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CHEBYSHEV TYPE INEQUALITY ON NABLA DISCRETE FRACTIONAL CALCULUS

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Abstract. In this paper, we establish some Chebyshev type inequalities on discrete fractional calculus with nabla operator (or backward difference operator).

1. Introduction

By the nineteenth century, efforts of a number of mathematicians, most notedly Riemann, Grünwald, Letnikov, and Liouville, lead to a consistent theory of fractional calculus for real variable functions. Although there are many definitions of fractional derivatives, the most known definitions are Riemann-Liouville and Caputo derivatives.

When it comes to the theory of discrete fractional calculus, we mention the paper presented by Diaz and Osler in 1974 [11]. In this paper, the authors introduced a fractional difference operator using an infinite series. In 1988, Gray and Zhang [15] introduced a new definition of a fractional difference operator and they proved a Leibniz formula, composition rule and power rule. Whereas Diaz et al. gave a definition for the delta (forward) difference operator, Gray et al. gave their definition for the nabla (backward) difference operator.

Mathematicians have began to pay attention to this theory for last three decades. As a pioneering work, Atici and Eloe [3] presented properties of a generalized falling function that plays a major role as an exponential function in difference calculus, power rule and commutativity of fractional sums. For more results we refer to [2, 4, 5, 6, 8, 14, 17].

Inequalities are useful tools in mathematics. In order to see the use of inequalities as mathematical tools, we refer to [13]. In this work, to show the continuous dependence of solutions of initial value problems on initial conditions author shows and uses fractional discrete analogue of Gronwall inequality. For more inequalities on discrete fractional calculus (delta or nabla case) see [1, 2, 7, 10, 12, 13, 16].

In this paper, our main purpose is to make a contribution to this area with establishing the discrete fractional analogue of Chebyshev's inequality using nabla operator.

Here, we will establish discrete fractional analogue of Chebyshev's inequality given below [9].

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Let f and g be two integrable functions in $[0,1]$. If both functions are simultaneously increasing or decreasing for same values of x in $[0,1]$, then

$$\int_0^1 f(x)g(x)dx \geqslant \int_0^1 f(x)dx \int_0^1 g(x)dx.$$

If one function is increasing and the other decreasing for the same values of x in $[0,1]$, then

$$\int_0^1 f(x)g(x)dx \leqslant \int_0^1 f(x)dx \int_0^1 g(x)dx.$$

2. Preliminaries on discrete nabla fractional calculus

In this section, we introduce the reader to basic concepts and results about discrete fractional calculus with nabla operator.

The rising function is defined by

$$t^{\overline{n}} = t(t+1)(t+2)\dots(t+n-1), \text{ for } n \in \mathbb{N}.$$

Using the Gamma function we can generalize the rising function as

$$t^{\overline{v}} = \frac{\Gamma(t+v)}{\Gamma(t)}, \quad v \in \mathbb{R} \text{ and } t \in \mathbb{R} \setminus \{\dots,-2,-1,0\}.$$

REMARK 1. Using the properties of the Gamma function, it is easily seen that for $t \geqslant 0$ and $v \geqslant 0$, we get $t^{\overline{v}} \geqslant 0$.

For $a \in \mathbb{R}$, we define the set $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$. Also, we use the notation $\rho(s) = s-1$ for the shift operator and $(\nabla f)(t) = f(t) - f(t-1)$ for the backward difference operator.

For a function $f : \mathbb{N}_a \to \mathbb{R}$, discrete fractional sum of order $v \geqslant 0$ is defined as

$$(\nabla_a^0 f)(t) = f(t), \quad t \in \mathbb{N},$$

$$(\nabla_a^{-v} f)(t) = \frac{1}{\Gamma(v)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{v-1}} f(s), \quad t \in \mathbb{N}_a, \quad v > 0.$$

REMARK 2. If $v = 1$, we get the summation operator

$$(\nabla_a^{-1} f)(t) = \sum_{s=a}^{t} f(s).$$

The following result will be used in the sequel.

LEMMA 1. (See [4, Lemma 2.1]) *If* $a \in \mathbb{R}$ *and* $\mu,\ \mu+\nu \in \mathbb{R}\setminus\{\dots,-2,-1\}$, *then*

$$\left(\nabla_a^{-\nu}(s-a+1)^{\overline{\mu}}\right)(t) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a+1)^{\overline{\mu+\nu}},\ t \in \mathbb{N}_a^{\nu}.$$

REMARK 3. The function $t \to (t-a)^{\overline{\nu}}$ defined on $\mathbb{N}_a$, $a \in \mathbb{R}$ and $\nu > 0$ is increasing. Indeed, we have that $\nabla(t-a)^{\overline{\nu}} = \nu(t-a)^{\overline{\nu-1}}$ and $(t-a)^{\overline{\nu-1}} \geqslant 0$.

DEFINITION 1. Two functions f and g are called synchronous, respectively asynchronous, on $\mathbb{N}_a$ if for all $\tau,\ s \in \mathbb{N}_a$, we have $(f(\tau)-f(s))(g(\tau)-g(s)) \geqslant 0$, respectively $(f(\tau)-f(s))(g(\tau)-g(s)) \leqslant 0$.

3. Discrete fractional Chebyshev type inequalities

We start by proving the main result of this paper.

THEOREM 1. *If* $\nu > 0$ *and* f,g *are two synchronous functions on* $\mathbb{N}_a$, *then*

$$(\nabla_a^{-\nu} fg)(t) \geqslant \frac{\Gamma(\nu+1)}{(t-a)^{\overline{\nu}}}(\nabla_a^{-\nu} f)(t)(\nabla_a^{-\nu} g)(t),\ t \in \mathbb{N}_a. \tag{1}$$

Proof. Since the functions f and g are synchronous on $\mathbb{N}_a$, then for all $\tau,\ s \in \mathbb{N}_a$, we have

$$(f(\tau)-f(s))(g(\tau)-g(s)) \geqslant 0,$$

i.e.

$$f(\tau)g(\tau)+f(s)g(s) \geqslant f(\tau)g(s)+f(s)g(\tau). \tag{2}$$

Now, multiplying both sides of (2) by $\frac{(t-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)}$, $t \in \mathbb{N}_a$ and $\tau \in \{a, a+1, \dots, t\}$, we obtain

$$\begin{aligned} &\frac{(t-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} f(\tau)g(\tau) + \frac{(t-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} f(s)g(s) \\ &\geqslant \frac{(t-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} f(\tau)g(s) + \frac{(t-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} f(s)g(\tau). \end{aligned} \tag{3}$$

Now, taking the sum of both sides of (3) for $\tau \in \{a, a+1, \dots, t\}$, we get

$$(\nabla_a^{-\nu} fg)(t) + f(s)g(s)(\nabla_a^{-\nu} 1)(t) \geqslant g(s)(\nabla_a^{-\nu} f)(t) + f(s)(\nabla_a^{-\nu} g)(t). \tag{4}$$

Multiplying both sides of (4) by $\frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)}$, $t \in \mathbb{N}_a$ and $s \in \{a, a+1, \dots, t\}$, we obtain

$$\begin{aligned} &\frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} (\nabla_a^{-\nu} fg)(t) + \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} f(s)g(s)(\nabla_a^{-\nu} 1)(t) \\ &\geqslant \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} g(s)(\nabla_a^{-\nu} f)(t) + \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} f(s)(\nabla_a^{-\nu} g)(t), \end{aligned} \tag{5}$$

and again, taking sum on both sides of (5) for $s \in \{a, a+1, \ldots, t\}$, and using Lemma 1, we get

$$\begin{aligned}&(\nabla_a^{-\nu}1)(t)(\nabla_a^{-\nu}fg)(t)+(\nabla_a^{-\nu}fg)(t)(\nabla_a^{-\nu}1)(t)\\ &\geqslant (\nabla_a^{-\nu}g)(t)(\nabla_a^{-\nu}f)(t)+(\nabla_a^{-\nu}f)(t)(\nabla_a^{-\nu}g)(t)\,,\end{aligned}$$

i.e.

$$\begin{aligned}(\nabla_a^{-\nu}f)(t)(\nabla_a^{-\nu}g)(t) &\leqslant (\nabla_a^{-\nu}1)(t)(\nabla_a^{-\nu}fg)(t)\\ &= \frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+1)}(\nabla_a^{-\nu}fg)(t).\end{aligned}$$

This shows (1). □

REMARK 4. The inequality sign in (1) is reversed if the functions are asynchronous on $\mathbb{N}_a$.

EXAMPLE 1. Let $\alpha,\beta \geqslant 0$ and consider the functions

$$f(t)=(t-a)^{\overline{\alpha}}, \quad f(t)=(t-a)^{\overline{\beta}},\ t\in\mathbb{N}_a.$$

From Remark 3, we say that the functions f and g are increasing, so f and g are synchronous. Therefore, we can apply Theorem 1 with functions f and g for $\nu \geqslant 0$. Using Lemma 1, we obtain

$$\begin{aligned}(\nabla_a^{-\nu}fg)(t) &\geqslant \frac{\Gamma(\nu+1)}{(t-a)^{\overline{\nu}}}(\nabla_a^{-\nu}f)(t)(\nabla_a^{-\nu}g)(t)\\ &= \frac{\Gamma(\nu+1)}{(t-a)^{\overline{\nu}}}(\nabla_a^{-\nu}(t-a)^{\overline{\alpha}})(\nabla_a^{-\nu}(t-a)^{\overline{\beta}})\\ &= \frac{\Gamma(\nu+1)}{(t-a)^{\overline{\nu}}}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\nu+1)}(t-a)^{\overline{\alpha+\nu}}\frac{\Gamma(\beta+1)}{\Gamma(\beta+\nu+1)}(t-a)^{\overline{\beta+\nu}}.\end{aligned}$$

THEOREM 2. *If $\nu,\ \mu>0$ and f,g are two synchronous functions on $\mathbb{N}_a$, then*

$$\begin{aligned}&\frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+1)}(\nabla_a^{-\mu}fg)(t)+\frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)}(\nabla_a^{-\nu}fg)(t)\\ &\geqslant (\nabla_a^{-\nu}f)(t)(\nabla_a^{-\mu}g)(t)+(\nabla_a^{-\mu}f)(t)(\nabla_a^{-\nu}g)(t),\ t\in\mathbb{N}_a.\end{aligned} \tag{6}$$

Proof. For the proof, we continue as in the proof of Theorem 1 and using inequality (4), we can write

$$\begin{aligned}&\frac{(t-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)}(\nabla_a^{-\nu}fg)(t)+\frac{(t-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)}f(s)g(s)(\nabla_a^{-\nu}1)(t)\\ &\geqslant \frac{(t-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)}g(s)(\nabla_a^{-\nu}f)(t)+\frac{(t-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)}f(s)(\nabla_a^{-\nu}g)(t).\end{aligned} \tag{7}$$

Now, taking the sum of both sides of (7) for $s \in \{a, a+1, \ldots, t\}$, we obtain the desired inequality (6). □

REMARK 5. If we let $v = \mu$ in Theorem 2, we obtain Theorem 1.

Finally, we give a generalization of Theorem 1.

THEOREM 3. *Assume that* f_i, $1 \leqslant i \leqslant n$, *are* $n \in \mathbb{N}$ *functions on* $\mathbb{N}_a$ *satisfying*

$$\prod_{i=1}^{k-1} f_i \text{ and } f_k \text{ are synchronous for all } k \in \{2, \ldots, n\}, \tag{8}$$

$$f_i \geqslant 0 \text{ for } 3 \leqslant i \leqslant n. \tag{9}$$

Suppose that $v > 0$. *Then, for all* $t \in \mathbb{N}_a$, *we have*

$$\left(\nabla_a^{-v} \prod_{i=1}^{n} f_i\right)(t) \geqslant \left(\frac{\Gamma(v+1)}{(t-a)^{\overline{v}}}\right)^{n-1} \prod_{i=1}^{n} \left(\nabla_a^{-v} f_i\right)(t). \tag{10}$$

Proof. In view of (8) and (9), applying Theorem 1 repeatedly, we have

$$\begin{aligned}
\left(\nabla_a^{-v} \prod_{i=1}^{n} f_i\right)(t) &\geqslant \frac{\Gamma(v+1)}{(t-a)^{\overline{v}}} \left(\nabla_a^{-v} \prod_{i=1}^{n-1} f_i\right)(t) (\nabla_a^{-v} f_n)(t) \\
&\geqslant \left(\frac{\Gamma(v+1)}{(t-a)^{\overline{v}}}\right)^2 \left(\nabla_a^{-v} \prod_{i=1}^{n-2} f_i\right)(t) \prod_{i=n-1}^{n} (\nabla_a^{-v} f_i)(t) \\
&\cdots \\
&\geqslant \left(\frac{\Gamma(v+1)}{(t-a)^{\overline{v}}}\right)^{n-1} \prod_{i=1}^{n} \left(\nabla_a^{-v} f_i\right)(t). \quad \square
\end{aligned}$$

REMARK 6. If the functions f_i, $1 \leqslant i \leqslant n$, in Theorem 3 are either all nonnegative increasing or nonnegative decreasing, then both (8) and (9) are satisfied.

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