Fractional Differential Calculus
Volume 7, Number 1 (2017), 123–134 doi:10.7153/fdc-07-02

GREEN FUNCTION'S NEW PROPERTY AND ITS APPLICATION TO THE SOLUTION FOR A HIGHER–ORDER FRACTIONAL BOUNDARY VALUE PROBLEM

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(*Communicated by S. Umarov*)

Abstract. We investigate the higher-order fractional boundary value problem:

$$\begin{cases} -D_{0+}^{v}u(t) = a(t)f(t,u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0, \quad \left[D_{0+}^{\alpha}u(t)\right]_{t=1} = 0, \end{cases}$$

where v and α are two given constants satisfying $n-1 < v \leqslant n$ with $n \geqslant 3$, $0 \leqslant \alpha \leqslant n-2$. Some results have been obtained in literature in case of $1 \leqslant \alpha \leqslant n-2$ or $\alpha = 0$, but no result is about $0 < \alpha < 1$. In this paper, new properties of the Green function associated with the higher-order fractional boundary value problem in case of $0 \leqslant \alpha \leqslant n-2$ are obtained which are the main contribution of the paper. As an application of these properties, the existence of positive solutions of the problem is then established. Our results improve on recent works in literature and fill in their gaps.

1. Introduction

The application of fractional calculus to dynamical system control has been getting ever increasing attention, which leads to the progress of research in this area. Among these researches, fractional differential equations have gained importance due to their applications in various sciences, we refer the reader to [1, 2, 3, 4, 5]. In recent years, there has been significant development in the existence of solutions [6, 7, 8, 9, 10] and positive solutions [11, 12, 13, 14, 15, 16, 17] to boundary value problems for the fractional differential equations.

Comparing with lower-order fractional differential equations, there are few results about higher-order fractional differential equations. In a recent paper [18], Zhang considered the following higher-order fractional boundary value problem (Hfbvp for short),

$$\begin{cases} -D_{0+}^{v}u(t) = a(t)f(t,u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0, \quad u^{(n-2)}(1) = 0, \end{cases} \tag{1}$$

where $n-1 < v \leqslant n$ with $n \geqslant 2$. By letting $u(t) = I_0^{n-2}v(t)$, Zhang modified Hfbvp (1) to a lower-order fractional differential equations, and by the properties of the Green

Mathematics subject classification (2010): 34A08, 34B18, 34B27, 26A33.
Keywords and phrases: Fractional differential equation, cone, positive solution, Green function.
Supported by the Fundamental Research Funds for the Central Universities (2014MS62).

Paper FDC-07-02

function corresponding to the modified problem, positive solutions for Hfbvp (1) were obtained.

In [19], Goodrich studied the following Hfbvp, which is more extensive than Hfbvp (1),

$$\begin{cases} -D_{0+}^{v}u(t)=a(t)f(t,u(t)), & 0<t<1,\\ u(0)=u'(0)=u''(0)=\cdots=u^{(n-2)}(0)=0, & \left[D_{0+}^{\alpha}u(t)\right]_{t=1}=0, \end{cases} \tag{2}$$

where $n-1<v\leqslant n$, $n\geqslant 3$, $1\leqslant\alpha\leqslant n-2$. A positive solution was obtained via Krasnosel'skii fixed point theorem in [19]. As presented in [19], the Green function $G(t,s)$ corresponding to Hfbvp (2) is

$$G(t,s)=\frac{1}{\Gamma(v)}\begin{cases} t^{v-1}(1-s)^{v-\alpha-1}-(t-s)^{v-1}, & 0\leqslant s\leqslant t\leqslant 1,\\ t^{v-1}(1-s)^{v-\alpha-1}, & 0\leqslant t\leqslant s\leqslant 1. \end{cases} \tag{3}$$

In [19], $1\leqslant\alpha\leqslant n-2$ was a vital condition in the proof of the properties of $G(t,s)$ which leads directly to the result that the value of $t^{v-1}(1-s)^{v-\alpha-1}-(t-s)^{v-1}$ increases about t for $0\leqslant s\leqslant t\leqslant 1$.

Very recently, Hfbvp (2) has been studied in [20]. With a new upper estimate for the Green function (3), new criteria for the existence of positive solutions were established. We can see that the new upper estimate is excellent and distinctive. But, just as in [19], $1\leqslant\alpha\leqslant n-2$ was still required (see the proof of Lemma 2.2 and Lemma 2.3) in [20].

If $0\leqslant\alpha<1$ in Hfbvp (2), both the monotonicity of $t^{v-1}(1-s)^{v-\alpha-1}-(t-s)^{v-1}$ and $\alpha-1\geqslant 0$ will disappear, and then the properties of the Green function $G(t,s)$ may be quite different. As far as we known, there was no result about this case up to now.

In fact, when $\alpha=0$, Hfbvp (2) turns to

$$\begin{cases} -D_{0+}^{v}u(t)=a(t)f(t,u(t)), & 0<t<1,\\ u(0)=u'(0)=u''(0)=\cdots=u^{(n-2)}(0)=0, & u(1)=0. \end{cases} \tag{4}$$

Hfbvp (4) has been studied in [21] ($\xi_i=0$ in [21]). Salem conclude in the proof of Lemma 2.1 in [21] that

$$t^{v-1}(1-s)^{v-1}-(t-s)^{v-1}$$

is decreasing with respect to t for $s\leqslant t$. But, we should point out that the conclusion is wrong. For example, choose $v=4$, and let $s=\frac{1}{2}$. Then, for $\frac{1}{2}\leqslant t\leqslant 1$, it is obvious that

$$G(t,s)=\frac{1}{\Gamma(v)}\left[t^{v-1}(1-s)^{v-1}-(t-s)^{v-1}\right]=\frac{1}{\Gamma(v)}\left[-\frac{7}{8}t^3+\frac{3}{2}t^2-\frac{3}{4}t+\frac{1}{8}\right]$$

is increasing in $[\frac{1}{2},\frac{4+\sqrt{2}}{7}]$, and decreasing in $[\frac{4+\sqrt{2}}{7},1]$. Therefore, the results in [21] should be reconsidered.

Above all, in this paper, we will still study the following Hfbvp:

$$\begin{cases} -D_{0+}^{v}u(t) = a(t)f(t,u(t)), & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0, & \left[D_{0+}^{\alpha}u(t)\right]_{t=1} = 0, \end{cases} \tag{5}$$

where v and α are two given constants satisfying $n-1 < v \leqslant n$ with $n \geqslant 3$, $0 \leqslant \alpha \leqslant n-2$, D_{0+}^{α} is the Riemann-Liouville fractional derivative, $a(t) \in L[0,1]$ is nonnegative, and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous. We will get the existence of positive solutions for the Hfbvp (5) in case of $0 \leqslant \alpha \leqslant n-2$. Our results fill in gaps of recent works in literature in case of $0 \leqslant \alpha < 1$, and improve on works in [18, 19, 20, 21] in case of $1 \leqslant \alpha \leqslant n-2$. The main results are based upon the properties of the corresponding Green function. In fact, we not only give but also unify the different properties of the Green function, which are the primary contribution of the paper.

2. Some preliminaries and the new estimate for the Green function

In this section, some notations and preliminary conclusions are given. Then, we show the new estimate for the Green function.

DEFINITION 2.1. ([11]) Let $v > 0$ with $v \in \mathbb{R}$. Suppose that $y : (0,+\infty) \to \mathbb{R}$. Then the v-th Riemann-Liouville fractional derivative is defined by

$$D_{0+}^{v}y(t) := \frac{1}{\Gamma(n-v)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{y(s)}{(t-s)^{v-n+1}} ds,$$

where $n = [v]+1$, provided that the right hand side term is pointwise defined on $t > 0$.

LEMMA 2.1. ([19]) *Suppose that $u(t)$ is the solution of Hfbvp (5), then*

$$u(t) = \int_0^1 G(t,s)a(s)f(t,u(s))ds,$$

where $G(t,s)$ is defined in (3).

LEMMA 2.2. *For $0 \leqslant \alpha \leqslant n-2$, we have the following properties.*
(i) $G(t,s)$ is continuous on $(t,s) \in I \times I$, where $I = [0,1]$;
(ii) $G(t,s) \geqslant 0$ for any $(t,s) \in I \times I$.

Proof. The conclusions are obvious by (3). □

THEOREM 2.1. *For $0 \leqslant \alpha < 1$, the following properties hold.*
(i) For any $s \in (0,1)$,

$$\max_{t \in I} G(t,s) = G(t_0,s) = \frac{s^{v-1}(1-s)^{v-\alpha-1}}{\Gamma(v)\left[1-(1-s)^{\frac{v-\alpha-1}{v-2}}\right]^{v-2}} > 0,$$

where $t_0 = \frac{s}{1-(1-s)^{\frac{\nu-\alpha-1}{\nu-2}}}$;

(ii) For any $(t,s) \in I \times I,\ G(t,s) \geqslant \rho(t,s)G(t_0,s) \geqslant \tilde{\rho}(t)G(t_0,s) = \tilde{\rho}(t)\max_{t\in I} G(t,s)$, *where*

$$\rho(t,s) = \begin{cases} t^{\nu-1}, & 0 \leqslant t < t_0 = \dfrac{s}{1-(1-s)^{\frac{\nu-\alpha-1}{\nu-2}}} \leqslant 1, \\ (1-t), & 1 \geqslant t \geqslant t_0 = \dfrac{s}{1-(1-s)^{\frac{\nu-\alpha-1}{\nu-2}}} \geqslant 0, \end{cases}$$

and

$$\tilde{\rho}(t) = \begin{cases} t^{\nu-1}, & 0 \leqslant t \leqslant t_1 \leqslant 1, \\ (1-t), & 1 \geqslant t \geqslant t_1 \geqslant 0, \end{cases}$$

where t_1 *is the unique solution to* $t^{\nu-1} = 1-t$.

Proof. (i) For any $s \in (0,1)$, when $t \geqslant s$, by (3),

$$\Gamma(\nu)G_t'(t,s) = (\nu-1)t^{\nu-2}\left[(1-s)^{\nu-\alpha-1} - \left(1-\frac{s}{t}\right)^{\nu-2}\right] \begin{cases} \leqslant 0, & t \geqslant t_0, \\ \geqslant 0, & t \leqslant t_0, \end{cases} \tag{6}$$

where $t_0 = \dfrac{s}{1-(1-s)^{\frac{\nu-\alpha-1}{\nu-2}}}$. Since $0 \leqslant \alpha < 1$ and $\nu > 3$, we get $\frac{\nu-\alpha-1}{\nu-2} > 1$ and thus

$$s < 1-(1-s)^{\frac{\nu-\alpha-1}{\nu-2}} < 1, \tag{7}$$

which means $s < t_0 < 1$. So

$$\max_{t\in[s,1]} G(t,s) = G(t_0,s) = \frac{s^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)\left[1-(1-s)^{\frac{\nu-\alpha-1}{\nu-2}}\right]^{\nu-2}} > 0. \tag{8}$$

When $t \leqslant s$, since $\left[1-(1-s)^{\frac{\nu-\alpha-1}{\nu-2}}\right]^{\nu-2} < 1$, we get by (3) that

$$\max_{t\in[0,s]} G(t,s) = \frac{s^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)} < \frac{s^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)\left[1-(1-s)^{\frac{\nu-\alpha-1}{\nu-2}}\right]^{\nu-2}} = G(t_0,s). \tag{9}$$

(8) and (9) show that (i) of Theorem 2.1 is true.

(ii) For any $s \in (0,1)$, when $t \in [t_0,1]\,(t_0 > s)$, we know $(1-s)^{\nu-\alpha-1} \leqslant \left(1-\frac{s}{t}\right)^{\nu-2}$ by (6). Then we have by (3) that

$$\begin{aligned} \Gamma(\nu)G_{tt}''(t,s) &= (\nu-1)(\nu-2)\left[t^{\nu-3}(1-s)^{\nu-\alpha-1} - (t-s)^{\nu-3}\right] \\ &= (\nu-1)(\nu-2)t^{\nu-3}\left[(1-s)^{\nu-\alpha-1} - (1-\frac{s}{t})^{\nu-3}\right] \\ &= (\nu-1)(\nu-2)t^{\nu-3}\left[(1-s)^{\nu-\alpha-1} - (1-\frac{s}{t})^{(\nu-2)\frac{\nu-3}{\nu-2}}\right] \\ &\leqslant (\nu-1)(\nu-2)t^{\nu-3}\left[(1-s)^{\nu-\alpha-1} - (1-s)^{(\nu-\alpha-1)\frac{\nu-3}{\nu-2}}\right] \\ &= (\nu-1)(\nu-2)t^{\nu-3}(1-s)^{\nu-\alpha-1}\left[1-\frac{1}{(1-s)^{\frac{\nu-\alpha-1}{\nu-2}}}\right] \leqslant 0, \end{aligned}$$

which means $G(t,s)$ is concave about t on $[t_0,1]$.

For any $t \in [t_0,1]$, by the concavity of $G(t,s)$ and the fact that $G(1,s) = (1-s)^{\nu-\alpha-1} - (1-s)^{\nu-1} > 0$, we have

$$\begin{aligned} G(t,s) &\geqslant \frac{G(t_0,s)-G(1,s)}{t_0-1}(t-1)+G(1,s) \\ &= \frac{G(t_0,s)}{1-t_0}(1-t)+\frac{t-t_0}{1-t_0}G(1,s) \\ &\geqslant G(t_0,s)(1-t). \end{aligned} \tag{10}$$

For any $t \in [s,t_0]$, we know $(1-s)^{\nu-\alpha-1} \geqslant (1-\frac{s}{t})^{\nu-2}$ by (6). Then we have by (3) that,

$$\begin{aligned} \Gamma(\nu)G(t,s) &= t^{\nu-1}(1-s)^{\nu-\alpha-1}-(t-s)^{\nu-1} \\ &= t^{\nu-1}\left[(1-s)^{\nu-\alpha-1}-\left(1-\frac{s}{t}\right)^{\nu-1}\right] \\ &= t^{\nu-1}\left[(1-s)^{\nu-\alpha-1}-\left(1-\frac{s}{t}\right)^{(\nu-2)\frac{\nu-1}{\nu-2}}\right] \\ &\geqslant t^{\nu-1}\left[(1-s)^{\nu-\alpha-1}-(1-s)^{(\nu-\alpha-1)\frac{\nu-1}{\nu-2}}\right] \\ &= t^{\nu-1}\Gamma(\nu)G(t_0,s)\left(\frac{1-(1-s)^{\frac{\nu-\alpha-1}{\nu-2}}}{s}\right)^{\nu-1}. \end{aligned} \tag{11}$$

From (7), we know $\frac{1-(1-s)^{\frac{\nu-\alpha-1}{\nu-2}}}{s} \geqslant 1$. Therefore,

$$\left(\frac{1-(1-s)^{\frac{\nu-\alpha-1}{\nu-2}}}{s}\right)^{\nu-1} \geqslant 1. \tag{12}$$

Substituting (12) into (11), we get

$$G(t,s) \geqslant t^{\nu-1}G(t_0,s), \quad t \in [s,t_0]. \tag{13}$$

For any $t \in [0,s]$, we have

$$\begin{aligned} \Gamma(\nu)G(t,s) &= t^{\nu-1}(1-s)^{\nu-\alpha-1} \\ &= t^{\nu-1}\Gamma(\nu)G(t_0,s)\frac{\left(1-(1-s)^{\frac{\nu-\alpha-1}{\nu-2}}\right)^{\nu-2}}{s^{\nu-1}} \\ &= t^{\nu-1}\Gamma(\nu)G(t_0,s)\left(\frac{1-(1-s)^{\frac{\nu-\alpha-1}{\nu-2}}}{s}\right)^{\nu-1} \cdot \frac{1}{1-(1-s)^{\frac{\nu-\alpha-1}{\nu-2}}}. \end{aligned} \tag{14}$$

Substituting (7) and (12) into (14), we get

$$G(t,s) \geqslant t^{\nu-1}G(t_0,s), \quad t \in [0,s]. \tag{15}$$

Let

$$\rho(t,s) = \begin{cases} t^{v-1}, & 0 \leqslant t < t_0 = \frac{s}{1-(1-s)^{\frac{v-\alpha-1}{v-2}}} < 1, \\ (1-t), & 1 \geqslant t \geqslant t_0 = \frac{s}{1-(1-s)^{\frac{v-\alpha-1}{v-2}}} > 0. \end{cases}$$

Considering the continuity of $G(t,s)$, we conclude from (10), (13) and (15) that

$$G(t,s) \geqslant \rho(t,s)G(t_0,s), (t,s) \in I \times I.$$

Obviously, $\rho(t,s)$ is a function of two variables. Since t^{v-1} is increasing and $1-t$ is decreasing on I respectively, we know that $t^{v-1} = 1-t$ has an unique solution t_1 on I. Let

$$\tilde{\rho}(t) = \begin{cases} t^{v-1}, & 0 \leqslant t \leqslant t_1 \leqslant 1, \\ (1-t), & 1 \geqslant t \geqslant t_1 \geqslant 0, \end{cases}$$

then $\rho(t,s) \geqslant \tilde{\rho}(t), \forall (t,s) \in I \times I$, which ends the proof. □

THEOREM 2.2. *For $1 \leqslant \alpha \leqslant n-2$, the following properties hold:*
(i) For any $s \in (0,1)$, $\max_{t \in I} G(t,s) = G(1,s)$;
(ii) For any $(t,s) \in I \times I$, $G(t,s) \geqslant \tilde{\rho}(t)G(1,s) = \tilde{\rho}(t) \max_{t \in I} G(t,s)$.

Proof. (i) For any $s \in (0,1)$, when $t \geqslant s$, by (3),

$$\Gamma(v)G(t,s) = t^{v-1}(1-s)^{v-\alpha-1} - (t-s)^{v-1},$$

and thus

$$\Gamma(v)G_t'(t,s) = (v-1)t^{v-2}\left[(1-s)^{v-\alpha-1} - \left(1-\frac{s}{t}\right)^{v-2}\right].$$

Considering $1 \leqslant \alpha \leqslant n-2$, i.e., $0 \leqslant v-\alpha-1 \leqslant v-2$ and $(1-s) \geqslant \left(1-\frac{s}{t}\right)$, we know

$$(1-s)^{v-\alpha-1} - \left(1-\frac{s}{t}\right)^{v-2} \geqslant 0,$$

which means that $G_t'(t,s) \geqslant 0$, and thus

$$\max_{t \in [s,1]} G(t,s) = G(1,s) = \frac{\left[(1-s)^{v-\alpha-1} - (1-s)^{v-1}\right]}{\Gamma(v)} > 0. \tag{16}$$

When $t \leqslant s$, since $\alpha \geqslant 1$ and $v \geqslant 3$, we get $(1-s)^{\alpha} + s^{v-1} \leqslant 1-s+s = 1$, i.e.,

$$s^{v-1} \leqslant 1-(1-s)^{\alpha},$$

multiplying $(1-s)^{v-\alpha-1}$ on both sides in above inequality, we get

$$s^{v-1}(1-s)^{v-\alpha-1} \leqslant (1-s)^{v-\alpha-1} - (1-s)^{v-1}.$$

Combining $G(t,s)=\frac{t^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)}\leqslant\frac{s^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)}$, we have

$$\max_{t\in[0,s]}G(t,s)\leqslant\frac{s^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)}\leqslant\frac{\left[(1-s)^{\nu-\alpha-1}-(1-s)^{\nu-1}\right]}{\Gamma(\nu)}=G(1,s). \tag{17}$$

From (16) and (17), we obtain that $\max_{t\in[0,1]}G(t,s)=G(1,s)$.

(ii) For any $s\in(0,1)$, when $t\in[s,1]$,

$$\begin{aligned}G(t,s)&=\frac{t^{\nu-1}(1-s)^{\nu-\alpha-1}-(t-s)^{\nu-1}}{\Gamma(\nu)}\\&=t^{\nu-1}\frac{(1-s)^{\nu-\alpha-1}-\left(1-\frac{s}{t}\right)^{\nu-1}}{\Gamma(\nu)G(1,s)}G(1,s)\\&=t^{\nu-1}\frac{(1-s)^{\nu-\alpha-1}-\left(1-\frac{s}{t}\right)^{\nu-1}}{(1-s)^{\nu-\alpha-1}-(1-s)^{\nu-1}}G(1,s)\\&\geqslant t^{\nu-1}G(1,s).\end{aligned} \tag{18}$$

When $t\in[0,s]$,

$$\begin{aligned}G(t,s)&=\frac{t^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)}\\&=t^{\nu-1}G(1,s)\frac{(1-s)^{\nu-\alpha-1}}{(1-s)^{\nu-\alpha-1}-(1-s)^{\nu-1}}\\&\geqslant t^{\nu-1}G(1,s).\end{aligned} \tag{19}$$

From (18) and (19), we have

$$G(t,s)\geqslant t^{\nu-1}G(1,s),\quad \forall(t,s)\in I\times I. \tag{20}$$

Furthermore, we easily see that $t^{\nu-1}\geqslant\tilde{\rho}(t)$, $\forall t\in I$, which is combined with (20) to obtain

$$G(t,s)\geqslant\tilde{\rho}(t)G(1,s)=\tilde{\rho}(t)\max_{t\in I}G(t,s),\quad (t,s)\in I\times I. \quad \square$$

THEOREM 2.3. *For $0\leqslant\alpha\leqslant n-2$, we have*

$$\min_{[\theta_1,\theta_2]}G(t,s)\geqslant k_0\max_{t\in I}G(t,s),\quad \forall s\in I,$$

where θ_1 and θ_2 are two any constants with $0<\theta_1<\theta_2<1$, $k_0=\min\{\theta_1^{\nu-1},1-\theta_2\}$.

Proof. For any two constants θ_1 and θ_2 with $0<\theta_1<\theta_2<1$, we have by (ii) of Theorem 2.1 and Theorem 2.2 that

$$\begin{aligned}\min_{t\in[\theta_1,\theta_2]}G(t,s)&\geqslant\min_{t\in[\theta_1,\theta_2]}\{\tilde{\rho}(t)\max_{t\in I}G(t,s)\}\\&=\min\{\theta_1^{\nu-1},1-\theta_2\}\max_{t\in I}G(t,s)=k_0\max_{t\in I}G(t,s). \quad \square\end{aligned}$$

Let $E = C[0,1]$ be endowed with the maximum norm $\|u\| = \max_{t\in I} |u(t)|$, then E is a Banach space. Let $C^+[0,1] = \{u | u \in E, u(t) \geqslant 0,\ t \in I\}$. u is called a positive solution of the Hfbvp (5) if $u \in C^+[0,1]$ satisfies the Hfbvp (5) and $\|u\| \neq 0$. For any two constants θ_1 and θ_2 with $0 < \theta_1 < \theta_2 < 1$, we define a cone $P \subseteq E$ by

$$P = \left\{ u \in E \,\middle|\, u(t) \geqslant 0,\ \min_{t\in[\theta_1,\theta_2]} u(t) \geqslant k_0 \|u\| \right\},$$

where $k_0 = \min\{\theta_1^{\nu-1}, 1-\theta_2\}$. We define an operator $T : C^+[0,1] \to E$ by

$$(Tu)(t) = \int_0^1 G(t,s)a(s)f(s,u(s))ds. \tag{21}$$

By Lemma 2.1, we know a fixed point $u \in P$ of T with $\|u\| \neq 0$ must be a positive solution of the Hfbvp (5).

LEMMA 2.3. *$T : C^+[0,1] \to C^+[0,1]$ is completely continuous and $T(C^+[0,1]) \subseteq P$.*

Proof. Let $u \in C^+[0,1]$, then $(Tu)(t) \geqslant 0$ is obvious. The operator $T : C^+[0,1] \to C^+[0,1]$ is completely continuous by an application of the Ascoli-Arzela theorem.

For $u \in C^+[0,1]$, we know by Theorem 2.3 that, for $0 < \theta_1 < \theta_2 < 1$,

$$\begin{aligned}
\min_{t\in[\theta_1,\theta_2]} (Tu)(t) &= \min_{t\in[\theta_1,\theta_2]} \int_0^1 G(t,s)a(s)f(s,u(s))ds \\
&\geqslant \int_0^1 \min_{t\in[\theta_1,\theta_2]} G(t,s)a(s)f(s,u(s))ds \\
&\geqslant k_0 \int_0^1 \max_{t\in I} G(t,s)a(s)f(s,u(s))ds \\
&\geqslant k_0 \max_{t\in I} \int_0^1 G(t,s)a(s)f(s,u(s))ds \\
&= k_0 \max_{t\in I} (Tu)(t) = k_0 \|Tu\|,
\end{aligned} \tag{22}$$

which means $(Tu) \in P$, and the proof ends. □

DEFINITION 2.2. A map θ is said to be a nonnegative continuous concave function on a cone P of a Banach space E, provided that $\theta : P \to [0,\infty)$ is continuous and

$$\theta(tx + (1-t)y) \geqslant t\theta(x) + (1-t)\theta(y),\ \ \forall x,y \in P,\ \ 0 \leqslant t \leqslant 1.$$

THEOREM 2.4. ([11]) *Let P be a cone of a Banach space E. Assume Ω_1 and Ω_2 are two bounded open balls of E at the origin with $\overline{\Omega_1} \subset \Omega_2$. Suppose $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Tu\| \leqslant \|u\|$, $\forall u \in P \cap \partial\Omega_1$ and $\|Tu\| \geqslant \|u\|$, $\forall u \in P \cap \partial\Omega_2$ or

(ii) $\|Tu\| \geqslant \|u\|$, $\forall u \in P \cap \partial\Omega_1$ and $\|Tu\| \leqslant \|u\|, \forall u \in P \cap \partial\Omega_2$ holds.

Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

THEOREM 2.5. ([11]) *Let P be a cone of a Banach space E. $P_c = \{x \in P \mid \|x\| < c\}$. θ is a nonnegative continuous concave function on P such that $\theta(x) \leqslant \|x\|$ for any $x \in \overline{P_c}$. Let $P(\theta,b,d) = \{x \in P | b \leqslant \theta(x), \|x\| \leqslant d\}$. Suppose $A : \overline{P_c} \longrightarrow \overline{P_c}$ is a completely continuous operator and there exist constants $0 < a < b < d \leqslant c$ such that*

(c1) $\{x \in P(\theta,b,d) \mid \theta(x) > b\} \neq \emptyset$ ($\emptyset$ means an empty set) and $\theta(Ax) > b$ for any $x \in P(\theta,b,d)$;

(c2) $\|Ax\| < a$ for any $x \leqslant a$;

(c3) For any $x \in P(\theta,b,c)$ with $\|Ax\| > d$, $\theta(Ax) > b$.

Then A has at least three fixed points x_1, x_2, x_3 with $\|x_1\| < a$, $b < \theta(x_2)$, $a < \|x_3\|$ with $\theta(x_3) < b$.

3. Existence of positive solutions for the Hfbvp (5)

In this section, we impose conditions on f which allow us to establish some existence results of positive solutions for the Hfbvp (5).

Throughout this section, we denote

$$M = \left(\int_0^1 \max_{t \in I} G(t,s)a(s)ds \right)^{-1} \quad \text{and} \quad N = \left(\int_{\theta_1}^{\theta_2} \max_{t \in I} G(t,s)a(s)ds \right)^{-1}.$$

THEOREM 3.1. *Assume that there exist constants ρ_2 and ρ_1 with $\rho_2 > \rho_1 > 0$ such that*

(B1) $\inf_{u \in S} \int_0^1 \max_{t \in I} G(t,s)a(s)f(s,u(s))ds \geqslant \rho_1$;

(B2) $\sup_{u \in S} \int_0^1 \max_{t \in I} G(t,s)a(s)f(s,u(s))ds \leqslant \rho_2$,

where $S = \{u \in C^+[0,1] | \tilde{\rho}(t)\rho_1 \leqslant u(t) \leqslant \rho_2, \forall t \in I\}$. Then the Hfbvp (5) has at least one positive solution $u \in S$.

Proof. For any $u \in S$, from (ii) of Theorem 2.1 and Theorem 2.2, combining $(B1)$ and $(B2)$, we see that

$$(Tu)(t) \geqslant \tilde{\rho}(t) \int_0^1 \max_{t \in I} G(t,s)a(s)f(s,u(s))ds \geqslant \tilde{\rho}(t)\rho_1$$

and

$$(Tu)(t) \leqslant \int_0^1 \max_{t \in I} G(t,s)a(s)f(s,u(s))ds \leqslant \rho_2,$$

which means that $T(S) \subseteq S$. The conclusion follows from Schauder$'$s fixed point theorem immediately. □

THEOREM 3.2. *Assume that there exist constants r_2, r_1, θ_1 and θ_2 with $r_2 > r_1 > 0$ and $0 < \theta_1 < \theta_2 < 1$ such that*

(H1) $f(t,u) \leqslant Mr_2$ for any $(t,u) \in [0,1] \times [0,r_2]$;

(H2) $f(t,u) \geqslant \frac{1}{k_0}Nr_1$ for any $(t,u) \in [\theta_1,\theta_2] \times [k_0 r_1, r_1]$,

where $k_0 = \min\{\theta_1^{\nu-1}, 1-\theta_2\}$. Then the Hfbvp (5) has at least one positive solution u with $r_1 \leqslant \|u\| \leqslant r_2$.

Proof. Let $\Omega_1 = \{u \in P | \|u\| < r_1\}$. For $u \in \partial\Omega_1$, we have $k_0 r_1 \leqslant u(s) \leqslant r_1, \forall s \in [\theta_1, \theta_2]$. It follows from Theorem 2.3 and (H_2) that, for any $t \in [\theta_1, \theta_2]$,

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)a(s)f(s,u(s))ds \\ &\geqslant \int_0^1 k_0 \max_{t\in I} G(t,s)a(s)f(s,u(s))ds \\ &\geqslant \int_{\theta_1}^{\theta_2} k_0 \max_{t\in I} G(t,s)a(s)f(s,u(s))ds \\ &\geqslant \int_{\theta_1}^{\theta_2} k_0 \max_{t\in I} G(t,s)a(s)\tfrac{1}{k_0}Nr_1 ds = r_1 = \|u\|,\end{aligned}$$

which means that

$$\|Tu\| \geqslant \|u\|, \ \forall\, u \in P \cap \partial\Omega_1.$$

Let $\Omega_2 = \{u \in P | \|u\| < r_2\}$. For $u \in \partial\Omega_2$, we have $0 \leqslant u(s) \leqslant r_2, \forall s \in [0,1]$. We have by (H_1) that

$$\begin{aligned}\|Tu\| &= \max_{0\leqslant t\leqslant 1} \int_0^1 G(t,s)a(s)f(s,u(s))ds \\ &\leqslant \int_0^1 \max_{t\in I} G(t,s)a(s)Mr_2 ds = r_2 = \|u\|,\end{aligned}$$

which means that

$$\|Tu\| \leqslant \|u\|, \ \forall\, u \in P \cap \partial\Omega_2.$$

Therefore, by (ii) of Theorem 2.4, we complete the proof. □

THEOREM 3.3. *Assume that there exist constants a, b, c, θ_1 and θ_2 with $0 < a < b < c$ and $0 < \theta_1 < \theta_2 < 1$ such that*

(A1) $f(t,u) < Ma$ for any $(t,u) \in [0,1] \times [0,a]$;

(A2) $f(t,u) > \frac{1}{k_0}Nb$ for any $(t,u) \in [\theta_1, \theta_2] \times [b,c]$;

(A3) $f(t,u) \leqslant Mc$ for any $(t,u) \in [0,1] \times [0,c]$,

where $k_0 = \min\{\theta_1^{\nu-1}, 1-\theta_2\}$. Then the Hfbvp (5) has at least three positive solutions u_1, u_2 and u_3 with

$$\begin{aligned}&\max_{t\in I}|u_1(t)| < a, \qquad b < \min_{t\in[\theta_1,\theta_2]}|u_2(t)| < \max_{t\in I}|u_2(t)| \leqslant c, \\ &a < \max_{t\in I}|u_3(t)| \leqslant c, \qquad \min_{t\in[\theta_1,\theta_2]}|u_3(t)| < b.\end{aligned}$$

Proof. We will show that all the conditions of Theorem 2.5 are satisfied.

We first define a function γ on the cone P by $\gamma(u) = \min_{t\in[\theta_1,\theta_2]} u(t)$. It is easy to verify that γ is a nonnegative continuous concave function. For any $u \in \overline{P_c}$, we get $0 \leqslant u(t) \leqslant \|u\| \leqslant c$, then by *(A3)*,

$$\begin{aligned}\|Tu\| &= \max_{t\in I} \int_0^1 G(t,s)a(s)f(s,u(s))ds \\ &\leqslant \int_0^1 \max_{t\in I} G(t,s)a(s)Mc ds = c.\end{aligned}$$

Hence $T : \overline{P_c} \to \overline{P_c}$ is completely continuous.

Next, we check condition *(c1)* of Theorem 2.5. We choose $u(t) = \frac{b+c}{2}$, $t \in [0,1]$. It is easy to see that $u(t) \in P(\gamma,b,c)$ and $\gamma(u) = \frac{b+c}{2} > b$, which means that $\{u \in P(\gamma,b,c) | \gamma(u) > b\} \neq \emptyset$. For any $u \in P(\gamma,b,c)$, we have $b \leqslant u(t) \leqslant c$, $t \in [\theta_1,\theta_2]$, and then we have by Theorem 2.3 and assumption *(A2)* that

$$\begin{aligned}\gamma(Au) &= \min_{t\in[\theta_1,\theta_2]} \int_0^1 G(t,s)a(s)f(s,u(s))ds \\ &\geqslant \int_0^1 \min_{t\in[\theta_1,\theta_2]} G(t,s)a(s)f(s,u(s))ds \\ &\geqslant \int_0^1 k_0 \max_{t\in I} G(t,s)a(s)f(s,u(s))ds \\ &\geqslant \int_{\theta_1}^{\theta_2} k_0 \max_{t\in I} G(t,s)a(s)f(s,u(s))ds \\ &> \int_{\theta_1}^{\theta_2} k_0 \max_{t\in I} G(t,s)a(s)\frac{1}{k_0}Nbds = b.\end{aligned}$$

(c2) of Theorem 2.5 is obvious by *(A1)*.

(c3) of Theorem 2.5 is also obvious since $d = c$, and thus *(c1)* implies *(c3)* here.

By Theorem 2.5, the Hfbvp (5) has at least three positive solutions u_1, u_2 and u_3 satisfying

$$\begin{aligned}&\max_{t\in I}|u_1(t)| < a, \quad b < \min_{t\in[\theta_1,\theta_2]}|u_2(t)| < \max_{t\in I}|u_2(t)| \leqslant c, \\ &a < \max_{t\in I}|u_3(t)| \leqslant c, \quad \min_{t\in[\theta_1,\theta_2]}|u_3(t)| < b. \quad \square\end{aligned}$$

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(Received July 4, 2016)

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