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ELASTICITY FOR ECONOMIC PROCESSES WITH MEMORY: FRACTIONAL DIFFERENTIAL CALCULUS APPROACH

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Abstract. Derivatives of non-integer orders are applied to generalize notion of elasticity in framework of economic dynamics with memory. Elasticity of Y with respect to X is defined for the case of a finite-interval fading memory of changes of X and Y. We define generalizations of point price elasticity of demand to the case of processes with memory. In these generalizations we take into account dependence of demand not only from current price (price at current time), but also all changes of prices for some time interval. For simplification, we will assume that there is one parameter, which characterizes a degree of damping memory over time. The properties of the suggested fractional elasticities and examples of calculations of these elasticities of demand are suggested.

1. Introduction

Theory of derivatives and integrals of non-integer (fractional) orders [1, 2, 3, 4] has a long history [5, 6]. Fractional calculus has wide applications in dynamical systems theory since it allows us to describe systems and media that are characterized by power-law non-locality and long-term memory (for example see [7, 8] and references therein). A variety of models, which are based on application of the fractional-order derivatives and integrals, have been proposed to describe behavior of financial and economical processes from different points of view [9]–[18].

Are known various types of fractional derivatives that are suggested by Riemann, Liouville, Riesz, Caputo, Grünwald, Letnikov, Sonin, Marchaud, Weyl and some others scientists [1, 2]. Derivatives and integrals of non-integer orders have unusual geometric [19], probabilistic [20, 21] and discrete interpretations [22, 23].

These fractional derivatives have a set of unusual properties [24]–[29] that should be satisfied for all type of derivatives of non-integer orders. For example such properties include a violation of the Leibniz rule (derivative of the product of two functions) and a violation of chain rules (derivative of the composition of two functions) [24, 27, 29]. It should be emphasized that the violation of the standard Leibniz rule [24, 29] is a characteristic property of fractional-order derivatives. The unusual properties of the fractional-order derivatives allow us to describe complex properties of dynamical systems.

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One of the most important areas of application of differential operators is a description of economic dynamics by using the concept of elasticity. Elasticity shows a relative change of an economic indicator under influence of change of an economic factor on which it depends at constant remaining factors acting on it. Usually effects of memory are ignored in the concept of elasticity. For example, the definition of the standard point-price elasticity of demand at time point $t = t_0$, which is expressed by the equation

$$E(Q(t);p(t);t_0) = \left(\frac{p(t)}{Q(t)} \frac{dQ/dt}{dp/dt} \right)_{t=t_0} = \left(\frac{p}{Q} \frac{dQ}{dp} \right)_{t=t_0}, \tag{1}$$

where Q is the quantity demanded and p is the price of a good. Equation (1) assumes that the elasticity depends only on the current price at $t = t_0$ an price at infinitesimal neighborhood of point t_0. In general, we should take into account that demand can depend on all changes of prices during a finite interval of time, since behavior of buyers can be determined by the presence of a memory of previous price changes. We can say that definition (1) can be used only if all buyers have a total amnesia.

In this paper we suggest a generalization of the point elasticity by using fractional-order derivatives to remove amnesia of buyers in the concept of elasticity. Derivatives of non-integer orders allow us to take into account a memory effect. Therefore the fractional generalization of point elasticity of demand cannot be considered as a point economic indicator only. Fractional elasticity depends on finite interval of time and/or price range, in addition to parameter of memory decay (forgetting). For simplification, we will assume that there is one parameter α, which characterizes a degree of memory decay during time interval.

In Section 2, we give definitions of fractional derivatives. Some important properties of fractional derivatives are described. In Section 3, fractional generalizations of elasticities of Y with respect to X are suggested. In Section 4, the properties of fractional elasticities are considered. In Section 5, we define generalizations of point elasticity of demand to the cases of memory. In Section 6, some simple examples of calculations of fractional elasticities are suggested.

2. Fractional-order derivatives

There is a lot of type fractional-order derivatives that are suggested by Riemann, Liouville, Riesz, Caputo, Grünwald, Letnikov, Sonin, Marchaud, Weyl [1, 2]. In this article, we use the Caputo fractional derivative. The main distinguishing feature of the Caputo fractional derivative is that the Caputo fractional derivative of a constant is zero. This type of derivatives is used in order to elasticity of constant demand will be equal to zero, that allows us to correspond it to a perfectly inelastic demand.

2.1. Definition of Caputo fractional derivatives

The left-sided Caputo fractional derivative of order α of the function $f(t)$, where $t \in [a;b]$, is defined by

$$ {}_{a}^{C}D_{t}^{\alpha}[\tau]f(\tau) := {}_{a}I_{t}^{n-\alpha}[\tau] \left(\frac{d}{d\tau}\right)^{n} f(\tau), \tag{2} $$

and the right-sided Caputo fractional derivative is

$$ {}_{t}^{C}D_{b}^{\alpha}[\tau]f(\tau) := {}_{t}I_{b}^{n-\alpha}[\tau] \left(-\frac{d}{d\tau}\right)^{n} f(\tau), \tag{3} $$

where $n-1 < \alpha < n$ and ${}_{a}^{C}I_{t}^{\alpha}[\tau]$ is the left-sided Riemann-Liouville integral of order $\alpha > 0$

$$ {}_{a}I_{t}^{\alpha}[\tau]f(\tau) := \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \frac{f(\tau)\,d\tau}{(t-\tau)^{1-\alpha}}, \quad (t > a), \tag{4} $$

and ${}_{t}^{C}I_{b}^{\alpha}[\tau]$ is the right-sided Riemann-Liouville integral of order $\alpha > 0$

$$ {}_{t}^{C}I_{b}^{\alpha}[\tau]f(\tau) := \frac{1}{\Gamma(\alpha)} \int_{t}^{b} \frac{f(\tau)\,d\tau}{(\tau-t)^{1-\alpha}}, \quad (t < b). \tag{5} $$

We also will use the simplified notations: ${}_{a}^{C}D_{t}^{\alpha}f(t)$ instead of ${}_{a}^{C}D_{t}^{\alpha}[\tau]f(\tau)$ and ${}_{t}^{C}D_{b}^{\alpha}f(t)$ instead of ${}_{t}^{C}D_{b}^{\alpha}[\tau]f(\tau)$.

2.2. Properties of Caputo fractional derivatives

To calculate the fractional elasticities we will use the following properties of the Caputo fractional derivatives.

1) The left-sided and right-sided Caputo derivatives are linear operators

$$ {}_{a}^{C}D_{t}^{\alpha}\Big(c_1 Y_1(t) + c_2 Y_2(t)\Big) = c_1\, {}_{a}^{C}D_{t}^{\alpha} Y_1(t) + c_2\, {}_{a}^{C}D_{t}^{\alpha} Y_2(t). \tag{6} $$

2) The Caputo fractional derivatives of power functions are given (see Property 2.16 of [2]) by the equations

$$ {}_{a}^{C}D_{t}^{\alpha}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\,(t-a)^{\beta-\alpha}, \quad (t > a,\ n-1 < \alpha < n,\ \beta > n-1) \tag{7} $$

$$ {}_{t}^{C}D_{b}^{\alpha}(b-t)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\,(b-t)^{\beta-\alpha}, \quad (t < b,\ n-1 < \alpha < n,\ \beta > n-1), \tag{8} $$

and

$$ {}_{a}^{C}D_{t}^{\alpha}(t-a)^{k} = 0, \quad {}_{t}^{C}D_{b}^{\alpha}(b-t)^{k} = 0 \quad (k = 0, 1, ..., n-1). \tag{9} $$

In particular, we have

$$ {}_{a}^{C}D_{t}^{\alpha}1 = 0, \quad {}_{t}^{C}D_{b}^{\alpha}1 = 0. \tag{10} $$

For the case $a = 0$, we can use

$$ {}_0^C D_t^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \quad (t > 0,\ n-1 < \alpha < n,\ \beta > n-1), \tag{11} $$

$$ {}_0^C D_t^{\alpha} t^{\alpha} = \Gamma(\alpha+1), \quad ({}_0^C D_t^{\alpha})^n t^{\alpha} = 0. \tag{12} $$

3) The fractional derivative of the exponential functions are given (see Property 2.17 of [2]) in the form

$$ {}_{-\infty}^{C} D_t^{\alpha}\, e^{\lambda t} = \lambda^{\alpha} e^{\lambda t}, \quad (\lambda > 0). \tag{13} $$

$$ {}_t^C D_{+\infty}^{\alpha}\, e^{-\lambda t} = \lambda^{\alpha} e^{-\lambda t}, \quad (\lambda > 0). \tag{14} $$

4) We also can use Lemma 2.23 of [2] in the form

$$ {}_a^C D_t^{\alpha} E_{\alpha}[\lambda\,(t-a)^{\alpha}] = \lambda\, E_{\alpha}[\lambda\,(t-a)^{\alpha}], \tag{15} $$

where $E_{\alpha}[\lambda\,(t-a)^{\alpha}]$ is the Mittag-Leffler function. Equation (15) means that the Mittag-Leffler function is invariant with respect to the left-sided Caputo fractional derivative ${}_a^C D_t^{\alpha}$, but it is not the case for the right-sided Caputo derivatives.

5) For example, the fractional-order derivatives of the composition of two functions violate the standard chain rule

$$ {}_a^C D_t^{\alpha}[\tau] Y(X(\tau)) \neq {}_{X(a)}^{C} D_{X(t)}^{\alpha}[X] Y(X) \cdot {}_a^C D_t^{\alpha}[\tau] X(\tau). \tag{16} $$

The violation the standard chain rule is one of the main properties of derivatives of non-integer orders [27].

6) The fractional-order derivatives of the product of two functions (the Leibniz rule) violate the usual rule

$$ {}_a^C D_t^{\alpha}[\tau] \Big(Y_1(\tau) Y_2 \tau) \Big) \neq \Big({}_a^C D_t^{\alpha}[\tau] Y_1(\tau) \Big) Y_2(\tau) + Y_1(\tau) \Big({}_a^C D_t^{\alpha}[\tau] Y_2(\tau) \Big). \tag{17} $$

This violation is a main characteristic property of all derivatives of non-integer orders [24, 29].

7) In general, an action of the fractional derivative on the a fractional derivative is not the same as the action of fractional derivative of order 2α, i.e.

$$ {}_a^C D_t^{\alpha}[t']\, {}_a^C D_{t'}^{\alpha}[\tau] Y(\tau) \neq {}_a^C D_t^{2\alpha}[\tau] Y(\tau). \tag{18} $$

This inequality means that the semi-group property cannot be realized for all type of functions. Equality instead of the inequality is obtained only for a narrow class of functions [2].

3. Fractional elasticities of Y with respect to X

The most important area of application of differential calculus is to describe the economics with the help of the concept of elasticity. The elasticity shows a relative change of an economic indicator under influence of change of economic factors on which it depends at constant remaining factors affecting it.

In this section, we define generalizations of point elasticity of Y with respect to X by taking into account a long-term memory and a finite-interval memory of changes of economic factor X and indicator Y. We will consider the following forms of memory.

(1) In the general case, the economic indicator and economic factors can depend on time, i.e. Y and X are functions of time $t \in [t_i;t_f]$. The absence of memory (amnesia) means that the value of $Y(t)$ is determined only by the values of $X(t)$ at the point $t = t_0 \in [t_i;t_f]$ and in infinitely small neighborhood of this point. The presence of a memory means that the value of $Y(t)$ depends on values of $X(t)$ at all points t of the finite interval $[t_i;t_f]$.

(2) The presence of a memory can also mean that the values of $Y(X_0)$ actually depends not only on X_0 but it also a depends on X from the intervals $[X_i;X_0)$ and $(X_0;X_f]$. In general, the time parameter cannot be excluded to have an explicit dependence of Y on X in the form of function. In a rigorous mathematical description of processes with memory, we should apply integro-differential equations. For simplification, we will assume that we have a solution of this equation in the form $Y = Y(X)$.

An important property of processes with memory is a decay property of memory, a "fading of memory" [31, 32, 33]. For simplification, we will assume that there is one parameter α, which characterizes a degree of damping memory over time.

DEFINITION 1. The fractional T-elasticity $E_\alpha(Y(t);X(t);[t_i,t_0])$ of order α at $t = t_0$ of $Y = Y(t)$ with respect to $X = X(t)$ is defined by the equation

$$E_\alpha(Y(t);X(t);[t_i,t_0]) := \frac{X(t_0)}{Y(t_0)} \frac{{}_{t_i}^{C}D_{t_0}^{\alpha}[t]Y(t)}{{}_{t_i}^{C}D_{t_0}^{\alpha}[t]X(t)}, \tag{19}$$

where $t \in [t_i,t_0]$.

The fractional T-elasticity $E_\alpha(Y(t);X(t);[t_i,t_0])$ describes an elasticity for the economic processes with a memory of the changes of economic factors and indicator. This type of memory describes the dependence of the economic indicator Y not only on $X(t_0)$ at the current time t_0 but also the economic factor $X(t)$ at all $t \in [t_i;t_0]$. The order α is the parameter that characterizes the degree of damping memory over time. In general, we can consider fractional elasticity with two different parameters α and β to describe fading memory of $Y(t)$ and $X(t$ respectively.

DEFINITION 2. Let us consider an economic indicator $Y = Y(X)$ as a function of an economic factor $X \in [X_i;X_f]$. The left-sided and right-sided fractional X-elasticities $E_{\alpha,l}(Y(X);[X_i,X_0])$ and $E_{\alpha,r}(Y(X);[X_0,X_f])$ of order α at $X_0 \in [X_i;X_f]$ of Y with respect to X are defined by the equations

$$E_{\alpha,l}(Y(X);[X_i,X_0]) := \frac{(X_0)^{\alpha}}{Y(X_0)} \, {}_{X_i}^{C}D_{X_0}^{\alpha}[X]\,Y(X), \tag{20}$$

$$E_{\alpha,r}(Y(X);[X_0,X_f]) := \frac{(X_0)^{\alpha}}{Y(X_0)} \, {}_{X_0}^{C}D_{X_f}^{\alpha}[X]\,Y(X), \tag{21}$$

where $X_i = X_{min}$ and $X_f = X_{max}$ are initial and final points of the investigated interval of the economic factor $X \in [X_i;X_f]$. Here ${}_{X_i}^{C}D_{X_0}^{\alpha}$ is the left-sided Caputo derivative and ${}_{X_0}^{C}D_{X_f}^{\alpha}$ is the right-sided Caputo derivative of order $\alpha > 0$.

Using that the standard (point) elasticity of Y with respect to X can be represented as a derivative of $f(t) = \ln(Y(t))$ by $g(t) = \ln(X(t))$ in the form

$$E(p,t_0) := \left(\frac{df(t)}{dg(t)}\right)_{t=t_0} = \left(\frac{d\ln(Y(t))}{d\ln(X(t))}\right)_{t=t_0}, \tag{22}$$

we can also define the corresponding fractional generalization by using the fractional derivative of function $f(t) = \ln(Y(t))$ by a function $g(t) = \ln(X(t))$ (see Section 18.2 of [1] and Section 2.5 of [2]).

DEFINITION 3. The fractional *Log*-elasticity $E_{\alpha,log}(Y(t);X(t);[t_i,t_0])$ of order α at $t = t_0 \in [t_i;t_f]$ is defined by the equation

$$E_{\alpha,log}(Y(t);X(t);[t_i,t_0]) :=$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_{t_i}^{t_0} d\tau \, \frac{dg(\tau)}{d\tau} \, \frac{f(\tau)}{(g(t)-g(\tau))^{\alpha+1-n}} \left(\frac{1}{dg(\tau)/d\tau}\,\frac{d}{d\tau}\right)^{n} f(\tau), \quad (t_0 > t_i), \tag{23}$$

where $n-1 \leqslant \alpha \leqslant n$, $f(t) = \ln(Y(t))$ and $g(t) = \ln(X(t))$.

REMARK 1. For the case $\alpha = 1$, equations (19), (20) and (21) take the forms

$$E_1(Y(t);X(t);[t_i,t_0]) = \frac{X(t_0)}{Y(X_0)} \left(\frac{dY(t)/dt}{dX(t)/dt}\right)_{t=t_0} \tag{24}$$

and

$$E_{1,l}(Y(X);[X_i,X_0]) = E_{1,r}(Y(X);[X_0,X_f]) = \frac{X_0}{Y(X_0)} \left(\frac{dY(X)}{dX}\right)_{X=X_0}, \tag{25}$$

where the elasticity does not depend on $t \neq t_0$ and $X \neq X_0$. This means that the case $\alpha = 1$ corresponds to the economic processes without memory.

REMARK 2. Using the chain rule

$$\frac{dY(X(t))}{dt} = \left(\frac{dY(X)}{dX}\right)_{X=X(t)} \frac{dX(t)}{dt}, \tag{26}$$

we have the equality of the fractional T-elasticity and the fractional X-elasticity for the case $\alpha = 1$ of the economic dynamics without memory,

$$E_1(Y(t);X(t);[t_i,t_0]) = E_{1,l}(Y(X);[X_i,X_0]) = E_{1,r}(Y(X);[X_0,X_f]). \tag{27}$$

This case corresponds to economic the case of total amnesia. The standard point elasticity of Y with respect to X describes economic processes, when market participants have amnesia.

Using the suggested fractional elasticities of Y with respect to X, which are suggested in Section 3, we consider generalizations of point-price elasticities of demand in Section 5.

4. Properties of fractional elasticities

Let us give main properties of the suggested fractional elasticities. For simplification, we describe these properties for $t_i = 0$ and $X_i = 0$.

1. The fractional elasticity is a dimensionless quantity,

$$E_\alpha(\lambda\, Y(t);X(t);[0,t_0]) = E_\alpha(Y(t);X(t);[0,t_0]), \tag{28}$$

$$E_\alpha(Y(t);\lambda\, X(t);[0,t_0]) = E_\alpha(Y(t);X(t);[0,t_0]). \tag{29}$$

These equations mean that its do not depend on units of the economic indicator Y and the economic factor X.

2. The fractional T-elasticity of inverse function is inverse

$$E_\alpha(X(t);Y(t);[0,t_0]) = \frac{1}{E_\alpha(Y(t);X(t);[0,t_0])}. \tag{30}$$

In general, the fractional X-elasticities of inverse functions are not inverse

$$E_{\alpha,l}(Y(X);[0,X_0]) \neq \frac{1}{E_\alpha(X(Y);[0,Y_0])}, \quad E_{\alpha,r}(Y(X);[X_0,X_f]) \neq \frac{1}{E_\alpha(X(Y);[Y_0,Y_f])}. \tag{31}$$

These inequalities become equalities for $\alpha = 1$.

3. In general, the fractional elasticity of the product of two functions, which depend on the same argument, does not equal to the sum of elasticities

$$E_\alpha(Y_1(t)\cdot Y_2(t);X(t);[0,t_0]) \neq E_\alpha(Y_1(t);X(t);[0,t_0]) + E_\alpha(Y_2(t);X(t);[0,t_0]) \tag{32}$$

for $\alpha \neq 1$. This inequality becomes an equality for $\alpha = 1$. Inequality (32) caused by the violation of the Liebniz rule (17).

4. The fractional elasticity of the sum of two functions, which depend on the same argument, is given by the equation

$$E_\alpha(Y_1(t)+Y_2(t);X(t);[0,t_0])$$

$$= \frac{1}{Y_1 + Y_2} \left(Y_1(t)\, E_\alpha(Y_1(t); X(t); [0,t_0]) + Y_2(t)\, E_\alpha(Y_2(t); X(t); [0,t_0]) \right). \tag{33}$$

5. The fractional elasticity of the power function is a constant

$$E_{\alpha,l}(X^\beta; [0,X_0]) = \frac{\beta}{\alpha}, \tag{34}$$

where $\beta > n-1$ and $n-1 < \alpha < n$ for all $n \in \mathbb{N}$.

6. The fractional elasticity of the exponential function is given by the equation

$$E_{\alpha,r}(e^{-\lambda X}; [X_0;\infty)) = (\lambda X)^\alpha, \tag{35}$$

where $\lambda > 0$

7. The fractional elasticity of the linear function is given by the equation

$$E_{\alpha,l}(a_0 + a_1 X; [0,X_0]) = \frac{1}{\Gamma(2-\alpha)} \frac{a_1 X}{a_0 + a_1 X}. \tag{36}$$

8. For derivatives of non-integer orders, the standard chain rule cannot be satisfied in general. For example, the chain rules for fractional derivatives of a composite function (see Eq. 2.209 in Section 2.7.3 of [3]) have the form that is similar to the following

$$\mathscr{D}_t^\alpha Y(X(t)) = \frac{t^\alpha Y(X(t))}{\Gamma(1-\alpha)} + \sum_{k=1}^{\infty} C_k^\alpha \frac{k!\, t^{k-\alpha}}{\Gamma(k-\alpha+1)} \times \sum_{m=1}^{k} (D_X^m Y(X))_{X=X(t)} \sum \prod_{r=1}^{k} \frac{1}{a_r!} \left(\frac{(D_t^r X)(t)}{r!} \right)^{a_r}, \tag{37}$$

where $t > 0$, $\sum$ extends over all combinations of non-negative integer values of a_1, $a_2, ..., a_k$ such that $\sum_{r=1}^{k} r a_r = k$ and $\sum_r^k a_r = m$. Obviously, that equation (37) is much more complicated than the chain rule (26) for the first order derivative . Note that equation (37) is a generalization of the chain rule for the derivative of integer order $n \in \mathbb{N}$ that is represented by the Faá di Bruno's formula of the form

$$D_t^n Y(X(t)) = n! \sum_{m=1}^{n} (D_X^m Y(X))_{X=X(t)} \sum \prod_{r=1}^{n} \frac{1}{a_r!} \left(\frac{D_t^r X(t)}{r!} \right)^{a_r}, \tag{38}$$

where $D_t^n = d^n/dt^n$. Therefore, we have the inequalities

$$E_\alpha(Y(t); X(t); [t_i, t_0]) \neq E_{\alpha,l}(Y(X); [X_i, X_0]), \tag{39}$$

$$E_\alpha(Y(t); X(t); [t_i, t_0]) \neq E_{\alpha,r}(Y(X); [X_0, X_f]) \tag{40}$$

for non-integer values of the order α. As a result, the fractional X-elasticities and the fractional T-elasticity should be considered as independent characteristics in the economic dynamics with memory.

9. The frational elasticities of constant demand are equal to zero.

$$E_\alpha(\text{const}; X(t); [t_i, t_0]) = 0, \quad E_{\alpha,l}(\text{const}; [X_i, X_0]) = E_{\alpha,r}(\text{const}; [X_0, X_f]) = 0, \tag{41}$$

that corresponds to perfectly inelastic demand.

These properties can be directly derived from the properties of the Caputo fractional derivative and the definition of the fractional elasticities.

5. Fractional elasticities of demand

In this section, we define generalizations of point-price elasticity of demand to the cases of memory. In these generalizations we take into account dependence of demand not only from the current price (price at current time), but also changes of prices in some interval (prices that were before this current price). For simplification, we will assume that there is one parameter α, which characterizes a degree of damping memory over time.

DEFINITION 4. Let demand $Q = Q(t)$ and price $p = p(t)$ be functions of time variable $t \in [t_i; t_f]$. The fractional T-elasticity $E_\alpha(Q(t); p(t); [t_i, t_0])$ of order α at $t = t_0$ of demand $Q(t)$ with respect to price $p(t)$ is defined by the equation

$$E_\alpha(Q(t); p(t); [t_i, t_0]) := \frac{p(t_0)}{Q(t_0)} \frac{{}_{t_i}^{C}D_{t_0}^{\alpha}[t]\, Q(t)}{{}_{t_i}^{C}D_{t_0}^{\alpha}[t]\, p(t)}, \tag{42}$$

where $t \in [t_i, t_0]$, and $t_i < t_0 < t_f$.

The fractional T-elasticity (42) describes an elasticity of demand for the processes in economic dynamical systems with the memory of price changes over time. This type of memory describes the dependence of demand Q not only from the price $p = p(t_0)$ at the current time t_0 but also the prices $p(t)$ that were before this price, i.e. all prices at $t \in [t_i; t_0]$. The order α is the parameter that characterizes the degree of damping memory over time.

DEFINITION 5. Let us consider a demand $Q = Q(p)$ as a function of price $p \in [p_l; p_h]$. The left-sided and right-sided fractional p-elasticities $E_{\alpha,l}(Q(p); [p_l, p_0])$ and $E_{\alpha,r}(Q(p); [p_0, p_h])$ of order α at $p_0 \in [p_l; p_h]$ of demand $Q = Q(p)$ is defined by the equations

$$E_{\alpha,l}(Q(p); [p_l, p_0]) := \frac{(p_0)^\alpha}{Q(p_0)}\, {}_{p_l}^{C}D_{p_0}^{\alpha}[p]\, Q(p), \tag{43}$$

$$E_{\alpha,r}(Q(p); [p_0, p_h]) := \frac{(p_0)^\alpha}{Q(p_0)}\, {}_{p_0}^{C}D_{p_h}^{\alpha}[p]\, Q(p), \tag{44}$$

where $p_l = p_{min}$ is a lowest price and $p_h = p_{max}$ is the highest price; ${}_{X_i}^{C}D_{X_0}^{\alpha}$ and ${}_{X_0}^{C}D_{X_f}^{\alpha}$ are the left-sided and right-sided Caputo derivatives of order $\alpha > 0$.

The fractional p-elasticities (43) and (44) describe an elasticity of demand for the processes in economic dynamical systems with price memory. The elasticity (43) takes into account a "memory of low prices". The "memory of high prices" is taken into account by the elasticity (44). These types of memory describe a dependence of demand Q not only on the current price p_0 but also all prices p of the given range ($p_l \leqslant p \leqslant p_h$). The order α characterizes a degree of damping memory over time.

Analogously to generalization of the price elasticities of demand, we can generalize of other types of elasticity. For example, we can give definitions of fractional income elasticity of demand. Using the demand function $Q = Q(t)$ and income function $I =$

$I(t)$ of time variable $t \in [t_i;t_0]$, the fractional income T-elasticity $E_\alpha(Q(t);I(t);[t_i,t_0])$ of order α at $t = t_0$ can be defined by the equation

$$E_\alpha(Q(t);I(t);[t_i,t_0]) := \frac{I(t_0)}{Q(t_0)} \frac{{}_{t_i}^{C}D_{t_0}^{\alpha}[t]\,Q(t)}{{}_{t_i}^{C}D_{t_0}^{\alpha}[t]\,I(t)}. \tag{45}$$

REMARK 3. In the definition of the fractional elasticities, we use the Caputo fractional derivatives instead of other types of derivatives. It is caused by that the Caputo fractional derivatives of a constant is equal to zero. This property leads us to zero velues of fractional elasticities elasticity for constant demand. Contrary to it the Riemann-Liouville fractional derivatives of a constant is not equal to zero (see equation (2.1.20) of [2]),

$${}_{0}^{RL}D_{p}^{\alpha}[p']\,Q(p') = \frac{p^{-\alpha}}{\Gamma(1-\alpha)}. \tag{46}$$

Therefore the fractional elasticities, which are defined by this type of derivatives, cannot be considered as a perfectly inelastic demand for the constant demand functions. For example, the corresponding left-sided fractional p-elasticity of the constant demand $Q(p) = q_0 = \mathrm{const}$ is the constant

$${}_{RL}E_{\alpha,l}(Q(p);[0,p]) := \frac{p^{\alpha}}{Q(p)}\, {}_{0}^{RL}D_{p}^{\alpha}[p']\,Q(p') = \frac{1}{\Gamma(1-\alpha)}, \tag{47}$$

where ${}_{0}^{RL}D_{p}^{\alpha}$ is the left-sided Riemann-Liouville derivative [2].

It should be noted that the fractional p-elasticities and the fractional T-elasticity should be considered as independent indicators of the economic dynamics with memory. This fact is based on the violation of the standard chain rule for derivatives of non-integer orders.

6. Examples of calculations

Let us consider simple examples of calculations of fractional elasticities. For simplification, we will use the demand equation

$$Q(p) = a_0 + a_1 p + a_2 p^2, \tag{48}$$

where p is the unit price and $Q(p)$ is the quantity demanded when the price is p.

Equation (48) is considered as a demand function for a product. Point-price elasticity is the elasticity of demand, which is defined by the equation

$$E(p) = (p/Q(p))(dQ(p)/dp).$$

To find the point elasticity of demand $E(p)$ for (48), we use

$$\frac{dQ(p)}{dp} = a_1 + 2a_2 p. \tag{49}$$

As a result, the standard (point-price) elasticity of demand is

$$E(p) = \frac{p}{Q(p)} \frac{dQ(p)}{dp} = \frac{a_1 p + 2a_2 p^2}{a_0 + a_1 p + a_2 p^2}. \tag{50}$$

Let us consider some examples of fractional elasticity for demand (48).

EXAMPLE 1. Let us consider the fractional p-elasticity (43) with $p_l = 0$ and $\alpha \in (0;1)$. Using (11), we get

$$\begin{aligned} {}_0^C D_p^\alpha Q(p) &= {}_0^C D_p^\alpha a_0 + a_1 {}_0^C D_p^\alpha p + a_2 {}_0^C D_p^\alpha p^2 \\ &= a_1 \frac{\Gamma(2)}{\Gamma(2-\alpha)} p^{1-\alpha} + a_2 \frac{\Gamma(3)}{\Gamma(3-\alpha)} p^{2-\alpha} \end{aligned} \tag{51}$$

Substitution of (51) into (43) gives the left-sided fractional p-elasticity in the form

$$\begin{aligned} E_{\alpha,l}(Q(p);[0,p]) &= \frac{(p)^\alpha}{Q(p)} {}_0^C D_p^\alpha [p] Q(p) = \frac{a_1 \frac{1}{\Gamma(2-\alpha)} p + a_2 \frac{2}{\Gamma(3-\alpha)} p^2}{a_0 + a_1 p + a_2 p^2} \\ &= \frac{1}{\Gamma(2-\alpha)} \frac{a_1 p + a_2 \frac{2}{2-\alpha} p^2}{a_0 + a_1 p + a_2 p^2}, \end{aligned} \tag{52}$$

where we use $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$ and $\Gamma(z+1) = z\Gamma(z)$. For $\alpha = 1$, equation (52) gives (50).

EXAMPLE 2. Let us consider the demand and price functions in the form

$$Q(t) = q_0 + q_1 t + q_2 t^2, \tag{53}$$

$$p(t) = p_0 t. \tag{54}$$

It is obvious that the substitution of (54) into (53) gives (48) with

$$a_0 = q_0, \quad a_1 = \frac{q_1}{p_0}, \quad a_2 = \frac{q_1}{p_0^2}. \tag{55}$$

Let us consider the fractional T-elasticity (42) with $t_i = 0$ and $\alpha \in (0;1)$. Using (11), we get

$${}_0^C D_t^\alpha Q(t) = q_1 \frac{\Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha} + q_2 \frac{\Gamma(3)}{\Gamma(3-\alpha)} t^{2-\alpha}. \tag{56}$$

and

$${}_0^C D_t^\alpha p(t) = p_0 \frac{\Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha}. \tag{57}$$

Substitution of (56) and (57) into (42) gives the fractional T-elasticity

$$\begin{aligned} E_\alpha(Q(t);p(t);[0,t_0]) &= \frac{p(t)}{Q(t)} \frac{0^C D_t^\alpha Q(t)}{{}_0^C D_t^\alpha p(t)} = \frac{p_0 t}{q_0 + q_1 t + q_2 t^2} \frac{q_1 \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} + q_2 \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha}}{p_0 \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha}} \\ &= \frac{p_0 t}{q_0 + q_1 t + q_2 t^2} \frac{q_1 t^{1-\alpha} + q_2 \frac{2}{(2-\alpha)} t^{2-\alpha}}{p_0 t^{1-\alpha}} \\ &= \frac{q_1 t + q_2 \frac{2}{(2-\alpha)} t^2}{q_0 + q_1 t + q_2 t^2} = \frac{a_1 p + a_2 \frac{2}{2-\alpha} p^2}{a_0 + a_1 p + a_2 p^2}, \end{aligned} \tag{58}$$

where we use $\Gamma(z+1) = z\Gamma(z)$ and equations (54), (55).

Because we have chosen simple equations of (53) and (54), the expressions (58) and (52) differ only by a factor

$$E_{\alpha,l}(Q(p);[0,p]) = \frac{1}{\Gamma(2-\alpha)} E_\alpha(Q(t);p(t);[0,t]). \tag{59}$$

In the general case, these fractional elasticity can be distinguished not only by a factor.

Example 3. Let us consider the demand and price functions in the following simple form

$$Q(t) = at^\beta, \tag{60}$$

$$p(t) = bt^\gamma. \tag{61}$$

Substitution of (61) into (60) gives

$$Q(p) = \frac{a}{b^{\beta/\gamma}} p^{\beta/\gamma}. \tag{62}$$

The standard (point-price) elasticity has the form

$$E(p) = \frac{p}{Q(p)} \frac{dQ(p)}{dp} = \frac{\beta}{\gamma}. \tag{63}$$

The fractional p-elasticity is given by the equation

$$E_{\alpha,l}(Q(p);[0,p]) = \frac{(p)^\alpha}{Q(p)} \, {}_0^C D_p^\alpha Q(p) = \frac{\Gamma(\beta/\gamma+1)}{\Gamma(\beta/\gamma+1-\alpha)}. \tag{64}$$

The fractional T-elasticity is written in the form

$$E_\alpha(Q(t);p(t);[0,t_0]) = \frac{p(t)}{Q(t)} \frac{{}_0^C D_t^\alpha Q(t)}{{}_0^C D_t^\alpha p(t)} = \frac{\Gamma(\beta+1)\Gamma(\gamma+1-\alpha)}{\Gamma(\gamma+1)\Gamma(\beta+1-\alpha)}. \tag{65}$$

For $\alpha = 1$, we get

$$E_\alpha(Q(t);p(t);[0,t_0]) = E_{\alpha,l}(Q(p);[0,p_0]) = E(p)$$

since $\Gamma(z+1) = z\Gamma(z)$. It is easy to see that the expression of the fractional p-elasticity (64) and the fractional T-elasticity (65) are different for $\alpha \neq 1$.

It is well-known the following conditions. If $E(p) < -1$, then demand is elastic and a percent increase in price yields a larger percent decrease in demand. If $-1 < E(p) < 0$, then demand is inelastic and a percent increase in price yields a smaller percent decrease in demand. It is evident that taking into account the effect of memory $(0 < \alpha < 1)$, we can get other inequalities for the price in comparison with the standard case ($\alpha = 1$).

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