riemann-Liouville derivatives of fractional order do not provide the needed physical interpretations in most of the cases for initial and boundary conditions.

Existence theory for real world problems which can be modeled by of fractional differential equations with multi-point boundary conditions have attracted the attention of many researchers and is a rapidly growing area of investigation, [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. El-Shahed and Shammakh [24], studied existence and multiplicity of positive solutions for the m-point boundary value problem

$$
\begin{gathered}
{}^{C}D^{\alpha}_{a+}u(t)+f(t,u(t))=0;\quad a\leqslant t\leqslant b,\quad n-1\leqslant\alpha<n,\quad n>2,\\
u^{'}(a)=\sum_{i=1}^{m-2}\beta_i u^{'}(\eta_i),\quad u^{''}(a)=u^{'''}(a)=\ldots=u^{n-1}(a)=0,\quad u(b)=\sum_{i=1}^{m-2}\gamma_i u(\eta_i)
\end{gathered}
$$

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where $a < \eta_1 < \eta_2 < \ldots < \eta_{m-2} < b$, $\sum\limits_{i=1}^{m-2} \beta_i < 1$, $\sum\limits_{i=1}^{m-2} \gamma_i$, and ${}^C D^{\alpha}_{a^+}$ is the caputo fractional derivative. In [26], Khan studied existence and approximation of solutions via the method of upper and lower solutions and the generalized quasilinearization techniques to three-point boundary value problem for higher order fractional differential equations of the form

$$\begin{aligned} &{}^c D^q u(t) + f(t,u(t)) = 0; \quad t \in (0,1), \quad q \in (n-1,n], \quad n \geqslant 2, \\ &u'(0) = 0, u''(0) = 0, \ldots, u^{n-1}(0) = 0, \quad u(1) = \xi u(\eta), \quad \xi, \eta \in (0,1) \end{aligned}$$

where ${}^c D^q$ is the Caputo fractional derivative. In these cited papers, the nonlinearity f is free from the fractional order derivative. The case where the nonlinearity f explicitly depends of fractional order derivative is important theoretically as well as in application point of view and requires more efforts to study existence results. Cui et al. [25], proved existence results via Schauder's fixed point theorem for fractional differential equation of the form

$$\begin{aligned} &{}^c D^{\alpha} u(t) + f(t,u(t), {}^c D^{\beta} u(t)) = 0; \quad 0 < t < 1, \quad 3 < \alpha \leqslant 4, \\ &u(0) = u'(0) = u''(0) = 0, \quad u(1) = u(\xi), \quad 0 < \xi < 1, \end{aligned}$$

where $\beta > 0$, $\alpha - \beta \geqslant 1$. El-Sayed and Bin-Taher [27], studied existence of positive solutions of nonlocal multi-point boundary value problem

$$\begin{aligned} &u''(t) = f(t, {}^c D^{\alpha} u(t)) = 0, \quad t \in (0,1), \ \alpha \in (0,1), \\ &u(0) = 0, \quad u(1) = \sum_{k=1}^{m} a_k u(\tau_k), \quad \tau_k \in (a,b) \subset (0,1). \end{aligned}$$

Motivated by the above mentioned recent work, our main focus in this article is to investigate existence and uniqueness of solutions of fractional order differential equations with m-points boundary conditions in the following form

$$\begin{aligned} &-{}^c D^q u(t) = f(t,u(t), {}^c D^{q-1} u(t)); \quad 0 < t < 1, \quad 1 < q \leqslant 2, \\ &u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \delta_i u(\eta_i), \end{aligned} \tag{1}$$

where ${}^c D^q$ is the Caputo fractional derivative; $\delta_i, \eta_i \in (0,1)$ with $\sum\limits_{i=1}^{m-2} \delta_i \eta_i < 1$, and $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ explicitly depends on the fractional order derivative.

We organized rest of the paper as follows: In section 2, we provide some preliminary results. Section 3 contains derivation of the Green's function, formulation of integral representation the problem, sufficient conditions for the existence and uniqueness of solutions via Schauder's fixed point theorem and Banach's contraction principle. At the end, an example is given to justify the application of our results.

2. Background materials and lemmas

In this section, we recall some basic definitions, lemmas and notations. The Banach space of all continuous functions from $J \to \mathbb{R}$ with the usual norm $\|u\|_\infty = sup\{|u(t)| : 0 \leqslant t \leqslant 1\}$, is denoted by $C(J,\mathbb{R})$. The Banach space of functions $u : J \to \mathbb{R}$ that are Lebesgue integrable with the norm $\|u\|_{L^1} = \int_0^1 |u(t)|dt$ is denoted by $L^1(J,\mathbb{R})$.

DEFINITION 2.1. [2, 3, 4, 5] The fractional integral of order $q \in \mathbb{R}_+$ of the function $h \in L^1([a,b],\mathbb{R})$ is defined as

$$I^q_{a+}h(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} h(s)\,ds.$$

When $a = 0$, we write $I^q h(t) = [h * \varphi_q](t)$, where $\varphi_q(t) = \frac{t^{q-1}}{\Gamma(q)}$ for $t > 0$, $\varphi_q(t) = 0$ for $t \leqslant 0$ and $\varphi_a \to \delta(t)$ as $q \to 0$, where δ is the delta function.

DEFINITION 2.2. [2, 3, 4, 5] The Caputo fractional order derivative of a function h on the interval $[a,b]$ is defined by

$${}^cD^q_{a+}h(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-s)^{n-q-1} h^{(n)}(s)\,ds,$$

where $n = [q]+1$.

Further details on fractional derivatives and integrals can be found in [2, 3, 4, 5].

LEMMA 2.3. [14] *The fractional order differential equation of order $q > 0$ of the form*

$${}^cD^q h(t) = 0, \ \ n-1 < q \leqslant n,$$

has a unique solution of the form $h(t) = C_0 + C_1 t + C_2 t^2 + \ldots + C_{n-1} t^{n-1}$, where $C_i \in R$, $i = 0,1,2,\ldots,n-1$.

LEMMA 2.4. [14] *The following result holds for a fractional derivative and integral of order q*

$$I^{q\,c}D^q h(t) = h(t) + C_0 + C_1 t + C_2 t^2 + \ldots + C_{n-1} t^{n-1},$$

for arbitrary $C_i \in R$, $i = 0,1,2,\ldots,n-1$.

LEMMA 2.5. [28] *The space $\widetilde{C}$ defined by*

$$\widetilde{C}(J,\mathbb{R}) = \{u \in C(J,\mathbb{R}) :^c D^{q-1} u \in C(J,\mathbb{R})\}$$

with the norm

$$\|u\|_{\widetilde{C}} = \max\{\|u\|_\infty, \|{}^cD^{q-1}u\|_\infty\},$$

is a Banach space.

3. Main results

In this section, we study existence and uniqueness of solution of the BVP (1).

LEMMA 3.1. *For $h \in L^1(J,\mathbb{R})$, the BVP for fractional differential equation*

$$\begin{aligned} &{}^cD^q u(t) + h(t) = 0; \quad 0 < t < 1, \ 1 < q \leqslant 2, \\ &u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \delta_i u(\eta_i), \end{aligned} \tag{2}$$

has a solution u of the form $u(t) = \int_0^1 G(t,s)h(s)\,ds$, where $G(t,s)$ is the Green function and is given by

$$G(t,s) = \frac{1}{\Gamma(q)} \begin{cases} \frac{t}{1-\Delta}\left[(1-s)^{q-1} - \sum_{j=i}^{m-2} \delta_j(\eta_j - s)^{q-1}\right] - (t-s)^{q-1}; & s \leqslant t, \ \eta_{i-1} < s \leqslant \eta_i, \\ & i = 1,2,\dots,m-1, \\ \frac{t}{1-\Delta}\left[(1-s)^{q-1} - \sum_{j=i}^{m-2} \delta_j(\eta_j - s)^{q-1}\right]; & t \leqslant s, \ \eta_{i-1} < s \leqslant \eta_i, \\ & i = 1,2,\dots,m-1. \end{cases}, \tag{3}$$

where $\Delta = \sum_{i=1}^{m-2} \delta_i \eta_i < 1$.

Proof. Applying I^q on $-{}^cD^q u(t) = h(t)$ and using Lemma (2.4), we have

$$u(t) = -I^q h(t) - C_0 - C_1 t \tag{4}$$

for some $C_0, C_1 \in R$. The boundary condition $u(0) = 0$ implies $C_0 = 0$ and the boundary condition $u(1) = \sum_{i=1}^{m-2} \delta_i u(\eta_i)$ yields $C_1 = \frac{1}{1-\Delta}\left[-I^q h(1) + \sum_{i=1}^{m-2} \delta_i I^q h(\eta_i)\right]$, It follows that

$$u(t) = -I^q h(t) + \frac{t}{1-\Delta}\left[I^q h(1) - \sum_{i=1}^{m-2} \delta_i I^q h(\eta_i)\right]. \tag{5}$$

For $0 \leqslant t \leqslant \eta_1$, (5) implies

$$\begin{aligned} u(t) = &\int_0^t \left[-\frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{t}{\Gamma(q)(1-\Delta)}\left((1-s)^{q-1} - \sum_{j=1}^{m-2} \delta_j(\eta_j - s)^{q-1}\right)\right] h(s)\,ds \\ &+ \frac{t}{\Gamma(q)(1-\Delta)} \int_t^{\eta_1} \left((1-s)^{q-1} - \sum_{j=1}^{m-2} \delta_j(\eta_j - s)^{q-1}\right) h(s)\,ds \\ &+ \sum_{i=2}^{m-2} \int_{\eta_{i-1}}^{\eta_i} \left((1-s)^{q-1} - \sum_{j=1}^{m-2} \delta_j(\eta_j - s)^{q-1}\right) h(s)\,ds + \int_{\eta_{m-2}}^1 (1-s)^{q-1} h(s)\,ds, \end{aligned}$$

and for $\eta_{l-1} \leqslant t \leqslant \eta_l$, $2 \leqslant l \leqslant m-2$, we obtain

$$\begin{aligned}
u(t) = &\int_0^{\eta_1} \left[-\frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{t}{\Gamma(q)(1-\Delta)} \left((1-s)^{q-1} - \sum_{j=1}^{m-2} \delta_j (\eta_j - s)^{q-1} \right) \right] h(s)\,ds \\
&+ \sum_{i=2}^{m-2} \int_{\eta_{i-1}}^{\eta_i} \Bigg[-\frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{t}{\Gamma(q)(1-\Delta)} \left((1-s)^{q-1} - \sum_{j=i}^{m-2} \delta_j (\eta_j - s)^{q-1} \right) \\
&+ (1-s)^{q-1} \Bigg] h(s)\,ds \\
&+ \int_{\eta_{l-1}}^{t} \left[-\frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{t}{\Gamma(q)(1-\Delta)} \left((1-s)^{q-1} - \sum_{j=l}^{m-2} \delta_j (\eta_j - s)^{q-1} \right) \right] h(s)\,ds \\
&+ \frac{t}{\Gamma(q)(1-\Delta)} \left[\int_t^{\eta_l} \left((1-s)^{q-1} - \sum_{j=l}^{m-2} \delta_j (\eta_j - s)^{q-1} \right) \right] h(s)\,ds \\
&+ \sum_{i=l+1}^{m-2} \int_{\eta_{i-1}}^{\eta_i} \left((1-s)^{q-1} - \sum_{j=i}^{m-2} \delta_j (\eta_j - s)^{q-1} \right) h(s)\,ds + \int_{\eta_{m-2}}^{1} (1-s)^{q-1} h(s)\,ds.
\end{aligned}$$

Further, for $\eta_{m-2} \leqslant t \leqslant 1$, (5) yields

$$\begin{aligned}
u(t) = &\int_0^{\eta_1} \left[-\frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{t}{\Gamma(q)(1-\Delta)} \left((1-s)^{q-1} - \sum_{j=1}^{m-2} \delta_j (\eta_j - s)^{q-1} \right) \right] h(s)\,ds \\
&+ \sum_{i=2}^{m-2} \int_{\eta_{i-1}}^{\eta_i} \left[-\frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{t}{\Gamma(q)(1-\Delta)} \left((1-s)^{q-1} - \sum_{j=i}^{m-2} \delta_j (\eta_j - s)^{q-1} \right) \right] h(s)\,ds \\
&+ \frac{t}{\Gamma(q)(1-\Delta)} \left[\int_{\eta_{m-2}}^{t} \left((1-\Delta)(t-s)^{q-1} + (1-s)^{q-1} \right) h(s)\,ds + \int_t^1 (1-s)^{q-1} h(s)\,ds \right],
\end{aligned}$$

and the desired result follows. □

Now we study existence of solution to the BVP (1). Our result is based on the Schauder's fixed point theorem. In view of Lemma (3.1), we write the BVP (1) is an equivalent integral equation

$$u(t) = \int_0^1 G(t,s) f(s, u(s), {}^c D^{q-1} u(s))\,ds, \quad t \in J \tag{6}$$

and define operator $T : \widetilde{C}(J,\mathbb{R}) \to \widetilde{C}(J,\mathbb{R})$ by

$$Tu(t) = \int_0^1 G(t,s) f(s, u(s), {}^c D^{q-1} u(s))\,ds. \tag{7}$$

Solutions of the BVP (1) means fixed points of T. We also have

$$\begin{aligned}
{}^c D^{q-1} Tu(t) = &\int_0^t f(s, u(s), {}^c D^{q-1} u(s))\,ds + \frac{t^{2-q}}{(1-\Delta)\Gamma(q)\Gamma(3-q)} \\
&\times \left[\int_0^1 (1-s)^{q-1} u(s)\,ds - \sum_{j=1}^{m-2} \delta_j \int_0^{\eta_j} (\eta_j - s)^{q-1} u(s)\,ds \right].
\end{aligned} \tag{8}$$

THEOREM 3.2. *Assume that* $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous and the following hold*

(a) *There exist* $p \in C(J, \mathbb{R}^+)$ *and a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *such that*

$$|f(t,u,z)| \leqslant p(t)\psi(|z|) \quad for \quad t \in J \quad and \quad each \quad u,z \in \mathbb{R};$$

(b) *There exists a constant* $\rho > 0$ *such that*

$$\rho \geqslant \max_{t \in J} \left\{ \psi(\rho)p^* G^*, b\psi(\rho)p^* \frac{\Gamma(3-q)\Gamma(q+1)+1}{\Gamma(3-q)\Gamma(q+1)} \right\}, \tag{9}$$

where $p^* = sup\{p(s),\ s \in J\}$, $G^* = \sup_{t\in J} \int_0^1 |G(t,s)|ds$,

then the BVP (1) *has at least one solution such that* $|u(t)| \leqslant \rho$ *for each* $t \in J$.

Proof. First we show that the operator T defined by (7) is continuous. Let $\{u_r\}$ be a sequence such that $u_r \to u \in \widetilde{C}(J,\mathbb{R})$, and let $\sigma > 0$ be such that $\|u_r\|_{\tilde{C}} \leqslant \sigma$, $\|u\|_{\tilde{C}} \leqslant \sigma$, then, for each $t \in J$, using (7) and (8), we obtain

$$|(Tu_r)(t) - (Tu)(t)| \leqslant \int_0^1 |G(t,s)f(s,u_r(s),{}^cD^{q-1}u_r(s)) - f(s,u(s),{}^cD^{q-1}u(s))|ds,$$

and

$$\begin{aligned}
&|({}^cD^{q-1}Tu_r)(t) - ({}^cD^{q-1}Tu)(t)| \\
\leqslant &\int_0^t |f(s,u_r(s),{}^cD^{q-1}u_r(s)) - f(s,u(s),{}^cD^{q-1}(s))|ds \\
&+ \frac{t^{2-q}}{(1-\Delta)\Gamma(q)\Gamma(3-q)}\Big[\Big(\int_0^1 (1-s)^{q-1}|f(s,u_r(s),{}^cD^{q-1}u_r(s)) - f(s,u(s),{}^cD^{q-1}(s))|\Big)ds \\
&+ \sum_{j=1}^{m-2}\delta_j\Big(\int_0^{\eta_j}(\eta_j - s)^{q-1}u(s)|f(s,u_r(s),{}^cD^{q-1}u_r(s)) - f(s,u(s),{}^cD^{q-1}(s))|\Big)\Big]ds.
\end{aligned}$$

In view of the continuity of f and Lebesgue dominated convergence theorem, we have

$$\|Tu_r - Tu\|_\infty \to 0 \quad as \quad r \to \infty,$$

and

$$\|({}^cD^{q-1}Tu_r)(t) - ({}^cD^{q-1}Tu)(t)\|_\infty \to 0, \quad \text{as} \quad r \to \infty,$$

implies that T ic continuous.

Define $\mathscr{D}$ a closed and convex subset of $\widetilde{C}(J,\mathbb{R})$ as follows

$$\mathscr{D} = \{u \in \tilde{C}(J,\mathbb{R}), \|u\|_{\tilde{C}} \leqslant \rho\}.$$

We show that $T(\mathscr{D}) \subset \mathscr{D}$. For $u \in \mathscr{D}$ and $t \in J$, using (a), (b), (7) and (8), we obtain

$$(Tu)(t)) \leqslant \int_0^1 |G(t,s)||f(s,u(s),^c D^{q-1}u(s)| \leqslant \psi(\|u\|_{\widetilde{C}} P^* G^*, \tag{10}$$

and

$$\begin{aligned}
|(^c D^{q-1}Tu)(t) \leqslant &||f(s,u(s),^c D^{q-1}u(s)|ds + \frac{t^{2-q}}{(1-\Delta)\Gamma(q)\Gamma(3-q)} \\
&\times \Big(\sum_{j=1}^{m-2} \delta_j \int_0^{\eta_j} (\eta_j - s)^{q-1}|u(s)|ds + \int_0^1 (1-s)^{q-1}|u(s)|\Big) \qquad (11)\\
&\leqslant b\psi(\rho)P^* \left(\frac{\Gamma(3-q)\Gamma(q+1)+1}{\Gamma(3-q)\Gamma(q+1)}\right).
\end{aligned}$$

From (10) and (11), it follows that $\|Tu\|_{\widetilde{C}} \leqslant \rho$ which implies that $T(\mathscr{D}) \subset \mathscr{D}$.

Finally, we show that T maps $\mathscr{D}$ into an equicontinuous set of $\widetilde{C}(J,\mathbb{R})$. Take $t_1, t_2 \in J$ with $t_1 < t_2$ and $u \in \mathscr{D}$, we have

$$\begin{aligned}
|(Tu)(t_2) - (Tu)(t_1)| &\leqslant \int_0^1 |G(t_2,s) - G(t_1,s)||f(s,u(s),^c D^{q-1}u(s))|ds \\
&\leqslant p^*\psi(\rho) \int_0^1 |G(t_2,s) - G(t_1,s)|ds
\end{aligned}$$

and from the continuity of $G(t,s)$, it follows that $|(Tu)(t_2) - (Tu)(t_1)| \to 0$ as $t_2 \to t_1$. Also,

$$\begin{aligned}
&|^c D^{q-1}Tu(t_2) -^c D^{q-1}Tu(t_1)| \\
&\leqslant |\int_0^{t_2} f(s,u(s),^c D^{q-1}u(s)ds - \int_0^{t_1} f(s,u(s),^c D^{q-1}u(s)ds| \\
&\quad + \frac{t_2^{2-q} - t_1^{2-q}}{(1-\Delta)\Gamma(q)\Gamma(3-q)} \Big(\sum_{j=1}^{m-2} \delta_j \int_0^{\eta_j} (\eta_j - s)^{q-1}|u(s)|ds + \int_0^1 (1-s)^{q-1}|u(s)|ds\Big) \\
&\leqslant p^*\psi\|^c D^{q-1}u(s)\| \left\{ t_2 - t_1 + \frac{t_2^{2-q} - t_1^{q-1}}{(1-\Delta)\Gamma(q)\Gamma(3-q)}(1 + \sum_{j=1}^{m-2} \delta_j \eta_i^q) \right\},
\end{aligned}$$

which implies

$$\|(^c D^{q-1}Tu)(t_2) - (^c D^{q-1}Tu)(t_1)\| \to 0 \quad as \quad t_2 \to t_1.$$

Consequently by Arzela Ascoli Theorem, T is completely continuous. By Schauder's fixed point theorem, T has a fixed point $u \in \mathscr{D}$ for $t \in J$. □

THEOREM 3.3. *Assuming that the following hold*

(i) $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(ii) *There exists constant $k>0$ such that for each $t\in J$ and all $x,y,\bar{x},\bar{y}\in\mathbb{R}$,*

$$|f(t,x,y)-f(t,\bar{x},\bar{y})|\leqslant k(|x-\bar{x}|+|y-\bar{y}|),$$

holds. Further, if

$$\left[\max\left\{2G^{*}k,\frac{2k\Gamma(3-q)\Gamma(q+1)+1}{\Gamma(3-q)\Gamma(q+1)}\right\}\right]<1, \tag{12}$$

then the BVP (1) *has a unique solution on J.*

Proof. For the uniqueness of solutions of (1), we use Banach contraction principle. The continuity of $Tu(t)$ and ${}^{c}D^{q-1}Tu(t)$ follow from the continuity of f and G. Let $u,\bar{u}\in\widetilde{C}(J,\mathbb{R})$, then for each $t\in J$, we have

$$\begin{aligned}|(Tu)(t)-(T\bar{u})(t)|&\leqslant\int_{0}^{1}G(t,s)|f(s,u(s),{}^{c}D^{q-1}u(s))-f(s,\bar{u}(s),{}^{c}D^{q-1}\bar{u}(s))|ds\\&\leqslant G^{*}k(\|u-\bar{u}\|_{\infty}+\|{}^{c}D^{q-1}u-{}^{c}D^{q-1}\bar{u}\|_{\infty})\\&\leqslant 2G^{*}k\|u-\bar{u}\|_{\widetilde{C}},\end{aligned}$$

which implies that

$$\|Tu-T\bar{u}\|\leqslant 2G^{*}k\|u-\bar{u}\|_{\widetilde{C}}. \tag{13}$$

Also,

$$\begin{aligned}&|{}^{c}D^{q-1}(Tu)(t)-{}^{c}D^{q-1}(T\bar{u})(t)|\\&\leqslant|f(s,u(s),{}^{c}D^{q-1}u(s))-f(s,\bar{u}(s),{}^{c}D^{q-1}\bar{u}(s))|ds+\frac{t^{2-q}}{(1-\Delta)\Gamma(q)\Gamma(3-q)}\\&\quad\times\Big[\Big(\int_{0}^{1}(1-s)^{q-1}|f(s,u(s),{}^{c}D^{q-1}u(s))-f(s,\bar{u}(s),{}^{c}D^{q-1}\bar{u}(s))|\Big)ds\\&\quad-\sum_{j=1}^{m-2}\delta_{j}\Big(\int_{0}^{\eta_{j}}(\eta_{j}-s)^{q-1}|f(s,u(s),{}^{c}D^{q-1}u(s))-f(s,\bar{u}(s),{}^{c}D^{q-1}\bar{u}(s))|\Big)\Big]ds,\end{aligned}$$

which implies that

$$\begin{aligned}|{}^{c}D^{q-1}(Tu)(t)-{}^{c}D^{q-1}(T\bar{u})(t)|&\leqslant 2k\|u-\bar{u}\|_{\infty}+\frac{2k}{\Gamma(3-q)\Gamma(q+1)}\|u-\bar{u}\|_{\infty}\\&\leqslant 2k\left(\frac{\Gamma(3-q)\Gamma(q+1)+1}{\Gamma(3-q)\Gamma(q+1)}\right)\|u-\bar{u}\|_{\widetilde{C}}.\end{aligned} \tag{14}$$

From (13) and (14), it follows that

$$\|Tu-T\bar{u}\|\leqslant\max\left\{2G^{*}k,\frac{2k\Gamma(3-q)\Gamma(q+1)+1}{\Gamma(3-q)\Gamma(q+1)}\right\}\|u-\bar{u}\|_{\widetilde{C}}.$$

Hence, by Banach contraction principle, T has a unique fixed point. □

4. Example

EXAMPLE 4.1. Consider the following m-points boundary value problem

$$
\begin{aligned}
&{}^cD^{\frac{3}{2}}u(t)+\frac{1}{20e^{2t}+7}\left(\frac{1}{1+2|u(t)|+3|{}^cD^{\frac{1}{2}}u(t)|}\right), \quad t\in J:=[0,1], \quad 1<q\leqslant 2,\\
&u(0)=0, \quad u(1)=\sum_{i=1}^{m-2}\delta_i u(\eta_i)=\frac{1}{5}.
\end{aligned} \tag{15}
$$

Here $f(t,x,y)=\frac{1}{20e^{2t}+7}\left(\frac{1}{1+2|u(t)|+3|{}^cD^{\frac{1}{2}}u(t)|}\right)$ and for $x,y,\bar{x},\bar{y}\in\mathbb{R}$ and $t\in J$, we obtain

$$
|f(t,x,y)-f(t,\bar{x},\bar{y}|\leqslant\frac{1}{9}\left[|x-\bar{x}|+|y-\bar{y}|\right],
$$

which is condition (ii) of Theorem (3.3) with $k=\frac{1}{9}$.

Further, we have

$$
G(t,s)=\frac{1}{\Gamma(\frac{3}{2})}\begin{cases}\frac{5t}{4}\left[(1-s)^{\frac{1}{2}}-\sum\limits_{j=i}^{m-2}\delta_j(\eta_j-s)^{\frac{1}{2}}\right]-(t-s)^{\frac{1}{2}}; & s\leqslant t,\eta_{i-1}<s\leqslant\eta_i,\\ & i=1,2,\dots,m-1,\\ \frac{5t}{4}\left[(1-s)^{\frac{1}{2}}-\sum\limits_{j=i}^{m-2}\delta_j(\eta_j-s)^{\frac{1}{2}}\right]; & t\leqslant s,\ \eta_{i-1}<s\leqslant\eta_i,\\ & i=1,2,\dots,m-1,\end{cases} \tag{16}
$$

$$
G^*=\sup_{t\in J}\int_0^1|G(t,s)|ds<\frac{5}{3\Gamma(\pi)}
$$

and clearly

$$
\max\left\{\frac{2}{9}G^*,\frac{2\Gamma(3-q)\Gamma(q+1)+1}{9\Gamma(3-q)\Gamma(q+1)}\right\}<1. \tag{17}
$$

Hence by Theorem (3.3) the BVP has a unique solution on $[0,1]$ for each value of $q\in(1,2]$.

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