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ON SOLVABILITY OF SOME NONLINEAR FRACTIONAL INTERVAL INTEGRAL EQUATIONS

SÜMEYYE ÇAKAN AND ÜMİT ÇAKAN

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Abstract. In this paper, firstly we deal with solvability of a first kind nonlinear interval integral equation of fractional order. Then we present a theorem giving sufficient conditions for existence of solution of a second kind nonlinear interval integral equation of fractional order in the space of continuous interval-valued functions on the interval $[a,b]$ by using Banach fixed point theorem. We give also some examples satisfying the conditions of our main theorems.

1. Introduction

As it is known, nonlinear integral equations and set-valued analysis constitute important branchs of nonlinear functional analysis. Particularly, integral equations are often used in characterization of several problems of engineering, mechanics, physics, economics, biology and so on, [4], [5]. On the other hand, set-valued maps provide a useful framework for control theory, optimization theory, game theory, robotics, chemical engineering and mathematical economics, [2], [8], [13]. For this reason $C([a,b],\Omega_C(\mathbb{R}))$ and $L^1([a,b],\Omega_C(\mathbb{R}))$ which are two classes of interval-valued maps have important places in set-valued analysis.

In recent years, some authors such as S. Arshad [1], V. Lupulescu [9], [10], Y. Shen [16], M.T. Malinowski [11], L. Stefanini and B. Bede [18], S. Salahshour and M. Khan [14] and references therein give various results about interval-valued differential and integral equations.

As one of these studies, the following interval integral equation (for short, IIE) has been considered in [10]

$$Y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} X(s)ds, \tag{1}$$

where Y and X are set-valued functions and $t \in [a,b]$.

In this paper, we consider the following nonlinear Volterra IIEs of fractional order in $L^1([a,b],\Omega_C(\mathbb{R}))$ and $C([a,b],\Omega_C(\mathbb{R}))$

$$Y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (TX)(s)ds \tag{2}$$

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and

$$X(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (TX)(s) ds, \tag{3}$$

respectively. Here $\alpha \in (0,1)$ and T transforms Z into itself ($Z = C([a,b], \Omega_C(\mathbb{R}))$ or $L^1([a,b], \Omega_C(\mathbb{R}))$).

It should be indicated that this paper is a generalization of several ones obtained up to now. If operator T is chosen as unit operator IIE (2) turns into IIE (1).

In Section 2, we present some definitions and preliminary results such as the interval-valued maps and Banach fixed point theorem. In Section 3, we give our main results concerning the existence of solutions of the IIEs (2) and (3). Also we establish two examples showing that our results are applicable.

2. Preliminaries

In this section, we give some definitions and results which will be needed in the next section (see [12] and [17] for more information).

Let us write $\Omega_C(\mathbb{R})$ to denote the family of all nonempty, closed, bounded and convex subsets of real numbers.

For $[\underline{A}, \overline{A}]$, $[\underline{B}, \overline{B}] \in \Omega_C(\mathbb{R})$ and $\lambda \in \mathbb{R}$, the usual interval operations, i.e. Minkowski addition and scalar multiplication are defined by

$$[\underline{A}, \overline{A}] + [\underline{B}, \overline{B}] = [\underline{A} + \underline{B}, \overline{A} + \overline{B}]$$

and

$$\lambda \cdot [\underline{A}, \overline{A}] = \begin{cases} [\lambda \underline{A}, \lambda \overline{A}], & \lambda > 0 \\ \{0\}, & \lambda = 0 \\ [\lambda \overline{A}, \lambda \underline{A}], & \lambda < 0 \end{cases},$$

respectively. Also $(-1) \cdot [\underline{A}, \overline{A}] = -[\underline{A}, \overline{A}] = [-\overline{A}, -\underline{A}]$.

A metric structure on $\Omega_C(\mathbb{R})$ is given by the Hausdorff-Pompeiu distance $H : \Omega_C(\mathbb{R}) \times \Omega_C(\mathbb{R}) \to \mathbb{R}^+ = [0, \infty)$ defined by

$$H(A,B) = \max\left\{ |\underline{A} - \underline{B}|, |\overline{A} - \overline{B}| \right\}.$$

It is well known that $(\Omega_C(\mathbb{R}), H)$ is a complete metric space.

We denote the width of an interval $A = [\underline{A}, \overline{A}]$ by $w(A) = \overline{A} - \underline{A}$.

Also we say that an interval-valued function $F : [a,b] \to \Omega_C(\mathbb{R})$ is w-increasing (w-decreasing) on $[a,b]$, if the real function $w_F : [a,b] \to \mathbb{R}^+$ defined by $w_F(t) = w(F(t))$ is increasing (decreasing) on $[a,b]$. Then we say that F is w-monotone on $[a,b]$, [12].

The generalized Hukuhara difference (or gH -difference) of two intervals $[\underline{A}, \overline{A}]$, $[\underline{B}, \overline{B}] \in \Omega_C(\mathbb{R})$ is defined as follows

$$[\underline{A}, \overline{A}] \ominus_g [\underline{B}, \overline{B}] = \left[\min\left\{\underline{A} - \underline{B}, \overline{A} - \overline{B}\right\}, \max\left\{\underline{A} - \underline{B}, \overline{A} - \overline{B}\right\}\right].$$

DEFINITION 1. [12] Let $F:[a,b]\to\Omega_C(\mathbb{R})$ be an interval-valued function, $t_0\in[a,b]$ and $F'(t_0)\in\Omega_C(\mathbb{R})$ be define as (if it exists)

$$F'(t_0)=\lim_{h\to 0}\frac{F(t_0+h)\ominus_g F(t_0)}{h}.$$

Then it is said that $F'(t_0)$ is generalized Hukuhara derivative (gH-derivative, for short) of F at t_0. Also we say that F is generalized Hukuhara differentiable (gH-differentiable, for short) on $[a,b]$ if $F'(t)\in\Omega_C(\mathbb{R})$ exists at each point $t\in[a,b]$. Then the interval-valued function $F':[a,b]\to\Omega_C(\mathbb{R})$ is called gH-derivative of F on $[a,b]$.

PROPOSITION 1. [12] Let $F:[a,b]\to\Omega_C(\mathbb{R})$ be an interval-valued function such that $F(t)=[\underline{F(t)},\overline{F(t)}]$ for $t\in[a,b]$. If the real-valued functions $\underline{F}$ and $\overline{F}$ are differentiable at $t\in[a,b]$ then F is gH-differentiable at $t\in[a,b]$ and

$$F'(t)=\left[\min\left\{\frac{d}{dt}\underline{F(t)},\frac{d}{dt}\overline{F(t)}\right\},\max\left\{\frac{d}{dt}\underline{F(t)},\frac{d}{dt}\overline{F(t)}\right\}\right].\tag{4}$$

The converse of Proposition 1 is not true, that is, the gH-differentiability of F does not imply the differentiability of $\underline{F}$ and $\overline{F}$, [12].

It is known that $C[a,b]$ which is the family of all real-valued and continuous functions defined on interval $[a,b]$ is a complete metric space with the standard metric

$$h(f,g)=\max\{|f(t)-g(t)|:t\in[a,b]\}.$$

Let $C([a,b],\Omega_C(\mathbb{R}))$ denote the set of all continuous interval-valued functions defined on $[a,b]$. Then $C([a,b],\Omega_C(\mathbb{R}))$ is a complete metric space with the following metric

$$H_C(F,G)=\sup_{a\leqslant t\leqslant b}H(F(t),G(t)).$$

DEFINITION 2. [12] An interval-valued function $F:[a,b]\to\Omega_C(\mathbb{R})$ is said to be absolutely continuous if for each $\varepsilon>0$ there exists $\delta>0$ such that for each family $\{(s_k,t_k):\ k=1,2,\ldots,n\}$ of disjoint open intervals in $[a,b]$ with $\sum_{k=1}^{n}(t_k-s_k)<\delta$, the inequality $\sum_{k=1}^{n}H(F(t_k),F(s_k))<\varepsilon$ holds.

Let $AC([a,b],\Omega_C(\mathbb{R}))$ denote the set of all absolutely continuous interval-valued functions defined on $[a,b]$.

REMARK 1. [10] Let F be an element of $AC([a,b],\Omega_C(\mathbb{R}))$ such that $F(t)=[\underline{F(t)},\overline{F(t)}]$ and $t\in[a,b]$. If F is w-monotone on $[a,b]$ then it is easy to check that F is absolutely continuous if and only if the real-valued functions both $\underline{F}$ and $\overline{F}$ are absolutely continuous. Therefore, if $F:[a,b]\to\Omega_C(\mathbb{R})$ is absolutely continuous and w-monotone on $[a,b]$ then F is gH-differentiable a.e. on $[a,b]$, and (4) holds for a.e. $t\in[a,b]$.

The Lebesgue integral for interval-valued functions is a special case of the Lebesgue integral for set-valued mappings, [3].

Let $F:[a,b]\to\Omega_C(\mathbb{R})$ be an interval-valued function such that $F(t)=[\underline{F(t)},\overline{F(t)}]$ and $\underline{F}$, $\overline{F}$ are measurable and Lebesgue integrable on $[a,b]$. Then it is said that F is Lebesgue integrable on $[a,b]$ and

$$\int_a^b F(t)dt=\left[\int_a^b \underline{F(t)}dt,\int_a^b \overline{F(t)}dt\right].$$

Let $1\leqslant p\leqslant\infty$ and $L^p([a,b],\Omega_C(\mathbb{R}))$ denote the set of all interval-valued functions $F:[a,b]\to\Omega_C(\mathbb{R})$ such that the real function $K:[a,b]\to\mathbb{R}$ defined as $K(t)=H(F(t),\theta)$ belong to $L^p[a,b]$. Then $L^p([a,b],\Omega_C(\mathbb{R}))$ is a complete metric space with respect to the following metric [6]

$$H_p(F,G)=\begin{cases}\left(\int_a^b H(F(t),G(t))^p\,dt\right)^{1/p}, & 1\leqslant p<\infty\\ \operatorname*{ess\,sup}_{t\in[a,b]} H(F(t),G(t)), & p=\infty\end{cases}.$$

PROPOSITION 2. [12] If $F:[a,b]\to\Omega_C(\mathbb{R})$ is Lebesgue integrable on $[a,b]$ then the interval-valued function $G:[a,b]\to\Omega_C(\mathbb{R})$ defined by $G(t)=\int_a^b F(s)ds$ for $t\in[a,b]$ is w-increasing, absolutely continuous and $G'(t)=F(t)$.

DEFINITION 3. [10] Let $x\in L^1[a,b]$ and $\alpha>0$. Then the Riemann-Liouville fractional integral of order α is defined by

$$I_{a^+}^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\frac{x(s)}{(t-s)^{1-\alpha}}ds.$$

Let $F\in L^1([a,b],\Omega_C(\mathbb{R}))$ and $\alpha>0$. Then the integral defined as

$$\mathfrak{I}_{a^+}^{\alpha}F(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\frac{F(s)}{(t-s)^{1-\alpha}}ds$$

is called the *interval-valued Riemann-Liouville fractional integral* of order $\alpha>0$. Also it is obviously that if $F\in L^1([a,b],\Omega_C(\mathbb{R}))$ then

$$\mathfrak{I}_{a^+}^{\alpha}F(t)=\left[I_{a^+}^{\alpha}\underline{F(t)},I_{a^+}^{\alpha}\overline{F(t)}\right]$$

for a.e. $t\in[a,b]$.

We will write $F_{1-\alpha}(\cdot)$ instead of $\mathfrak{I}_{a^+}^{1-\alpha}F(\cdot)$.

LEMMA 1. [10] *The Riemann-Liouville integral of order $\alpha>0$ is a bounded operator from $L^1([a,b],\Omega_C(\mathbb{R}))$ into $L^1([a,b],\Omega_C(\mathbb{R}))$.*

Furthermore the reader can benefit from the papers [9], [15] and references therein for the more informations about the fractional calculus.

THEOREM 1. (Banach Fixed Point Theorem) [7] *Let X be a complete metric space. If $T:X\to X$ is a contraction on X then T has precisely one fixed point in X.*

3. The Main Results

In this section, we firstly consider IIE (2) under the following conditions:

($\mathbf{a}_1$) $Y \in L^1([a,b],\Omega_C(\mathbb{R}))$ and $Y_{1-\alpha}$ is $w-$increasing and AC on $[a,b]$. Also $Y_{1-\alpha}(a) = \{0\}$.

($\mathbf{b}_1$) T is an injective operator on $L^1([a,b],\Omega_C(\mathbb{R}))$.

THEOREM 2. *If the assumptions* ($\mathbf{a}_1$) *and* ($\mathbf{b}_1$) *are satisfied then IIE* (2) *has a unique solution* $X(t) = T^{-1}\left(Y'_{1-\alpha}(t)\right)$ *for* $t \in [a,b]$.

Proof. Let us define

$$Z(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} Y'_{1-\alpha}(s)ds \tag{5}$$

Taking ($\mathbf{a}_1$), ($\mathbf{b}_1$) and Remark 1 into account, we say that $Y'_{1-\alpha}(\cdot)$ exists a.e. on $[a,b]$ and $Y'_{1-\alpha}(\cdot) \in L^1([a,b],\Omega_C(\mathbb{R}))$. Therefore we conclude that $Z(\cdot) \in L^1([a,b],\Omega_C(\mathbb{R}))$ from Lemma 1.

We must to show that $Z(t) = Y(t)$ for a.e. $t \in [a,b]$. To do this, multiplying with $\Gamma(\alpha)(t-s)^{\alpha-1}$ both sides of IIE (5), we write

$$\Gamma(\alpha)(t-s)^{-\alpha} Z(s) = (t-s)^{-\alpha} \int_a^s (s-\tau)^{\alpha-1} Y'_{1-\alpha}(\tau)d\tau$$

and after integrating from a to t, we get

$$\Gamma(\alpha) \int_a^t (t-s)^{-\alpha} Z(s)ds = \int_a^t (t-s)^{-\alpha} \left(\int_a^s (s-\tau)^{\alpha-1} Y'_{1-\alpha}(\tau)d\tau \right) ds$$

and so

$$\begin{aligned} &\Gamma(\alpha) \left[\int_a^t (t-s)^{-\alpha} \underline{Z(s)}ds,\ \int_a^t (t-s)^{-\alpha} \overline{Z(s)}ds \right] \\ &= \left[\int_a^t (t-s)^{-\alpha} \left(\int_a^s (s-\tau)^{\alpha-1} \underline{Y'_{1-\alpha}(\tau)}d\tau \right) ds, \right. \\ &\quad \left. \int_a^t (t-s)^{-\alpha} \left(\int_a^s (s-\tau)^{\alpha-1} \overline{Y'_{1-\alpha}(\tau)}d\tau \right) ds \right]. \end{aligned} \tag{6}$$

By using Dirichlet Formula, we obtain

$$\begin{aligned} \int_a^t (t-s)^{-\alpha} \left(\int_a^s (s-\tau)^{\alpha-1} \underline{Y'_{1-\alpha}(\tau)}d\tau \right) ds &= \int_a^t \underline{Y'_{1-\alpha}(\tau)}d\tau \int_\tau^t (t-s)^{-\alpha}(s-\tau)^{\alpha-1}ds \\ &= \Gamma(\alpha)\Gamma(1-\alpha) \int_a^t \underline{Y'_{1-\alpha}(\tau)}d\tau \end{aligned}$$

and similarly

$$\int_a^t (t-s)^{-\alpha}\left(\int_a^s (s-\tau)^{\alpha-1}\overline{Y'_{1-\alpha}(\tau)}d\tau\right)ds = \int_a^t \overline{Y'_{1-\alpha}(\tau)}d\tau \int_\tau^t (t-s)^{-\alpha}(s-\tau)^{\alpha-1}ds$$
$$= \Gamma(\alpha)\Gamma(1-\alpha)\int_a^t \overline{Y'_{1-\alpha}(\tau)}d\tau.$$

Taking these last two equations and (6) into account, we have

$$\Gamma(\alpha)\int_a^t (t-s)^{-\alpha}Z(s)ds = \Gamma(\alpha)\Gamma(1-\alpha)\int_a^t Y'_{1-\alpha}(\tau)d\tau$$

and so

$$\frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}Z(s)ds = \int_a^t Y'_{1-\alpha}(\tau)d\tau,$$

$$Z_{1-\alpha}(t) = \int_a^t Y'_{1-\alpha}(\tau)d\tau.$$

Since $Y'_{1-\alpha}(\cdot) \in L^1([a,b],\Omega_C(\mathbb{R}))$, the interval-valued function $Z_{1-\alpha}$ is AC and w-monotone on $[a,b]$ from Proposition 2. Also $Z_{1-\alpha}$ is differentiable a.e. on $[a,b]$ from Remark 1. Therefore, by using again Proposition 2 we have

$$Z'_{1-\alpha}(t) = Y'_{1-\alpha}(t) \tag{7}$$

for a.e. on $[a,b]$. On the other hand, it can be seen that (7) implies $Z(t) = Y(t)$ for a.e. $t \in [a,b]$, (see [10]).
So we infer that $X = X(t)$ is a solution of IIE (2) if and only if X satisfies the equation

$$(TX)(t) = Y'_{1-\alpha}(t)$$

for a.e. $t \in [a,b]$. Also, since T is an injective operator then we can apply inverse operator T^{-1} to both sides of the last equality. Then we get that

$$X(t) = T^{-1}\left(Y'_{1-\alpha}(t)\right)$$
$$= T^{-1}\left(\frac{1}{\Gamma(1-\alpha)}\left(\int_a^t (t-s)^{-\alpha}Y(s)ds\right)'\right)$$

for a.e. $t \in [a,b]$ and $T^{-1}\left(Y'_{1-\alpha}(t)\right)$ is the unique solution of IIE (2).

REMARK 2. It can be easily seen that if T is chosen as unit operator then IIE (2) returns to IIE (1) considering in [10]. Then the solution of (2) is $T^{-1}\left(Y'_{1-\alpha}(t)\right) = Y'_{1-\alpha}(t)$. Also, if $Y_{1-\alpha}$ is not w-increasing then $Y'_{1-\alpha}(t)$ may not be a solution of IIE (2) (see Example 1 in [10]).

EXAMPLE 1. Let us consider the following first kind nonlinear Volterra IIE in $L^1([0,1],\Omega_C(\mathbb{R}))$:

$$\left[\sqrt{t},\sqrt{t}+t^2\right] = \frac{1}{\Gamma(1/2)}\int_0^t (t-s)^{-1/2}\left[e^{\underline{X(s)}}, e^{\overline{X(s)}}\right]ds. \tag{8}$$

Here the operator $T : L^1([0,1],\Omega_C(\mathbb{R})) \to L^1([0,1],\Omega_C(\mathbb{R}))$ and the interval-valued function $Y : [0,1] \to \Omega_C(\mathbb{R})$ are defined by

$$(TX)(t) = e^{X(t)} = \left[e^{\underline{X(t)}}, e^{\overline{X(t)}}\right]$$

and

$$Y(t) = \left[\sqrt{t}, \sqrt{t} + t^2\right],$$

respectively. Since $f(x) = e^x$ is an increasing real-valued function, the notation $e^{X(t)} = \left[e^{\underline{X(t)}}, e^{\overline{X(t)}}\right]$ is meaningful for all $X(t) \in \Omega_C(\mathbb{R})$. On the other hand

$$\begin{aligned} Y_{1-\frac{1}{2}}(t) &= \frac{1}{\Gamma(1/2)} \int_a^t (t-s)^{-1/2} \left[\sqrt{s}, \sqrt{s} + s^2\right] ds \\ &= \left[\frac{\sqrt{\pi}}{2} t, \frac{\sqrt{\pi}}{2} t + \frac{16\sqrt{\pi} t^{5/2}}{15\pi}\right] \end{aligned}$$

for $t \in [0,1]$ and so $Y_{1-\frac{1}{2}} \in AC([0,1],\Omega_C(\mathbb{R}))$.

Also

$$w\left(Y_{1-\frac{1}{2}}(t)\right) = \frac{16\sqrt{\pi} t^{5/2}}{15\pi}$$

and

$$w'\left(Y_{1-\frac{1}{2}}(t)\right) = \frac{8\sqrt{\pi} t^{3/2}}{3\pi} \geqslant 0.$$

Then $Y_{1-\frac{1}{2}}$ is w-increasing on $[0,1]$ and

$$Y'_{1-\frac{1}{2}}(t) = \left[\frac{\sqrt{\pi}}{2}, \frac{\sqrt{\pi}}{2} + \frac{8\sqrt{\pi} t^{3/2}}{3\pi}\right],$$

for $t \in [0,1]$. So

$$(TX)(t) = e^{X(t)} = \left[e^{\underline{X(t)}}, e^{\overline{X(t)}}\right] = Y'_{1-\frac{1}{2}}(t) = \left[\frac{\sqrt{\pi}}{2}, \frac{\sqrt{\pi}}{2} + \frac{8\sqrt{\pi} t^{3/2}}{3\pi}\right]$$

and we obtain that

$$X(t) = T^{-1}\left(Y'_{1-\frac{1}{2}}(t)\right) = \left[\ln\frac{\sqrt{\pi}}{2}, \ln\left(\frac{\sqrt{\pi}}{2} + \frac{8\sqrt{\pi} t^{3/2}}{3\pi}\right)\right]$$

is solution of IIE (8). Figure 1 shows the graph of this solution.

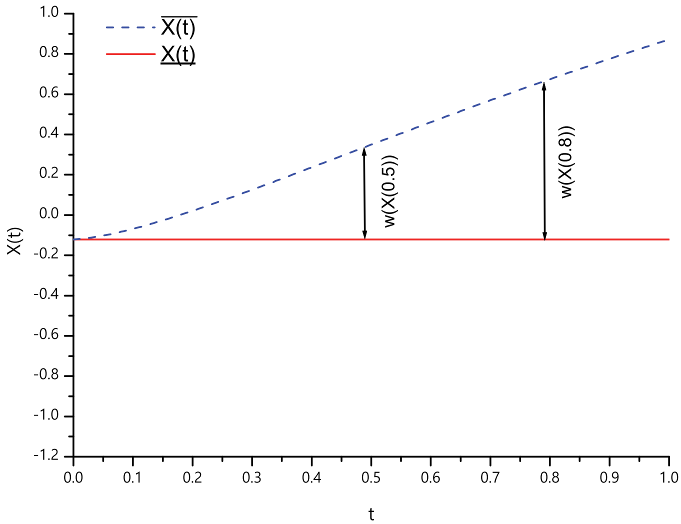

Figure 1. $X(t) = \left[\underline{X}(t), \overline{X}(t)\right]$. For example, $X(0.5) = [-0.120..., 0.349...]$ and $X(0.8) = [-0.120..., 0.674...]$.

The main tool used in next theorem depends on the following lemma. Before giving the our other principal result, let us prove this fact.

LEMMA 2. *The equality*

$$H_C(X,Y) = \sup_{t\in[a,b]} A_{(X,Y)}(t)$$

holds for all $X,Y \in C([a,b],\Omega_C(\mathbb{R}))$, *where*

$$A_{(X,Y)}(t) = \sup_{s\in[a,t]} \left(\max\left\{\left|\underline{X}(s) - \underline{Y}(s)\right|, \left|\overline{X}(s) - \overline{Y}(s)\right|\right\}\right)$$

and

$$H_C(X,Y) = \sup_{t\in[a,b]} \left(\max\left\{\left|\underline{X}(t) - \underline{Y}(t)\right|, \left|\overline{X}(t) - \overline{Y}(t)\right|\right\}\right).$$

Proof. First of all, we should indicate that the function $A_{(X,Y)} : [a,b] \to \mathbb{R}$ is well defined and

$$0 \leqslant A_{(X,Y)}(t) \leqslant H_C(X,Y)$$

holds for all $t \in [a,b]$. Then we can write

$$\sup_{t\in[a,b]} A_{(X,Y)}(t) \leqslant H_C(X,Y) \tag{9}$$

and

$$A_{(X,Y)}(b) = H_C(X,Y) \leqslant \sup_{t\in[a,b]} A_{(X,Y)}(t) \tag{10}$$

from the properties of supremum. So, we obtain

$$H_C(X,Y) = \sup_{t\in[a,b]} A_{(X,Y)}(t)$$

by using the inequalities (9) and (10).

Now, we consider the second kind nonlinear IIE (3) in $C([a,b],\Omega_C(\mathbb{R}))$ under the following conditions:

($\mathbf{a}_2$) There exists a nonnegative constant M such that the operator $T : D \subseteq C([a,b],\Omega_C(\mathbb{R})) \to C([a,b],\Omega_C(\mathbb{R}))$ satisfies the inequalities

$$\left|\underline{(TX)(t)} - \underline{(TY)(t)}\right| \leqslant M\left|\underline{X(t)} - \underline{Y(t)}\right|,$$
$$\left|\overline{(TX)(t)} - \overline{(TY)(t)}\right| \leqslant M\left|\overline{X(t)} - \overline{Y(t)}\right|$$

for all $X,Y \in D \subseteq C([a,b],\Omega_C(\mathbb{R}))$ and $t \in [a,b]$, where D is a closed subset of $C([a,b],\Omega_C(\mathbb{R}))$.

($\mathbf{b}_2$) The inequality

$$\frac{M(b-a)^\alpha}{\Gamma(\alpha+1)} < 1$$

holds.

THEOREM 3. *Under the assumptions* ($\mathbf{a}_2$) *and* ($\mathbf{b}_2$), *there exists a solution* $X = X(t)$ *of IIE* (3).

Proof. Theorem 1 will be used as the main tool in this proof. Let us define the operator F by

$$(FX)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(TX)(s)ds$$

for every $X \in D \subseteq C([a,b],\Omega_C(\mathbb{R}))$ and $t \in [a,b]$.

Then by using Lemma 2 and conditions $(\mathbf{a}_2)$ and $(\mathbf{b}_2)$, we get

$$
\begin{aligned}
& H_C(FX,FY) \\
= & \sup_{t\in[a,b]} H((FX)(t),(FY)(t)) \\
= & \sup_{t\in[a,b]} \left(\max\left\{\left|\underline{(FX)(t)}-\underline{(FY)(t)}\right|,\left|\overline{(FX)(t)}-\overline{(FY)(t)}\right|\right\}\right) \\
= & \sup_{t\in[a,b]} \left(\max\left\{\left|\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\underline{(TX)(s)}ds-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\underline{(TY)(s)}ds\right|,\right.\right. \\
& \left.\left.\left|\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\overline{(TX)(s)}ds-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\overline{(TY)(s)}ds\right|\right\}\right) \\
\leqslant & \frac{1}{\Gamma(\alpha)}\sup_{t\in[a,b]} \left(\max\left\{\int_a^t (t-s)^{\alpha-1}\left|\underline{(TX)(s)}-\underline{(TY)(s)}\right|ds,\right.\right. \\
& \left.\left.\int_a^t (t-s)^{\alpha-1}\left|\overline{(TX)(s)}-\overline{(TY)(s)}\right|ds\right\}\right) \\
\leqslant & \frac{1}{\Gamma(\alpha)}\sup_{t\in[a,b]} \left(\max\left\{\int_a^t (t-s)^{\alpha-1}M\left|\underline{X(s)}-\underline{Y(s)}\right|ds,\int_a^t (t-s)^{\alpha-1}M\left|\overline{X(s)}-\overline{Y(s)}\right|ds\right\}\right) \\
\leqslant & \frac{M}{\Gamma(\alpha)}\sup_{t\in[a,b]} \left(\int_a^t (t-s)^{\alpha-1}\max\left\{\left|\underline{X(s)}-\underline{Y(s)}\right|,\left|\overline{X(s)}-\overline{Y(s)}\right|\right\}ds\right) \\
\leqslant & \frac{M}{\Gamma(\alpha)}\sup_{t\in[a,b]} \left(\int_a^t (t-s)^{\alpha-1}A_{(X,Y)}(t)\,ds\right) \\
= & \frac{M}{\Gamma(\alpha)}\sup_{t\in[a,b]} \left(A_{(X,Y)}(t)\int_a^t (t-s)^{\alpha-1}ds\right) \\
\leqslant & \frac{M(b-a)^{\alpha}}{\Gamma(\alpha+1)}\sup_{t\in[a,b]} A_{(X,Y)}(t) \\
= & \frac{M(b-a)^{\alpha}}{\Gamma(\alpha+1)}H_C(X,Y)
\end{aligned}
$$

for all $X,Y\in D\subseteq C([a,b],\Omega_C(\mathbb{R}))$.

Since D is a complete subspace of $C([a,b],\Omega_C(\mathbb{R}))$ and $\frac{M(b-a)^{\alpha}}{\Gamma(\alpha+1)}<1$ by assumption $(\mathbf{b}_2)$, the operator F is a contraction on $D\subseteq C([a,b],\Omega_C(\mathbb{R}))$. Therefore, from Theorem 1 we say that F has precisely one fixed point in $D\subseteq C([a,b],\Omega_C(\mathbb{R}))$. Consequently, the nonlinear IIE (3) has a solution in $C([a,b],\Omega_C(\mathbb{R}))$. This completes the proof.

EXAMPLE 2. Let us consider the nonlinear Volterra IIE of the second kind

$$
X(t)=\frac{1}{\Gamma(1/2)}\int_0^t (t-s)^{-1/2}\frac{\sin X(s)}{2}ds \tag{11}
$$

in $B\left[\theta, \frac{\pi}{2}\right] \subset C([0,\frac{\pi}{2}],\Omega_C(\mathbb{R}))$.

It should be noted that the closed ball $B\left[\theta, \frac{\pi}{2}\right]$ is a complete subspace of $C([0,\frac{\pi}{2}],\Omega_C(\mathbb{R}))$. So Banach fixed point theorem can be used on $B\left[\theta, \frac{\pi}{2}\right]$.

On the other hand we have $X(t) \subseteq [-\frac{\pi}{2}, \frac{\pi}{2}]$ for all $X \in B\left[\theta, \frac{\pi}{2}\right]$ and $f(x) = \sin x$ is an increasing real-valued function on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore the notation $\sin X(t) = \left[\sin \underline{X(t)}, \sin \overline{X(t)}\right]$ is meaningful for all $X \in B\left[\theta, \frac{\pi}{2}\right] \subset C([0,\frac{\pi}{2}],\Omega_C(\mathbb{R}))$.

For this equation,

$$(TX)(t) = \frac{\sin X(t)}{2} = \frac{1}{2}\left[\sin \underline{X(t)}, \sin \overline{X(t)}\right]$$

and $a = 0,\ b = \frac{\pi}{2},\ \alpha = \frac{1}{2}$. On the other hand,

$$\begin{aligned}
&\left|\underline{(TX)(t)} - \underline{(TY)(t)}\right| \\
&= \frac{1}{2}\left|\sin \underline{X(t)} - \sin \underline{Y(t)}\right| \\
&= \left|\sin\left(\frac{\underline{X(t)} - \underline{Y(t)}}{2}\right)\right| \left|\cos\left(\frac{\underline{X(t)} + \underline{Y(t)}}{2}\right)\right| \\
&\leqslant \left|\sin\left(\frac{\underline{X(t)} - \underline{Y(t)}}{2}\right)\right| \\
&\leqslant \frac{1}{2}\left|\underline{X(t)} - \underline{Y(t)}\right|
\end{aligned}$$

and similarly

$$\left|\overline{(TX)(t)} - \overline{(TY)(t)}\right| \leqslant \frac{1}{2}\left|\overline{X(t)} - \overline{Y(t)}\right|.$$

It is easy to see that the operator T satisfies the condition $(\mathbf{a}_2)$ with $M = 1/2$. Also,

$$\frac{M(b-a)^{\alpha}}{\Gamma(\alpha+1)} = \frac{\sqrt{2}}{2} < 1.$$

Therefore, Theorem 3 guarantees that IIE (11) has a solution $X = X(t)$ in $B\left[\theta, \frac{\pi}{2}\right] \subset C([0,\frac{\pi}{2}],\Omega_C(\mathbb{R}))$.

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Sümeyye Çakan
İnönü University
Faculty of Science and Arts
Department of Mathematics
44280, Malatya, Turkey
e-mail: sumeyye.tay@gmail.com

Ümit Çakan
İnönü University
Faculty of Science and Arts
Department of Mathematics
44280, Malatya, Turkey
e-mail: umitcakan@gmail.com