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EXISTENCE OF GLOBAL SOLUTIONS OF IMPULSIVE IVPS OF SINGULAR FRACTIONAL DIFFERENTIAL SYSTEMS ON HALF LINE

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Abstract. A impulsive boundary value problem of fractional differential equation is proposed. By constructing a novel transformation, the considered impulsive system is convert into a continuous system. We construct a weighted function space, by employing a fixed point theorem, we establish existence results for global solutions for a system of impulsive singular fractional differential equations. Am example is presented to illustrate the efficiency of the results obtained.

1. Introduction

Fractional differential equation is a generalization of ordinary differential equation to arbitrary non-integer orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventeenth century. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation [17]. Fractional differential equations therefore find numerous applications in different branches of physics, chemistry and biological sciences such as visco-elasticity, feed back amplifiers, electrical circuits, electro analytical chemistry, fractional multipoles and neuron modelling [20]. The reader may refer to the books and monographs [7, 18, 19, 21] for fractional calculus and developments on fractional differential and fractional integro-differential equations with applications.

On the other hand, the theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such characteristics arise naturally and often, for example, phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. For an introduction of the basic theory of impulsive differential equation, we refer the reader to [14].

Recently, the authors in papers [1, 3, 4, 5, 8, 22, 23] and the survey paper [2] studied the existence of solutions for different initial value problems involving impulsive fractional differential equations. In [6], Furati and Tatar studied the following family of fractional differential problems of weighted Cauchy type

$$\begin{cases} D_{0^+}^{\alpha}u(t) = f(t,u(t)) + \int_0^1 g(t,s,u(s))ds, \quad t>0, \\ t^{1-\alpha}u(t)|_{t=0} = b \end{cases} \tag{1.1}$$

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where $0<\alpha<1$, $b\in\mathbb{R}$, f and g are continuous functions. $D^{\alpha}_{0^+}$ denotes the Riemann-Liouville fractional derivative of order α. Using Schauder fixed point theorem, it is proved that (1.1) admits at least one solution on a sufficiently small interval under some assumptions imposed on f and g. Further, in [16, 17] the authors investigated the initial value problem for the fractional differential equation

$$\begin{cases} D^{\alpha}_{0^+}u(t)=f(t,u(t)), \quad t\in(0,T], \\ t^{1-\alpha}u(t)|_{t=0}=u_0. \end{cases} \tag{1.2}$$

Using the monotone iterative method, the existence and uniqueness of solution of (1.2) is established under the existence of upper and lower solutions of (1.2).

In recent paper [10], Liu studied the existence of solutions of a initial value problem of singular impulsive fractional differential system on half line involving the Caputo type fractional derivative with a single starting point. While in [11, 12], existence of solutions of periodic type BVPs for singular fractional differential systems imposed impulse effects (single and multiple impulse points respectively) involving Riemann-Liouville type fractional derivatives with multiple starting points were investigated.

We note that there has been no paper concerned with solvability of singular fractional differential system involving Riemann-Liouville type fractional derivatives with a single starting point and with infinitely many impulse effect points.

Motivated by mentioned papers and reason, in this paper, we discuss the following initial value problem of nonlinear singular impulsive fractional differential system on half line

$$\begin{cases} D^{\alpha}_{0^+}u(t)=m(t)f(t,u(t),v(t)), \quad t\in(0,+\infty),\ t\neq t_s,\ s=0,1,2,\cdots, \\ D^{\beta}_{0^+}v(t)=n(t)g(t,u(t),v(t)), \quad t\in(0,+\infty),\ t\neq t_s,\ s=0,1,2,\cdots, \\ \lim\limits_{t\to0}t^{1-\alpha}u(t)=\int_0^{\infty}\phi(s)F(s,u(s),v(s))ds, \\ \lim\limits_{t\to0}t^{1-\beta}v(t)=\int_0^{\infty}\psi(s)G(s,u(s),v(s))ds, \\ \lim\limits_{t\to t_s^+}(t-t_s)^{1-\alpha}u(t)=I(t_s,u(t_s),v(t_s)),\ s=1,2,\cdots, \\ \lim\limits_{t\to t_s^+}(t-t_s)^{1-\beta}v(t)=J(t_s,u(t_s),v(t_s)),\ s=1,2,\cdots \end{cases} \tag{1.3}$$

where

(a) $0<\alpha,\beta<1$, $D^{\alpha}_{0^+}$ and $D^{\beta}_{0^+}$ are the Riemann-Liouville fractional derivatives of orders α and β respectively with single starting point 0,

(b) $0=t_0<t_1<\cdots<t_s<\cdots$ with $\lim\limits_{s\to\infty}t_s=+\infty$,

(c) $m,n:(0,+\infty)\to\mathbb{R}$ satisfy $m|_{(t_s,t_{s+1}]}$, $n|_{(t_s,t_{s+1}]}\in C^0(t_s,t_{s+1}]$ $(s=0,1,2,\cdots)$, both m and n may be singular at $t=0$, there exist constants $L_1,L_2>0$ and $k,l>-1$ such that

$$|m(t)|\leqslant L_1t^k \quad \text{and} \quad |n(t)|\leqslant L_2t^l, \quad t\in(0,+\infty),$$

(d) $\phi,\psi:(0,+\infty)\to\mathbb{R}$ satisfy $\phi,\psi\in L^1(0,+\infty)$, and

(e) f, g, F, G are Carathéodory functions defined on $(0,+\infty)\times\mathbb{R}\times\mathbb{R}$, I, J are discrete Carathéodory functions defined on $\{t_s:s=1,2,\cdots\}\times\mathbb{R}\times\mathbb{R}$.

A pair of functions (x,y) with $x:(0,+\infty)\to\mathbb{R}$ and $y:(0,+\infty)\to\mathbb{R}$ is said to be a *global solution* of (1.3) if

$$x|_{(t_s,t_{s+1}]},\ y|_{(t_s,t_{s+1}]}\in C^0(t_s,t_{s+1}],\ s=0,1,2,\cdots,$$

$$\lim_{t\to t_s^+}(t-t_s)^{1-\alpha}u(t),\ \lim_{t\to t_s^+}(t-t_s)^{1-\beta}v(t)\text{ are finite, } s=0,1,2,\cdots,$$

$$D_{0^+}^{\alpha}x\text{ is }\alpha-\text{integrable on }(0,+\infty),\ ,\ D_{0^+}^{\beta}y\text{ is }\beta-\text{integrable on }(0,+\infty)$$

and (x,y) satisfies all equations in (1.3).

We shall construct a weighted Banach space and apply the Leray-Schauder nonlinear alternative to obtain the existence of at least one global solution of (1.3). Our results are new and naturally complement the literature on fractional differential equations.

The paper is outlined as follows. Section 2 contains some preliminary results. The main results are presented in Section 3. Finally, in Section 4 we give an example to illustrate the efficiency of the results obtained.

2. Preliminaries

For convenience of readers, we state some necessary definitions from fractional calculus theory.

For $\phi\in L^1(0,+\infty)$, denote $\|\phi\|_1=\int_0^{+\infty}|\phi(s)|ds$. Let the Gamma and Beta functions $\Gamma(\alpha)$ and $\mathbf{B}(p,q)$ be defined by

$$\Gamma(\alpha)=\int_0^{+\infty}x^{\alpha-1}e^{-x}dx,\qquad \mathbf{B}(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx.$$

DEFINITION 2.1. [21] The *Riemann-Liouville fractional integral of order* $\alpha>0$ of a function $g:(0,+\infty)\to\mathbb{R}$ is given by

$$I_{0^+}^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s)ds,$$

provided that the right-hand side exists.

DEFINITION 2.2. [21] The *Riemann-Liouville fractional derivative of order* $\alpha>0$ of a continuous function $g:(0,+\infty)\to\mathbb{R}$ is given by

$$D_{0^+}^{\alpha}g(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t\frac{g(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n-1<\alpha<n$, provided that the right-hand side exists.

DEFINITION 2.3. [9] An odd homeomorphism Φ of the real line $\mathbb{R}$ onto itself is called a *sup-multiplicative-like function* if there exists a homeomorphism ω of $[0,+\infty)$ onto itself which *supports* Φ in the sense that for all $v_1, v_2 \geqslant 0$,

$$\Phi(v_1 v_2) \geqslant \omega(v_1)\Phi(v_2). \tag{2.1}$$

ω is called the *supporting function* of Φ.

REMARK 2.1. Note that any sup-multiplicative function is sup-multiplicative-like function. Also any function of the form

$$\Phi(u) := \sum_{j=0}^{k} c_j |u|^j u, \quad u \in \mathbb{R}$$

is sup-multiplicative-like, provided that $c_j \geqslant 0$. Here a supporting function is defined by $\omega(u) := \min\{u^{k+1}, u\}$, $u \geqslant 0$.

REMARK 2.2. It is clear that a sup-multiplicative-like function Φ and any corresponding supporting function ω are increasing functions vanishing at zero. Moreover, their inverses Φ^{-1} and ν respectively are increasing and such that for all $w_1, w_2 \geqslant 0$,

$$\Phi^{-1}(w_1 w_2) \leqslant \nu(w_1)\Phi^{-1}(w_2). \tag{2.2}$$

ν is called the *supporting function* of Φ^{-1}.

In this paper we always suppose that Φ is a sup-multiplicative-like function with its supporting function ω. The inverse function Φ^{-1} has its supporting function ν.

Let $\sigma > k+1$ and $\delta > l+1$. Denote

$$\rho(t) = \frac{(t-t_s)^{1-\alpha}}{1+t^{\sigma}}, \quad t \in (t_s, t_{s+1}], \quad s = 0,1,2,\cdots,,$$

$$\varrho(t) = \frac{(t-t_s)^{1-\beta}}{1+t^{\delta}}, \quad t \in (t_s, t_{s+1}], \quad s = 0,1,2,\cdots.$$

DEFINITION 2.4. We say $K : (0,+\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a *Carathéodory function* if it satisfies the following:

(i) $t \to K\left(t, \frac{x}{\rho(t)}, \frac{y}{\varrho(t)}\right)$ is continuous on $(t_s, t_{s+1}]$ $(s = 0,1,2,\cdots)$, and for any $(x,y) \in \mathbb{R}^2$ there exist the limits

$$\lim_{t \to t_s^+} K\left(t, \frac{x}{\rho(t)}, \frac{y}{\varrho(t)}\right), \quad s = 0,1,2,\cdots;$$

(ii) $(x,y) \to K\left(t, \frac{x}{\rho(t)}, \frac{y}{\varrho(t)}\right)$ is continuous on $\mathbb{R}^2$ for all $t \in (0,+\infty)$;

(iii) for each $r > 0$ there exists a constant $A_r > 0$ such that

$$\left|K\left(t, \frac{x}{\rho(t)}, \frac{y}{\varrho(t)}\right)\right| \leqslant A_r, \quad t \in (0,+\infty), \quad |x|,|y| \leqslant r.$$

DEFINITION 2.5. $L:\{t_s\}\times\mathbb{R}^2$ is called a discrete Carathéodory function if

(i) $(x,y)\to L\left(t_s,\ \frac{x}{\rho(t_s)},\ \frac{y}{\varrho(t_s)}\right)$ is continuous on $\mathbb{R}^2$ for all $s=1,2,\cdots$;

(ii) for each $r>0$ there exists $A_{r,s}\geqslant 0$ such that

$$\left|L\left(t_s,\ \frac{x}{\rho(t_s)},\ \frac{y}{\varrho(t_s)}\right)\right|\leqslant A_{r,s},\quad s=1,2,\cdots,\quad |x|,|y|\leqslant r$$

and

$$\sum_{s=1}^{+\infty}A_{r,s}<+\infty.$$

DEFINITION 2.6. Let $\theta>0$. $h:(0,+\infty)\to R$ is called a $\theta-$integral on $(0,+\infty)$ if $\int_0^t(t-s)^{\theta-1}h(s)ds$ is well defined on $(0,+\infty)$.

To obtain the main results, we need the Leray-Schauder nonlinear alternative.

LEMMA 2.1. (Leray-Schauder Nonlinear Alternative) [16] *Let X be a Banach space and $T:X\to X$ be a completely continuous operator. Suppose Ω is a nonempty open subset of X centered at zero. Then, either there exists $x\in\partial\Omega$ and $\lambda\in(0,1)$ such that $x=\lambda Tx$, or there exists $x\in\overline{\Omega}$ such that $x=Tx$.*

3. Main results

In this section we shall establish the existence of at least one solution of system (1.3). Throughout, we assume that the functions and parameters in (1.3) satisfy (a)–(e) (stated in Section 1).

Let

$$X=\left\{x:(0,\infty)\to\mathbb{R}:\begin{array}{l}x|_{(t_s,t_{s+1}]}\in C^0(t_s,t_{s+1}]\ (s=0,1,2,\cdots),\\ \text{there exist the limits }\lim\limits_{t\to t_s^+}(t-t_s)^{1-\alpha}x(t)\ (s=0,1,2,\cdots),\\ \lim\limits_{t\to+\infty}\rho(t)x(t)\text{ is finite,}\end{array}\right\}$$

and

$$Y=\left\{y:(0,\infty)\to\mathbb{R}:\begin{array}{l}y|_{(t_s,t_{s+1}]}\in C^0(t_s,t_{s+1}]\ (s=0,1,2,\cdots),\\ \text{there exist the limits }\lim\limits_{t\to t_s^+}(t-t_s)^{1-\beta}y(t)\ (s=0,1,2,\cdots),\\ \lim\limits_{t\to+\infty}\varrho(t)y(t)\text{ is finite}\end{array}\right\}.$$

For $x\in X$ and $y\in Y$, define the norms by

$$\|x\|=\|x\|_X=\sup_{t\in(0,+\infty)}\rho(t)|x(t)|\qquad\text{and}\qquad\|y\|=\|y\|_Y=\sup_{t\in(0,+\infty)}\varrho(t)|y(t)|.$$

It is easy to show that X and Y are real Banach spaces. Thus, $(X\times Y,\|\cdot\|)$ is a Banach space with the norm defined by

$$\|(x,y)\|=\max\{\|x\|_X,\ \|y\|_Y\},\quad (x,y)\in X\times Y.$$

Let $x\in X$ and $y\in Y$. Then, there exists $r>0$ such that

$$\|y\|=\sup_{t\in(0,+\infty)}|y(t)|\varrho(t)\leqslant r\qquad\text{and}\qquad \|x\|=\sup_{t\in(0,+\infty)}|x(t)|\varrho(t)\leqslant r. \tag{3.1}$$

From (A), f is a Carathéodory function, thus there exists $A_r>0$ such that

$$|f(t,x(t),y(t))|=\left|f\left(t,\ \frac{\rho(t)x(t)}{\rho(t)},\ \frac{\varrho(t)y(t)}{\varrho(t)}\right)\right|\leqslant A_r,\quad t\in(0,+\infty). \tag{3.2}$$

Similarly, since F is a Carathéodory function and I a discrete Carathéodory function, there exist positive constants A'_r and $A_{r,s}$ $(s=1,2,\cdots)$ such that

$$\begin{aligned}&|F(t,x(t),y(t))|\leqslant A'_r,\quad t\in(0,+\infty),\\ &|I(t_s,x(t_s),y(t_s))|\leqslant A_{r,s}\ (s=1,2,\cdots),\quad \sum_{s=1}^{\infty}A_{r,s}<+\infty.\end{aligned} \tag{3.3}$$

Likewise, g, G and J are also Carathéodory functions and discrete Carathéodory function, so there exist positive constants B_r, B'_r and $B_{r,s}$ $(s=1,2,\cdots)$ such that

$$\begin{aligned}&|g(t,x(t),y(t))|\leqslant B_r,\quad |G(t,x(t),y(t))|\leqslant B'_r,\quad t\in(0,\infty),\\ &|J(t_s,x(t_s),y(t_s))|\leqslant B_{r,s}\ (s=1,2,\cdots),\quad \sum_{s=1}^{\infty}B_{r,s}<\infty.\end{aligned} \tag{3.4}$$

LEMMA 3.1. *Suppose that $x\in X$ and $y\in Y$. Then, $u\in X$ is a solution of*

$$\begin{cases}D^{\alpha}_{0^+}u(t)=m(t)f(t,x(t),y(t)),\quad t\in(0,+\infty),\ t\neq t_s,\ s=1,2,\cdots,\\ \lim\limits_{t\to 0}t^{1-\alpha}u(t)=\displaystyle\int_0^{\infty}\phi(s)F(s,x(s),y(s))ds,\\ \lim_{t\to t_s^+}(t-t_s)^{1-\alpha}u(t)=I(t_s,x(t_s),y(t_s)),\ s=1,2,\cdots\end{cases} \tag{3.5}$$

if and only if u satisfies the integral equation

$$\begin{aligned}u(t)=&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s),y(s))ds\\ &+t^{\alpha-1}\int_0^{\infty}\phi(s)F(s,x(s),y(s))ds+\sum_{j=1}^{i}(t-t_j)^{\alpha-1}I(t_j,x(t_j),y(t_j)),\\ &t\in(t_i,t_{i+1}],\quad i=0,1,2,\cdots.\end{aligned} \tag{3.6}$$

Proof. Let $u\in X$ be a solution of (3.5). We firstly prove that there exists numbers $c_j\in\mathbb{R}$ such that for $t\in(t_i,t_{i+1}]$ $(i=0,1,2,\cdots)$

$$\begin{aligned}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}m(s)f(s,x(s),y(s))ds+\sum_{j=0}^{i}c_j(t-t_j)^{\alpha-1},\\ t\in(t_i,t_{i+1}],\quad i=0,1,2,\cdots.\end{aligned} \tag{3.7}$$

In fact, for $t \in (t_0,t_1]$, we have from $u \in X$ that

$$\begin{aligned}
&I_{0^+}^{\alpha} m(t) f(t,x(t),y(t)) \\
&= I_{0^+}^{\alpha} D_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{1}{\Gamma(1-\alpha)} \left[\int_0^s (s-v)^{-\alpha} u(v)dv\right]' ds \\
&= \frac{1}{\alpha\Gamma(\alpha)} \left[\int_0^t (t-s)^{\alpha} \frac{1}{\Gamma(1-\alpha)} \left(\int_0^s (s-v)^{-\alpha} u(v)dv\right)' ds\right]' \\
&= \frac{1}{\alpha\Gamma(\alpha)} \left[(t-s)^{\alpha} \frac{1}{\Gamma(1-\alpha)} \int_0^s (s-v)^{-\alpha} u(v)dv\Big|_0^t \right. \\
&\quad \left. +\alpha \int_0^t (t-s)^{\alpha-1} \frac{1}{\Gamma(1-\alpha)} \int_0^s (s-v)^{-\alpha} u(v)dvds\right]' \\
&= \frac{1}{\alpha\Gamma(\alpha)} \left[-t^{\alpha} \frac{1}{\Gamma(1-\alpha)} \lim \lim_{t\to 0^+} \int_0^t (t-v)^{-\alpha} u(v)dv \right. \\
&\quad \left. +\alpha \int_0^t \int_s^t (t-s)^{\alpha-1} \frac{1}{\Gamma(1-\alpha)} (s-v)^{-\alpha} dsu(v)dv\right]'.
\end{aligned}$$

Since $\lim_{t\to 0^+} t^{1-\alpha}u(t) = A$ exists, for $\varepsilon > 0$, we have $A-\varepsilon < t^{1-\alpha}u(t) < A+\varepsilon$ for sufficiently small $t \in (t_0,t_1]$. So

$$\begin{aligned}
&(A-\varepsilon)\int_0^1 (1-w)^{-\alpha} w^{\alpha-1} dw \\
&= (A-\varepsilon)\int_0^t (t-v)^{-\alpha} v^{\alpha-1} dv \leqslant \int_0^t (t-v)^{-\alpha} u(v)dv \\
&\leqslant (A+\varepsilon)\int_0^t (t-v)^{-\alpha} v^{\alpha-1} dv = (A+\varepsilon)\int_0^1 (1-w)^{-\alpha} w^{\alpha-1} dw.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(A-\varepsilon)\int_0^1 (1-w)^{-\alpha} w^{\alpha-1} dw \\
&\leqslant \frac{\lim}{t \to 0^+} \int_0^t (t-v)^{-\alpha} v^{\alpha-1} dv \leqslant \overline{\lim_{t\to 0^+}} \int_0^t (t-v)^{-\alpha} u(v)dv \\
&\leqslant (A+\varepsilon)\int_0^1 (1-w)^{-\alpha} w^{\alpha-1} dw.
\end{aligned}$$

Then $\varepsilon \to 0$ implies that $\lim_{t\to 0^+} \int_0^t (t-v)^{-\alpha} v^{\alpha-1} dv = A$ exists. Hence

$$\begin{aligned}
&I_{0^+}^{\alpha} m(t) f(t,x(t),y(t)) \\
&= \frac{1}{\alpha\Gamma(\alpha)} \left[-t^{\alpha} \frac{1}{\Gamma(1-\alpha)} A + \alpha \int_0^t \int_s^t (t-s)^{\alpha-1} \frac{1}{\Gamma(1-\alpha)} (s-v)^{-\alpha} dsu(v)dv\right]' \\
&= \frac{1}{\alpha\Gamma(\alpha)} \left[-t^{\alpha} \frac{1}{\Gamma(1-\alpha)} A + \alpha \int_0^t \int_0^1 (1-w)^{\alpha-1} \frac{1}{\Gamma(1-\alpha)} w^{-\alpha} dwu(v)dv\right]' \\
&= -c_0 t^{\alpha-1} + u(t).
\end{aligned}$$

Thus

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)f(s,x(s),y(s))ds+c_0t^{\alpha-1},\quad t\in(t_0,t_1].$$

We find that (3.7) holds for $i=0$. Now we suppose that (3.7) holds for $i=0,1,2,\cdots,n$. We will prove that (3.7) holds for $i=n+1$. By mathematical induction method, we complete the proof of (3.7).

Suppose that

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)f(s,x(s),y(s))ds+\sum_{j=0}^{n}c_j(t-t_j)^{\alpha-1}+\Phi(t),$$
$$t\in(t_{n+1},t_{n+2}]. \tag{$*$}$$

Then

$$\begin{aligned}
&m(t)f(t,x(t),y(t))\\
&=D_{0^+}^{\alpha}u(t)=\frac{1}{\Gamma(1-\alpha)}\Big[\int_0^t (t-s)^{\alpha}u(s)ds\Big]'\\
&=\frac{1}{\Gamma(1-\alpha)}\Big[\sum_{v=0}^{n}\int_{t_v}^{t_{v+1}}(t-s)^{-\alpha}\Big(\frac{1}{\Gamma(\alpha)}\int_0^s (s-v)^{\alpha-1}m(v)f(v,x(v),y(v))dv\\
&\quad+\sum_{j=0}^{v}c_j(s-t_j)^{\alpha-1}\Big)ds+\int_{t_{n+1}}^{t}(t-s)^{-\alpha}\Big(\frac{1}{\Gamma(\alpha)}\int_0^s (s-v)^{\alpha-1}m(v)f(v,x(v),y(v))dv\\
&\quad+\sum_{j=0}^{n}c_j(s-t_j)^{\alpha-1}+\Phi(s)\Big)ds\Big]'\\
&=\frac{1}{\Gamma(1-\alpha)}\Big[\int_0^t (t-s)^{-\alpha}\frac{1}{\Gamma(\alpha)}\int_0^s (s-v)^{\alpha-1}m(v)f(v,x(v),y(v))dvds\\
&\quad+\sum_{v=0}^{n}\int_{t_v}^{t_{v+1}}(t-s)^{-\alpha}\sum_{j=0}^{v}c_j(s-t_j)^{\alpha-1}ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_{n+1}}^{t}(t-s)^{-\alpha}\sum_{j=0}^{n}c_j(s-t_j)^{\alpha-1}ds+\int_{t_{n+1}}^{t}(t-s)^{-\alpha}\Phi(s)ds\Big]'\\
&=D_{t_{n+1}^+}^{\alpha}\Phi(t)+\frac{1}{\Gamma(1-\alpha)}\Big[\int_0^t\int_s^t (t-s)^{-\alpha}\frac{1}{\Gamma(\alpha)}(s-v)^{\alpha-1}dsm(v)f(v,x(v),y(v))dv\\
&\quad+\sum_{v=0}^{n}\sum_{j=0}^{v}c_j\int_{\frac{t_v-t_j}{t-t_j}}^{\frac{t_{v+1}-t_j}{t-t_j}}(1-w)^{-\alpha}w^{\alpha-1}dw+\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n}c_j\int_{\frac{t_{n+1}-t_j}{t-t_j}}^{1}(1-w)^{-\alpha}w^{\alpha-1}dw\Big]'\\
&=D_{t_{n+1}^+}^{\alpha}\Phi(t)+\frac{1}{\Gamma(1-\alpha)}\Big[\int_0^t\int_0^1 (1-w)^{-\alpha}\frac{1}{\Gamma(\alpha)}w^{\alpha-1}dwm(v)f(v,x(v),y(v))dv\\
&\quad+\sum_{j=0}^{n}c_j\sum_{v=j}^{n}\int_{\frac{t_v-t_j}{t-t_j}}^{\frac{t_{v+1}-t_j}{t-t_j}}(1-w)^{-\alpha}w^{\alpha-1}dw+\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n}c_j\int_{\frac{t_{n+1}-t_j}{t-t_j}}^{1}(1-w)^{-\alpha}w^{\alpha-1}dw\Big]'\\
&=m(t)f(t,x(t),y(t)).
\end{aligned}$$

It follows that $D^{\alpha}_{t^+_{n+1}}\Phi(t)=0$. Then similarly to $(*)$ we know that there exists $c_{n+1}\in R$ such that $\Phi(t)=c_{n+1}(t-t_{n+1})^{\alpha-1}$. Hence

$$u(t)=\tfrac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}m(s)f(s,x(s),y(s))ds+\sum_{j=0}^{n+1}c_j(t-t_j)^{\alpha-1},\quad t\in(t_{n+1},t_{n+2}]. \tag{$*$}$$

Thus (3.7) holds for $i=n+1$. This completes the proof of (3.7).

From $\lim\limits_{t\to 0}t^{1-\alpha}u(t)=\int_0^\infty\phi(s)F(s,x(s),y(s))ds$, we get

$$c_0=\int_0^\infty\phi(s)F(s,x(s),y(s))ds.$$

From $\lim\limits_{t\to t_i^+}(t-t_i)^{1-\alpha}u(t)=I(t_i,x(t_i),y(t_i))$, we have

$$c_i=I(t_i,x(t_i),y(t_i)).$$

On substituting c_i into (3.7), we obtain for $t\in(t_i,t_{i+1}]$ $(i=0,1,2,\cdots)$,

$$\begin{aligned}u(t)=&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,m(s),y(s))ds+t^{\alpha-1}\int_0^\infty\phi(s)F(s,x(s),y(s))ds\\&+\sum_{j=1}^{i}(t-t_j)^{\alpha-1}I(t_j,x(t_j),y(t_j)),\quad t\in(t_i,t_{i+1}],\quad i=0,1,2,\cdots\end{aligned}$$

which is simply the same as (3.6).

Moreover, since $x\in X$ and $y\in Y$, we have (3.1)–(3.3) which will lead to the expression of u in (3.6) is well defined on $(0,+\infty)$. We will prove that $u\in X$ and u satisfies (3.5).

In fact, by (3.1)–(3.3) and (c), we have

$$\begin{aligned}\left|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s),y(s))ds\right|&\leqslant A_rL_1\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^kds\\&=A_rL_1t^{\alpha+k}\int_0^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^kdw\ \text{ by }\ \frac{s}{t}=w\\&=A_rL_1t^{\alpha+k}\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}.\end{aligned}$$

We have

$$\begin{aligned}\lim_{t\to 0^+}\rho(t)u(t)&=\frac{t^{1-\alpha}}{1+t^\sigma}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s),y(s))ds+\frac{\int_0^\infty\phi(s)F(s,x(s),y(s))ds}{1+t^\sigma}\\&=\int_0^\infty\phi(s)F(s,x(s),y(s))ds\end{aligned}$$

and for $s = 1, 2, \cdots$, we have

$$\begin{aligned}
\lim_{t \to t_s^+} \rho(t)u(t) &= \frac{(t-t_s)^{1-\alpha}}{1+t^\sigma} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s) f(s,x(s),y(s))ds \\
&\quad + \frac{t^{\alpha-1}(t-t_s)^{1-\alpha} \int_0^\infty \phi(s)F(s,x(s),y(s))ds}{1+t^\sigma} \\
&\quad + \frac{(t-t_s)^{1-\alpha} \sum_{j=1}^{s} (t-t_j)^{\alpha-1} I(t_j,x(t_j),y(t_j))}{1+t^\sigma} \\
&= I(t_s,x(t_s),y(t_s)).
\end{aligned}$$

One can get for $t \in (t_i, t_{i+1}]$ that

$$\begin{aligned}
D_{0^+}^\alpha u(t) &= \frac{1}{\Gamma(1-\alpha)} \Big[\int_0^t (t-s)^{-\alpha} u(s)ds \Big]' \\
&= \frac{1}{\Gamma(1-\alpha)} \Big[\sum_{n=0}^{i-1} \int_{t_n}^{t_{n+1}} (t-s)^{-\alpha} u(s)ds + \int_{t_i}^t (t-s)^{-\alpha} u(s)ds \Big] \\
&= \frac{1}{\Gamma(1-\alpha)} \Big[\sum_{n=0}^{i-1} \int_{t_n}^{t_{n+1}} (t-s)^{-\alpha} \Big(\int_0^s \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)} m(v) f(v,m(v),y(v))dv \\
&\quad + s^{\alpha-1} \int_0^\infty \phi(v)F(v,x(v),y(v))dv + \sum_{j=1}^{n} (s-t_j)^{\alpha-1} I(t_j,x(t_j),y(t_j)) \Big) ds \\
&\quad + \int_{t_i}^t (t-s)^{-\alpha} \Big(\int_0^s \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)} m(v) f(v,m(v),y(v))dv \\
&\quad + s^{\alpha-1} \int_0^\infty \phi(v)F(v,x(v),y(v))dv + \sum_{j=1}^{i} (s-t_j)^{\alpha-1} I(t_j,x(t_j),y(t_j)) \Big) ds \Big]' \\
&= \frac{1}{\Gamma(1-\alpha)} \Big[\int_0^t (t-s)^{-\alpha} \int_0^s \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)} m(v) f(v,m(v),y(v))dvds \\
&\quad + \sum_{n=0}^{i-1} \int_{t_n}^{t_{n+1}} (t-s)^{-\alpha} s^{\alpha-1} ds \int_0^\infty \phi(v)F(v,x(v),y(v))dv \\
&\quad + \sum_{n=0}^{i-1} \int_{t_n}^{t_{n+1}} (t-s)^{-\alpha} \sum_{j=1}^{n} (s-t_j)^{\alpha-1} dsI(t_j,x(t_j),y(t_j)) \\
&\quad + \int_{t_i}^t (t-s)^{-\alpha} s^{\alpha-1} ds \int_0^\infty \phi(v)F(v,x(v),y(v))dv \\
&\quad + \int_{t_i}^t (t-s)^{-\alpha} \sum_{j=1}^{i} (s-t_j)^{\alpha-1} dsI(t_j,x(t_j),y(t_j)) \Big]' \\
&= \frac{1}{\Gamma(1-\alpha)} \Big[\int_0^t \int_s^t (t-s)^{-\alpha} \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)} dsm(v) f(v,m(v),y(v))dv
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=0}^{i-1}\int_{\frac{t_n}{t}}^{\frac{t_{n+1}}{t}}(1-w)^{-\alpha}w^{\alpha-1}dw\int_0^{\infty}\phi(v)F(v,x(v),y(v))dv\\
&+\sum_{n=0}^{i-1}\sum_{j=1}^{n}\int_{\frac{t_n-t_j}{t-t_j}}^{\frac{t_{n+1}-t_j}{t-t_j}}(1-w)^{-\alpha}w^{\alpha-1}dwI(t_j,x(t_j),y(t_j))\\
&+\int_{\frac{t_i}{t}}^{1}(1-w)^{-\alpha}w^{\alpha-1}dw\int_0^{\infty}\phi(v)F(v,x(v),y(v))dv\\
&+\sum_{j=1}^{i}\int_{\frac{t_i-t_j}{t-t_j}}^{1}(1-w)^{-\alpha}w^{\alpha-1}dwI(t_j,x(t_j),y(t_j))\Big]'\\
=&\frac{1}{\Gamma(1-\alpha)}\Big[\int_0^t\int_0^1(1-w)^{-\alpha}\frac{w^{\alpha-1}}{\Gamma(\alpha)}dwm(v)f(v,m(v),y(v))dv\\
&+\sum_{n=0}^{i-1}\int_{\frac{t_n}{t}}^{\frac{t_{n+1}}{t}}(1-w)^{-\alpha}w^{\alpha-1}dw\int_0^{\infty}\phi(v)F(v,x(v),y(v))dv\\
&+\sum_{j=0}^{i-1}\sum_{n=j}^{i-1}\int_{\frac{t_n-t_j}{t-t_j}}^{\frac{t_{n+1}-t_j}{t-t_j}}(1-w)^{-\alpha}w^{\alpha-1}dwI(t_j,x(t_j),y(t_j))\\
&+\int_{\frac{t_i}{t}}^{1}(1-w)^{-\alpha}w^{\alpha-1}dw\int_0^{\infty}\phi(v)F(v,x(v),y(v))dv\\
&+\sum_{j=1}^{i}\int_{\frac{t_i-t_j}{t-t_j}}^{1}(1-w)^{-\alpha}w^{\alpha-1}dwI(t_j,x(t_j),y(t_j))\Big]'\\
=&\frac{1}{\Gamma(1-\alpha)}\Big[\int_0^t\int_0^1(1-w)^{-\alpha}\frac{w^{\alpha-1}}{\Gamma(\alpha)}dwm(v)f(v,m(v),y(v))dv\\
&+\int_0^1(1-w)^{-\alpha}w^{\alpha-1}dw\int_0^{\infty}\phi(v)F(v,x(v),y(v))dv\\
&+\sum_{j=0}^{i-1}\int_0^1(1-w)^{-\alpha}w^{\alpha-1}dwI(t_j,x(t_j),y(t_j))\\
&+\int_0^1(1-w)^{-\alpha}w^{\alpha-1}dwI(t_i,x(t_i),y(t_i))\Big]'\\
=&\,m(t)f(t,x(t),y(t)),\quad t\in(t_i,t_{i+1}],\quad i=0,1,2,\cdots.
\end{aligned}
$$

It is easy to see that $u|_{t_i,t_{i+1}]}\in C^0(t_i,t_{i+1}]$ and $\lim\limits_{t\to t_i^+}\rho(t)u(t)$ exists for all $i=1,2,\cdots$. Furthermore, we have for $t\in(t_i,t_{i+1}]$ that

$$
\begin{aligned}
\rho(t)|u(t)|\leqslant&\frac{(t-t_i)^{1-\alpha}}{1+t^{\sigma}}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|m(s)f(s,m(s),y(s))|ds\\
&+\frac{(t-t_i)^{1-\alpha}}{1+t^{\sigma}}t^{\alpha-1}\int_0^{\infty}|\phi(s)||F(s,x(s),y(s))|ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}\sum_{j=1}^{i}(t-t_j)^{\alpha-1}|I(t_j,x(t_j),y(t_j))| \\
&\leqslant A_rL_1\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^kds \\
&\quad +A_r'\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}t^{\alpha-1}\int_0^\infty|\phi(s)|ds+\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}\sum_{j=1}^{i}(t-t_j)^{\alpha-1}A_{r,j} \\
&\leqslant A_rL_1\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}t^{\alpha+k}\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}+A_r'\frac{1}{1+t^\sigma}\int_0^\infty|\phi(s)|ds+\frac{1}{1+t^\sigma}\sum_{j=1}^{+\infty}A_{r,j}.
\end{aligned}
$$

It follows that $\lim_{t\to+\infty}\rho(t)u(t)=0$. So $u\in X$ and u is a solution of (3.5). The proof is complete. □

LEMMA 3.2. *Suppose that $x\in X$ and $y\in Y$. Then, $v\in Y$ is a solution of*

$$
\begin{cases}
D_{0^+}^{\beta}v(t)=n(t)g(t,x(t),y(t)), \quad t\in(0,+\infty),\ t\neq t_s,\ s=1,2,\cdots, \\
\lim\limits_{t\to 0}t^{1-\beta}v(t)=\int_0^\infty\psi(s)G(s,x(s),y(s))ds, \\
\lim\limits_{t\to t_s^+}(t-t_s)^{1-\beta}v(t)=J_s(t_s,x(t_s),y(t_s)),\ s=1,2,\cdots
\end{cases}
\tag{3.8}
$$

if and only if v satisfies the integral equation

$$
\begin{aligned}
v(t)=&\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}n(s)g(s,x(s),y(s))ds \\
&+t^{\beta-1}\int_0^\infty\psi(s)G(s,x(s),y(s))ds+\sum_{j=1}^{i}(t-t_j)^{\beta-1}J(t_j,x(t_j),y(t_j)), \\
&\qquad t\in(t_i,t_{i+1}],\quad i=0,1,2,\cdots.
\end{aligned}
\tag{3.9}
$$

Proof. The proof is similar to that of Lemma 3.1. □

Now, we define the operator T on $X\times Y$ by

$$T(x,y)(t)=(T_1(x,y)(t),T_2(x,y)(t))$$

where

$$
\begin{aligned}
T_1(x,y)(t)=&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s),y(s))ds \\
&+t^{\alpha-1}\int_0^\infty\phi(s)F(s,x(s),y(s))ds+\sum_{j=1}^{i}(t-t_j)^{\alpha-1}I(t_j,x(t_j),y(t_j)), \\
&\qquad t\in(t_i,t_{i+1}],\quad i=0,1,2,\cdots
\end{aligned}
\tag{3.10}
$$

and

$$T_2(x,y)(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} n(s)g(s,x(s),y(s))ds$$
$$+t^{\beta-1}\int_0^{\infty} \psi(s)G(s,x(s),y(s))ds + \sum_{s=1}^{i} t_s^{\beta-1}J(t_j,x(t_j),y(t_j)),$$
$$t\in(t_i,t_{i+1}],\quad i=0,1,2,\cdots. \tag{3.11}$$

REMARK 3.1. By Lemmas 3.1 and 3.2, $(x,y)\in X\times Y$ is a solution of system (1.3) if and only if $(x,y)\in X\times Y$ is a fixed point of the operator T.

LEMMA 3.3. *The operator $T: X\times Y \to X\times Y$ is well defined and is completely continuous.*

Proof. The proof is long and will be divided into parts. First, we prove that T is well defined. Next, we show that T is continuous, and finally we prove that T is compact. Hence, T is completely continuous. We omit some of details, one may see [13].

Step 1. We shall prove that $T: X\times Y\to X\times Y$ is well defined. For $(x,y)\in X\times Y$, we have $\|(x,y)\| = r > 0$. Then, (3.1)–(3.4) hold. We can shown similarly to the proof of Lemma 3.1 that $T_1(x,y)\in X$. Similarly we can show that $T_2(x,y)\in Y$. Hence, $(T_1(x,y),T_2(x,y))\in X\times Y$ and $T: X\times Y\to X\times Y$ is well defined.

Step 2. We shall prove that T is continuous. Let $(x_n,y_n)\in X\times Y$ with $(x_n,y_n)\to(x_0,y_0)$ as $n\to\infty$. We shall show that $T(x_n,y_n)\to T(x_0,y_0)$ as $n\to\infty$, i.e., $T_1(x_n,y_n)\to T_1(x_0,y_0)$ and $T_2(x_n,y_n)\to T_2(x_0,y_0)$ as $n\to\infty$. The proof is similar to (a) in the proof of Lemma 10 in [13].

Step 3. We shall prove that T is compact, i.e., for each nonempty open bounded subset Ω of $X\times Y$, we shall prove that $T(\overline{\Omega})$ is relatively compact. For this, we shall show that $T(\overline{\Omega})$ is uniformly bounded, equi-continuous on each subinterval $[a,b]\subseteq(t_i,t_{i+1}]$ $(i=0,1,2,\cdots)$, $T(\overline{\Omega})$ is equi-convergent as $t\to t_i$ $(i=0,1,2,\cdots)$ and $t\to\infty$.

Let Ω be an open bounded subset of $X\times Y$. There exists $r>0$ such that (3.1) holds for all $(x,y)\in\overline{\Omega}$. Hence, (3.2)–(3.4) also hold for all $(x,y)\in\overline{\Omega}$. We need to do the following three substeps similarly to (b) of the proof of Lemma 10 in [13].

Step 3*a*. We shall show that $T(\overline{\Omega})$ is uniformly bounded.

Step 3*b*. We shall prove that $T(\overline{\Omega})$ is equi-continuous on each subinterval $[a,b]\subseteq(t_i,t_{i+1}]$ $(i=0,1,2,\cdots)$.

Step 3*c*. We shall show that $T(\overline{\Omega})$ is equi-convergent as $t\to t_i$ $(i=0,1,2,\cdots)$ and $t\to\infty$.

We have established that $T(\overline{\Omega})$ is relatively compact. So T is completely continuous. This completes the proof. □

We are now ready to present the main theorem.

THEOREM 3.1. *Let (a)–(e) hold, $\Phi : \mathbb{R} \to \mathbb{R}$ be a sup-multiplicative-like function with supporting function ω, and its inverse function $\Phi^{-1} : \mathbb{R} \to \mathbb{R}$ with supporting function ν. Furthermore, suppose that*

(i) *there exist nonnegative numbers c_f, b_f, a_f, C_F, B_F, A_F, $C_{I,s}$, $B_{I,s}$ and $A_{I,s}$ such that $\sum_{s=1}^{\infty} C_{I,s}$, $\sum_{s=1}^{\infty} B_{I,s}$ and $\sum_{s=1}^{\infty} A_{I,s}$ are convergent, and the following hold for all $(U,V) \in \mathbb{R}^2$ and $t \in (0,+\infty)$:*

$$\left| f\left(t, \frac{U}{\rho(t)}, \frac{V}{\varrho(t)}\right)\right| \leqslant c_f + b_f|U| + a_f\Phi^{-1}(|V|),$$
$$\left| F\left(t, \frac{U}{\rho(t)}, \frac{V}{\varrho(t)}\right)\right| \leqslant C_F + B_F|U| + A_F\Phi^{-1}(|V|),$$
$$\left| I\left(t_s, \frac{U}{\rho(t_s)}, \frac{V}{\varrho(t_s)}\right)\right| \leqslant C_{I,s} + B_{I,s}|U| + A_{I,s}\Phi^{-1}(|V|);$$

(ii) *there exist nonnegative numbers c_g, b_g, a_g, C_G, B_G, A_G, $C_{J,s}$, $B_{J,s}$ and $A_{J,s}$ such that $\sum_{s=1}^{\infty} C_{J,s}$, $\sum_{s=1}^{\infty} B_{J,s}$ and $\sum_{s=1}^{\infty} A_{J,s}$ are convergent, and the following hold for all $(U,V) \in \mathbb{R}^2$ and $t \in (0,\infty)$:*

$$\left| g\left(t, \frac{U}{\rho(t)}, \frac{V}{\varrho(t)}\right)\right| \leqslant c_g + b_g\Phi(|U|) + a_g|V|,$$
$$\left| G\left(t, \frac{U}{\rho(t)}, \frac{V}{\varrho(t)}\right)\right| \leqslant C_G + B_G\Phi(|U|) + A_G|V|,$$
$$\left| J\left(t_s, \frac{U}{\rho(t_s)}, \frac{V}{\varrho(t_s)}\right)\right| \leqslant C_{J,s} + B_{J,s}\Phi(|U|) + A_{J,s}|V|.$$

Then, the system (1.3) has at least one solution in $X \times Y$ if

$$\Sigma_3 < 1, \qquad \Theta_2 < 1, \qquad \frac{\Theta_3}{1-\Theta_2}\nu\left(\frac{2\Sigma_2}{1-\Sigma_3}\right) < 1 \tag{3.15}$$

or

$$\Sigma_3 < 1, \qquad \Theta_2 < 1, \qquad \frac{\Sigma_2}{1-\Sigma_3}\frac{1}{w\left(\frac{1-\Theta_2}{2\Theta_3}\right)} < 1, \tag{3.16}$$

where

$$\begin{aligned}
\Theta_2 &= L_1\frac{\sigma-k-1}{\sigma}\left(\frac{k+1}{\sigma-k-1}\right)^{(k+1)/\sigma}\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}b_f + \|\phi\|_1 B_F + \sum_{s=1}^{\infty} B_{I,s},\\
\Theta_3 &= L_1\frac{\sigma-k-1}{\sigma}\left(\frac{k+1}{\sigma-k-1}\right)^{(k+1)/\sigma}\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}a_f + \|\phi\|_1 A_F + \sum_{s=1}^{\infty} A_{I,s},\\
\Sigma_2 &= L_2\frac{\delta-l-1}{\delta}\left(\frac{l+1}{\delta-l-1}\right)^{(l+1)/\delta}\frac{\mathbf{B}(\beta,l+1)}{\Gamma(\beta)}b_g + \|\psi\|_1 B_G + \sum_{s=1}^{\infty} B_{J,s},\\
\Sigma_3 &= L_2\frac{\delta-l-1}{\delta}\left(\frac{l+1}{\delta-l-1}\right)^{(l+1)/\delta}\frac{\mathbf{B}(\beta,l+1)}{\Gamma(\beta)}a_g + \|\psi\|_1 A_G + \sum_{s=1}^{\infty} A_{J,s}.
\end{aligned} \tag{3.17}$$

Proof. We shall apply Lemma 2.1. From Lemma 3.3 we note that T is completely continuous. Let us consider the operator equation

$$(x,y) = \lambda T(x,y) \tag{3.18}$$

where $\lambda \in (0,1)$. We shall show that any solution (x,y) of (3.18) satisfies

$$\|(x,y)\| \leqslant M \tag{3.19}$$

where M is a constant independent of λ. Now, in the context of Lemma 2.1, let

$$\Omega = \{(x,y) \in X \times Y : \|(x,y)\| < M+1\}.$$

In view of (3.19), it is not possible to have $(x,y) \in \partial\Omega$ satisfying $(x,y) = \lambda T(x,y)$, hence we conclude by Lemma 2.1 that there exists $(x,y) \in \overline{\Omega}$ such that $(x,y) = T(x,y)$, i.e., the system (1.3) has a solution in $X \times Y$. This completes the proof.

We shall now proceed to prove (3.19). Let (x,y) be a solution of the operator equation (3.18). It follows that $x = \lambda T_1(x,y)$ and $y = \lambda T_2(x,y)$, i.e.,

$$\begin{aligned} x(t) &= \lambda T_1(x,y)(t) \\ &= \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s) f(s,x(s),y(s))ds \\ &\quad + \lambda t^{\alpha-1} \int_0^\infty \phi(s) F(s,x(s),y(s))ds + \sum_{j=1}^{i} (t-t_j)^{\alpha-1} I(t_j,x(t_j),y(t_j)), \\ &\qquad t \in (t_i,t_{i+1}], \quad i = 0,1,2,\cdots \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} y(t) &= \lambda T_2(x,y)(t) \\ &= \lambda \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} n(s) g(s,x(s),y(s))ds \\ &\quad + \lambda t^{\beta-1} \int_0^\infty \psi(s) G(s,x(s),y(s))ds + \sum_{j=1}^{i} (t-t_j)^{\beta-1} J(t_j,x(t_j),y(t_j)), \\ &\qquad t \in (t_i,t_{i+1}], \quad i = 0,1,2,\cdots. \end{aligned} \tag{3.21}$$

It is easy to see from condition (i) that

$$\begin{aligned} |f(t,x(t),y(t))| &= \left| f\left(t, \frac{\rho(t)x(t)}{\rho(t)}, \frac{\varrho(t)y(t)}{\varrho(t)}\right)\right| \\ &\leqslant c_f + b_f \rho(t)|x(t)| + a_f \Phi^{-1}\left(\varrho(t)|y(t)|\right) \\ &\leqslant c_f + b_f ||x|| + a_f \Phi^{-1}\left(||y||\right). \end{aligned} \tag{3.22}$$

Similarly, we get

$$\begin{aligned} &|F(t,x(t),y(t))| \leqslant C_F + B_F ||x|| + A_F \Phi^{-1}\left(||y||\right), \\ &|I(t_s,x(t_s),y(t_s))| \leqslant C_{I,s} + B_{I,s}\, ||x|| + A_{I,s} \Phi^{-1}\left(||y||\right). \end{aligned} \tag{3.23}$$

From (3.20), using (3.22) and (3.23), we find for $t \in (t_i, t_{i+1}]$ $(i = 0, 1, 2, \cdots)$,

$$\begin{aligned}
\rho(t)|x(t)| &\leqslant \frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}|T_1(x,y)(t)| \\
&\leqslant \frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|m(s)f(s,x(s),y(s))|ds \\
&\quad + \frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}t^{\alpha-1}\int_0^\infty |\phi(s)F(s,x(s),y(s))|ds \\
&\quad + \frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}\sum_{j=1}^{i}(t-t_j)^{\alpha-1}|I(t_j,x(t_j),y(t_j))| \\
&\leqslant L_1\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^k ds[c_f + b_f||x|| + a_f\Phi^{-1}(||y||)] \\
&\quad + \frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}t^{\alpha-1}\int_0^\infty |\phi(s)|ds[C_F + B_F||x|| + A_F\Phi^{-1}(||y||)] \\
&\quad + \frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}\sum_{j=1}^{i}(t-t_j)^{\alpha-1}[C_{I,s} + B_{I,s}\,||x|| + A_{I,s}\Phi^{-1}(||y||)] \\
&\leqslant L_1\frac{t^{1+k}}{1+t^\sigma}\int_0^1 \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw[c_f + b_f||x|| + a_f\Phi^{-1}(||y||)] \\
&\quad + ||\phi||_1[C_F + B_F||x|| + A_F\Phi^{-1}(||y||)] + \sum_{j=1}^{+\infty}[C_{I,s} + B_{I,s}\,||x|| + A_{I,s}\Phi^{-1}(||y||)] \\
&= L_1\frac{\sigma-k-1}{\sigma}\left(\frac{k+1}{\sigma-k-1}\right)^{(k+1)/\sigma}\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}[c_f + b_f||x|| + a_f\Phi^{-1}(||y||)] \\
&\quad + ||\phi||_1[C_F + B_F||x|| + A_F\Phi^{-1}(||y||)] + \sum_{j=1}^{+\infty}[C_{I,s} + B_{I,s}\,||x|| + A_{I,s}\Phi^{-1}(||y||)].
\end{aligned}$$

So

$$\rho(t)|x(t)| \leqslant \Theta_1 + \Theta_2||x|| + \Theta_3\Phi^{-1}(||y||),$$

where

$$\Theta_1 = L_1\frac{\sigma-k-1}{\sigma}\left(\frac{k+1}{\sigma-k-1}\right)^{(k+1)/\sigma}\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}c_f + \|\phi\|_1 C_F + \sum_{s=1}^{\infty}C_{I,s}.$$

It follows that

$$\|x\| = \sup_{t\in(0,\infty)}\rho(t)|x(t)| \leqslant \Theta_1 + \Theta_2\|x\| + \Theta_3\Phi^{-1}(\|y\|),$$

or equivalently

$$\|x\| \leqslant \frac{\Theta_1}{1-\Theta_2} + \frac{\Theta_3}{1-\Theta_2}\Phi^{-1}(\|y\|). \tag{3.24}$$

Similarly, from (3.21) we can show that

$$\|y\| \leqslant \frac{\Sigma_1}{1-\Sigma_3} + \frac{\Sigma_2}{1-\Sigma_3}\Phi(\|x\|) \tag{3.25}$$

where

$$\Sigma_1 = L_2 \frac{\delta - l - 1}{\delta}\left(\frac{l+1}{\delta - l - 1}\right)^{(l+1)/\delta} \frac{\mathbf{B}(\beta, l+1)}{\Gamma(\beta)} c_g + \|\psi\|_1 C_G + \sum_{s=1}^{\infty} C_{J,s}.$$

Case 1. Suppose (3.15) holds. Without loss of generality, suppose that

$$\|x\| \geqslant \Phi^{-1}\left(\frac{\Sigma_1}{\Sigma_2}\right). \tag{3.26}$$

Then, using (3.25) in (3.24) as well as (3.26) and (2.2), we get

$$\begin{aligned} \|x\| &\leqslant \frac{\Theta_1}{1-\Theta_2} + \frac{\Theta_3}{1-\Theta_2}\Phi^{-1}\left(\frac{\Sigma_1}{1-\Sigma_3} + \frac{\Sigma_2}{1-\Sigma_3}\Phi(\|x\|)\right) \\ &\leqslant \frac{\Theta_1}{1-\Theta_2} + \frac{\Theta_3}{1-\Theta_2}\Phi^{-1}\left(\frac{2\Sigma_2}{1-\Sigma_3}\Phi(\|x\|)\right) \\ &\leqslant \frac{\Theta_1}{1-\Theta_2} + \frac{\Theta_3}{1-\Theta_2}\nu\left(\frac{2\Sigma_2}{1-\Sigma_3}\right)\Phi^{-1}\left(\Phi(\|x\|)\right) \\ &= \frac{\Theta_1}{1-\Theta_2} + \frac{\Theta_3}{1-\Theta_2}\nu\left(\frac{2\Sigma_2}{1-\Sigma_3}\right)\|x\|. \end{aligned} \tag{3.27}$$

From (3.15) we have $\frac{\Theta_3}{1-\Theta_2}\nu\left(\frac{2\Sigma_2}{1-\Sigma_3}\right) < 1$, therefore it follows from (3.27) that

$$\|x\| \leqslant \frac{\Theta_1}{1-\Theta_2}\left[1 - \frac{\Theta_3}{1-\Theta_2}\nu\left(\frac{2\Sigma_2}{1-\Sigma_3}\right)\right]^{-1} \equiv W. \tag{3.28}$$

From the above discussion, we have either $\|x\| \leqslant W$ or $\|x\| < \Phi^{-1}\left(\frac{\Sigma_1}{\Sigma_2}\right)$. Therefore,

$$\|x\| \leqslant \max\left\{W,\ \Phi^{-1}\left(\frac{\Sigma_1}{\Sigma_2}\right)\right\} \equiv M_1. \tag{3.29}$$

Substituting (3.29) into (3.25) yields

$$\|y\| \leqslant \frac{\Sigma_1}{1-\Sigma_3} + \frac{\Sigma_2}{1-\Sigma_3}\Phi\left(\max\left\{W,\ \Phi^{-1}\left(\frac{\Sigma_1}{\Sigma_2}\right)\right\}\right) \equiv M_2. \tag{3.30}$$

Combining (3.29) and (3.30), we have proved that (3.19) holds with $M = \max\{M_1, M_2\}$.

Case 2. Suppose (3.16) holds. Without loss of generality, suppose that

$$\|y\| \geqslant \Phi\left(\frac{\Theta_1}{\Theta_3}\right). \tag{3.31}$$

Then, using (3.24) in (3.25) and together with (3.31) and (2.1), we find

$$\begin{aligned}\|y\| &\leqslant \frac{\Sigma_1}{1-\Sigma_3}+\frac{\Sigma_2}{1-\Sigma_3}\Phi\left(\frac{\Theta_1}{1-\Theta_2}+\frac{\Theta_3}{1-\Theta_2}\Phi^{-1}(\|y\|)\right)\\ &\leqslant \frac{\Sigma_1}{1-\Sigma_3}+\frac{\Sigma_2}{1-\Sigma_3}\Phi\left(\frac{2\Theta_3}{1-\Theta_2}\Phi^{-1}(\|y\|)\right)\\ &\leqslant \frac{\Sigma_1}{1-\Sigma_3}+\frac{\Sigma_2}{1-\Sigma_3}\frac{\Phi\left(\Phi^{-1}(\|y\|)\right)}{w\left(\frac{1-\Theta_2}{2\Theta_3}\right)}\\ &= \frac{\Sigma_1}{1-\Sigma_3}+\frac{\Sigma_2}{1-\Sigma_3}\frac{1}{w\left(\frac{1-\Theta_2}{2\Theta_3}\right)}\|y\|.\end{aligned} \tag{3.32}$$

Since $\frac{\Sigma_2}{1-\Sigma_3}\left[w\left(\frac{1-\Theta_2}{2\Theta_3}\right)\right]^{-1}<1,$ it is clear from (3.32) that

$$\|y\| \leqslant \frac{\Sigma_1}{1-\Sigma_3}\left[1-\frac{\Sigma_2}{1-\Sigma_3}\frac{1}{w\left(\frac{1-\Theta_2}{2\Theta_3}\right)}\right]^{-1} \equiv W'. \tag{3.33}$$

From the above discussion, we have either $\|y\| \leqslant W'$ or $\|y\| < \Phi\left(\frac{\Theta_1}{\Theta_3}\right)$. Hence, we get

$$\|y\| \leqslant \max\left\{W',\ \Phi\left(\frac{\Theta_1}{\Theta_3}\right)\right\} \equiv M_3, \tag{3.34}$$

which on substituting into (3.24) gives

$$\|x\| \leqslant \frac{\Theta_1}{1-\Theta_2}+\frac{\Theta_3}{1-\Theta_2}\Phi^{-1}\left(\max\left\{W,\ \Phi\left(\frac{\Theta_1}{\Theta_3}\right)\right\}\right) \equiv M_4. \tag{3.35}$$

Coupling (3.34) and (3.35), we have shown that (3.19) holds with $M=\max\{M_3,M_4\}$.
The proof is complete. □

THEOREM 3.2. *Let (a)–(e) hold. Furthermore, suppose that*

(i) *there exist nonnegative numbers* c_f, C_F *and* $C_{I,s}$ *such that* $\sum_{s=1}^{\infty} C_{I,s}$ *is convergent, and the following hold for all* $(U,V)\in \mathbb{R}^2$ *and* $t\in(0,+\infty)$:

$$\begin{aligned}\left|f\left(t,\ \frac{U}{\rho(t)},\ \frac{V}{\varrho(t)}\right)\right| &\leqslant c_f,\\ \left|F\left(t,\ \frac{U}{\rho(t)},\ \frac{V}{\varrho(t)}\right)\right| &\leqslant C_F,\\ \left|I\left(t_s,\ \frac{U}{\rho(t_s)},\ \frac{V}{\varrho(t_s)}\right)\right| &\leqslant C_{I,s};\end{aligned}$$

(ii) *there exist nonnegative numbers* c_g, C_G *and* $C_{J,s}$ *such that* $\sum_{s=1}^{\infty} C_{J,s}$ *is convergent, and the following hold for all* $(U,V) \in \mathbb{R}^2$ *and* $t \in (0,\infty)$:

$$\left|g\left(t,\ \frac{U}{\rho(t)},\ \frac{V}{\varrho(t)}\right)\right| \leqslant c_g,$$
$$\left|G\left(t,\ \frac{U}{\rho(t)},\ \frac{V}{\varrho(t)}\right)\right| \leqslant C_G,$$
$$\left|J\left(t_s,\ \frac{U}{\rho(t_s)},\ \frac{V}{\varrho(t_s)}\right)\right| \leqslant C_{J,s}.$$

Then, the system (1.3) has at least one solution in $X \times Y$.

Proof. In Theorem 3.1, choose $a_f = a_g = b_f = b_g = 0$, $A_F = A_G = B_F = B_G = 0$ and $A_{I,s} = A_{J,s} = B_{I,s} = B_{J,s} = 0$. It is easy to see that (3.15) and (3.16) hold. We get Theorem 3.2. The proof is completed. □

4. Examples

To illustrate the usefulness of our main result, we present an example that Theorem 3.1 can readily apply.

EXAMPLE 4.1. Consider the following impulsive boundary value problem

$$\begin{cases} D_{0^+}^{\frac{2}{5}} u(t) = t^{-\frac{1}{2}} \left[c_0 + b_0 \dfrac{(t-s)^{3/5}}{1+t^{2/3}} u(t) + a_0 \dfrac{(t-s)^{6/5}}{(1+t^{2/3})^3} (v(t))^3\right], \\ \qquad\qquad t \in (s, s+1], \quad s = 0,1,2,\cdots, \\ D_{0^+}^{\frac{3}{5}} v(t) = t^{-\frac{1}{2}} \left[c_1 + b_1 \dfrac{(t-s)^{1/5}}{(1+t^{2/3})^{1/3}} (u(t))^{\frac{1}{3}} + a_1 \dfrac{(t-s)^{2/5}}{1+t^{2/3}} v(t)\right], \\ \qquad\qquad t \in (s, s+1], \quad s = 0,1,2,\cdots, \\ \lim\limits_{t\to 0} t^{\frac{3}{5}} u(t) = B_0 \displaystyle\int_0^{\infty} e^{-s} \frac{1}{1+s^{2/3}} u(s)\,ds, \\ \lim\limits_{t\to 0} t^{\frac{2}{5}} v(t) = B_1 \displaystyle\int_0^{\infty} e^{-s} \frac{1}{(1+s^{2/3})^{1/3}} (u(s))^{\frac{1}{3}}\,ds, \\ \lim\limits_{t\to s^+} (t-s)^{\frac{3}{5}} u(t) = 2^{-s}, \quad s = 1,2,\cdots, \\ \lim\limits_{t\to s^+} (t-s)^{\frac{2}{5}} v(t) = 3^{-s}, \quad s = 1,2,\cdots \end{cases} \tag{4.1}$$

where c_0, b_0, a_0, c_1, b_1, a_1, B_0 and B_1 are constants.

Corresponding to system (1.3) we have

(a) $\alpha = \frac{2}{5}$, $\beta = \frac{3}{5}$,

(b) $0 = t_0 < t_1 = 1 < \cdots < t_s = s < \cdots$ with $\lim\limits_{s\to\infty} s = \infty$,

(c) $m(t) = t^{-\frac{1}{2}} = n(t)$ are singular at $t = 0$, $|m(t)| = |n(t)| \leqslant L_1 t^k = L_2 t^l$ with $L_1 = L_2 = 1$ and $k = l = -\frac{1}{2}$,

(d) $\phi(t) = e^{-t} = \psi(t)$ satisfy $\phi, \psi \in L^1(0,\infty)$, and

(e) f, g, F, G, I and J are defined by

$$\rho(t) = \frac{(t-s)^{3/5}}{1+t^{2/3}}, \quad t \in (s, s+1], \quad s = 0, 1, 2, \cdots,$$

$$\varrho(t) = \frac{(t-s)^{2/5}}{1+t^{2/3}}, \quad t \in (s, s+1], \quad s = 0, 1, 2, \cdots,$$

$$\begin{aligned}
&f\left(t, \frac{x}{\rho(t)}, \frac{y}{\varrho(t)}\right) = c_0 + b_0\, x + a_0\, y^3,\\
&g\left(t, \frac{x}{\rho(t)}, \frac{y}{\varrho(t)}\right) = c_1 + b_1 x^{1/3} + a_1\, y,\\
&F\left(t, \frac{x}{\rho(t)}, \frac{y}{\varrho(t)}\right) = B_0\, x, \qquad G\left(t, \frac{x}{\rho(t)}, \frac{y}{\varrho(t)}\right) = B_1\, x^{\frac{1}{3}},\\
&I\left(s, \tfrac{x}{\rho(s)}, \tfrac{y}{\varrho(s)}\right) = 2^{-s}, \quad J\left(s, \tfrac{x}{\rho(s)}, \tfrac{y}{\varrho(s)}\right) = 3^{-s}, \quad s = 1, 2, \cdots.
\end{aligned}$$

Choose $\sigma = \delta = \frac{2}{3}$. Then, $\sigma > k+1$ and $\delta > l+1$. It is easy to show that

(A) f, g, F, G are Carathéodory functions,

(B) I and J are discrete Carathéododory functions.

Furthermore, in the context of Theorem 3.1, we have $\Phi^{-1}(x) = x^3$ with supporting function $w(x) = x^{\frac{1}{3}}$, and $\Phi(x) = x^{\frac{1}{3}}$ with supporting function $\nu(x) = x^3$. It is easy to see that conditions (i) and (ii) in Theorem 3.1 are satisfied with

$$\begin{aligned}
&c_f = |c_0|, \quad b_f = |b_0|, \quad a_f = |a_0|,\\
&C_F = 0, \quad B_F = |B_0|, \quad A_F = 0, \quad C_{I,s} = 2^{-s}, \quad B_{I,s} = A_{I,s} = 0,\\
&c_g = |c_1|, \quad b_g = |b_1|, \quad a_g = |a_1|,\\
&C_G = 0, \quad B_G = |B_1|, \quad A_G = 0, \quad C_{J,s} = 3^{-s}, \quad B_{J,s} = A_{J,s} = 0.
\end{aligned}$$

By direct computation, we get

$$\begin{aligned}
\Theta_2 &= \frac{\sqrt[4]{27}}{4} \frac{\mathbf{B}(2/5, 1/2)}{\Gamma(2/5)} |b_0| + |B_0|,\\
\Theta_3 &= \frac{\sqrt[4]{27}}{4} \frac{\mathbf{B}(2/5, 1/2)}{\Gamma(2/5)} |a_0|,\\
\Sigma_2 &= \frac{\sqrt[4]{27}}{4} \frac{\mathbf{B}(3/5, 1/2)}{\Gamma(3/5)} |b_1| + |B_1|,
\end{aligned}$$

$$\Sigma_3 = \frac{\sqrt[4]{27}}{4}\frac{\mathbf{B}(3/5,1/2)}{\Gamma(3/5)}|a_1|.$$

Applying Theorem 3.1, we see that system (4.1) has at least one solution if (3.15) or (3.16) holds, i.e., if

$$\frac{\sqrt[4]{27}}{4}\frac{\mathbf{B}(3/5,1/2)}{\Gamma(3/5)}|a_1| < 1, \quad \frac{\sqrt[4]{27}}{4}\frac{\mathbf{B}(2/5,1/2)}{\Gamma(2/5)}|b_0| + |B_0| < 1,$$

and

$$\frac{\frac{\sqrt[4]{27}}{4}\frac{\mathbf{B}(2/5,1/2)}{\Gamma(2/5)}|a_0|}{1-\frac{\sqrt[4]{27}}{4}\frac{\mathbf{B}(2/5,1/2)}{\Gamma(2/5)}|b_0| - |B_0|}\left[\frac{\frac{\sqrt[4]{27}}{2}\frac{\mathbf{B}(3/5,1/2)}{\Gamma(3/5)}|b_1| + 2|B_1|}{1-\frac{\sqrt[4]{27}}{4}\frac{\mathbf{B}(3/5,1/2)}{\Gamma(3/5)}|a_1|}\right]^3 < 1$$

or

$$\frac{\frac{\sqrt[4]{27}}{4}\frac{\mathbf{B}(3/5,1/2)}{\Gamma(3/5)}|b_1| + |B_1|}{1-\frac{\sqrt[4]{27}}{4}\frac{\mathbf{B}(3/5,1/2)}{\Gamma(3/5)}|a_1|}\left[\frac{1-\frac{\sqrt[4]{27}}{4}\frac{\mathbf{B}(2/5,1/2)}{\Gamma(2/5)}|b_0| - |B_0|}{\frac{\sqrt[4]{27}}{2}\frac{\mathbf{B}(2/5,1/2)}{\Gamma(2/5)}|a_0|}\right]^{-\frac{1}{3}} < 1. \tag{4.2}$$

REMARK 4.1. It is easy to see from (4.2) that system (4.1) has at least one solution for sufficiently small $|a_0|$, $|b_0|$, $|a_1|$, $|b_1|$, $|B_0|$ and $|B_1|$.

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