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EXISTENCE, UNIQUENESS AND CONTINUATION OF SOLUTION OF A SUB DIFFUSION FUNCTIONAL DIFFERENTIAL EQUATIONS WITH AN INTEGRAL CONDITION

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Abstract. In this work, we consider a sub-diffusion functional differential equation with an integral condition. We apply the method of semidiscretization in time, to establish the existence and uniqueness of solutions. We also study the continuation of the solution to the maximal interval of existence. By using Rothe's sequence and values of the fractional integrals over time steps, the results are obtained.

1. Introduction

A sub diffusion model describing the dynamics of a single species can be written as follows,

$$\frac{\partial u}{\partial t}(x,t) - \partial_t^{1-\alpha}\Big(\frac{\partial^2 u}{\partial x^2}(x,t)\Big) = -du(x,t) + bu(x,t-\tau)), \quad (x,t) \in (0,1)\times[0,T], \tag{1.1}$$

$$u(x,t) = \Phi(x,t), \quad t \in [-\tau, 0], \quad x \in (0,1),$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad t \in [0,T],$$

$$\int_0^1 u(x,t)dx = \psi(t), \quad t \in [0,T],$$

where the coefficients b and d are positive constants. The term $u(x,t)$ denotes the size of a population at time t and at the point $x \in [0,1]$. The second partial derivative $\frac{\partial^2 u}{\partial x^2}$ represents the internal migration. Here we assume that the migration can be slower and faster sometimes. Hence the fractional derivative plays a crucial role to describe such behaviour. The positive constants b and d denote the birth and death rates respectively. The history function $\psi(t)$ may be treated as a function which control the average population size at time t. Hence, we have no-flux condition at the left end. Moreover the right end is free, so there may be a flux at this end. The integral condition control the average population size.

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The above problem can be treated as a general sub-diffusion equations. So motivated by the above model, this paper is concerned with the following sub diffusion equation of order $\alpha \in (0,1)$ with integral condition

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} - \partial_t^{1-\alpha}\Big(\frac{\partial^2 u(x,t)}{\partial x^2}\Big) &= f(x,t,u(x,t),u_t) \quad \text{in} \quad (0,1)\times[0,T],\\
u(x,t) &= \Phi(x,t) \quad \text{in} \quad (0,1)\times[-\tau,0],\\
\frac{\partial u(0,t)}{\partial x} &= 0, \quad \text{on} \quad [0,T],\\
\int_0^1 u(x,t)dx &= 0 \quad \text{on} \quad [0,T].
\end{aligned} \tag{1.2}
$$

It is required that the function $u:[-\tau,T]\to \mathbf{H}$ to be in $\mathbf{C}_T$. The nonlinear forcing term f is defined from $(0,1)\times[0,T]\times\mathbb{R}\times\mathbf{D}$ into $\mathbb{R}$, with a suitably chosen space $\mathbf{D}\subset\mathbf{C}_0$. The history function Φ is defined from $[-\tau,0]$ to $\mathbf{H}$ with the property that $\Phi\in\mathbf{C}_T$. We also define the space $B_2^1(0,1)$ which actually reduces the second partial derivative into the inner product to a simple standard inner product, so it is a wonderful space for problems involving second order partial derivatives. These spaces are defined in details in the preliminaries section.

The operator $A=-\frac{\partial^2}{\partial x^2}$ is a symmetric and uniformly elliptic operator. The Riemann-Liouville fractional derivative ∂_t^α is defined by

$$
\partial_t^\alpha h(t) := \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{d\tau^n}\int_0^t (t-\tau)^{n-\alpha-1}h(\tau)d\tau, \quad n-1<\alpha<n,\ n\in N, \tag{1.3}
$$

for $\alpha>0$. For more detail on fractional and fractional partial differential equations, we refer to [1, 7, 8, 12, 19, 21, 22]. For more details on fractional derivatives and integrals, we refer the reader to [16, 18]. The case when $\alpha=1$ in the equation (1.2) gives classical diffusion equation. Moreover equation (1.2) with $0<\alpha<1$ is called fractional diffusion equation and regarded as a model of anomalous diffusion in heterogeneous media.

Anomalous diffusion is one of the most ubiquitous phenomena in nature. It is observed in various fields of physics, for instance, transport of fluid in porous media, surface growth, diffusion of plasma, diffusion at liquid surfaces, and two-dimensional rotating flow. Due to such anomalies, the dynamics cannot be studied with the help of classical diffusion models. Here, fractional derivatives play key role in characterizing anomalous diffusion and the corresponding macroscopic model is a fractional partial differential equation. From continuous time random walk model, Metzler and Klafter [15] derived equation with fractional derivative of order $0<\alpha<1$ as a macroscopic model.

2. Preliminaries

In this section, we define various notations used in this paper and state the existence and uniqueness theorem. The proof is given at the end after obtaining the bounds of approximate solution. We denote by $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, a usual L^p-space. In particular, $L^2(\Omega)$ denotes the L^2-space equipped with the scalar product $(\cdot,\cdot)$.

In the present work, in order to establish the existence, uniqueness of a solution we apply Rothe's method or method of lines [10]. Furthermore, the unique continuation of a solution to the maximal interval of existence is also studied. We note that there is no loss of generality in considering the homogeneous conditions in (1.2) as the more general problem can be transformed in the problem with homogeneous conditions. For more details, we refer to [2, 3]. The model (1.2) with $\alpha = 1$ is studied in [2]. Several authors have studied heat equations with integral conditions ([6, 9, 11, 17]).

This paper is motivated by the papers of Bouziani and Merazga [5, 13] and Bahuguna et. al., Bahuguna and Shukla [2, 4]. In [5, 13] the authors have used the method of semi-discretization without delays. In [4] the method of semigroups of bounded linear operators in a Banach space is used to study a partial differential equation involving delays arising in the population dynamics. We use the method of semi-discretizaion in time first to establish the local existence of a unique solution on a subinterval $[-\tau, T_0]$, $0 < T_0 \leqslant T$ and then extend it either to the whole interval $[-\tau, T]$ or to the maximal subinterval $[-\tau, T_{max}) \subset [-\tau, T]$ of existence with $\lim_{t \to T_{max}-} \|u(t)\| = +\infty$.

Our model may be treated as an abstract equation in the real Hilbert space $\mathbf{H} = L^2(0,1)$ of square-integrable functions defined from $(0,1)$ into $\mathbb{R}$. The space $\mathbf{H} = L^2(0,1)$ is induced by the inner product

$$(u,v) = \int_0^1 u(x)v(x)\,dx, \quad u,v \in \mathbf{H},$$

with the corresponding norm given by

$$\|u\|^2 = \int_0^1 |u(x)|^2\,dx.$$

The Sobolev space $\mathbf{H}^k$ for $k \in \mathbb{N}$ is the Hilbert space of all functions $u \in \mathbf{H}$ such that the distributional derivative $u^{(j)} \in \mathbf{H}$ with the inner product

$$(u,v)_k = \sum_{j=0}^{k} (u^{(j)}, v^{(j)}), \quad u,v \in \mathbf{H}^k.$$

The corresponding norm is given by

$$\|u\|_k^2 = \sum_{j=0}^{k} \|u^{(j)}\|^2.$$

The integral condition in the model may be incorporated with the space itself by taking a new space $\mathbf{V} \subset \mathbf{H}$, which is defined by

$$\mathbf{V} = \{u \in \mathbf{H} : \int_0^1 u(x)\,dx = 0\}.$$

It is not difficult to see that the space $\mathbf{V}$ is a closed subspace of $\mathbf{H}$ and hence is a Hilbert space itself with the inner product $(.,.)$. and the corresponding norm $\|.\|$.

The symbol $C(I;X)$ represents the space of all continuous functions u from $[a,b]$ into a Banach space X with the following norm

$$\|u\|_{C(I;X)} = \max_{a\leqslant t\leqslant b} \|u(t)\|_X.$$

Further, the space $L^2(I;X)$ consists of all square-Bochner integrable functions (equivalent classes) u. The norm is given by

$$\|u\|^2_{L^2(I;X)} = \int_a^b \|u(t)\|^2_X \, dt.$$

Similarly the space $L^\infty(I;X)$ is the Banach space of all essentially bounded functions from I into X with the following norm

$$\|u\|_{L^\infty(I;X)} = \operatorname*{ess\,sup}_{t\in I} \|u(t)\|_X.$$

The Banach space $C^{0,1}(I;X)$ is the space of all Lipschitz continuous functions from I into X. In this case the norm is given by

$$\|u\|_{C^{0,1}(I;X)} = \|u\|_{C(I;X)} + \sup_{t,s\in I;\, t\neq s} \frac{\|u(t)-u(s)\|}{|t-s|}.$$

The most important space we need for this work is the space $B_2^1(0,1)$, which is introduced by Merazga and A. Bouziani [13]. It is the completion of the space $\mathbf{C}_0(0,1)$ of all real continuous functions having compact supports in (0,1) with the following inner product

$$(u,v)_{B_2^1} = \int_0^1 \Im_x u . \Im_x v dx,$$

where $\Im_x v = \int_0^x v(\xi)d\xi$ for every fixed $x \in (0,1)$. The norm is given by

$$\|u\|^2_{B_2^1} = \int_0^1 (\Im_x u)^2 dx.$$

One can easily verify the following inequality

$$\|v\|^2_{B_2^1} \leqslant \frac{1}{2}\|v\|^2$$

for every $v \in \mathbf{H}$. Moreover the embedding $\mathbf{H} \to B_2^1(0,1)$ is continuous.

It is not difficult to observe that the space of all continuous functions from $[-\tau,t]$ into $B_2^1(0,1)$ i.e. $\mathbf{C}_t = C([-\tau,t];B_2^1(0,1))$ for $t \in [0,T]$ is a Banach space endowed with the supremum norm

$$\|\chi\|_t = \sup_{-\tau\leqslant\eta\leqslant t} \|\chi(\eta)\|_{B_2^1}, \quad \chi \in \mathbf{C}_t.$$

The spaces $\mathbf{C}_0$ and $\mathbf{C}_T$ are the spaces corresponding to $t=0$ and $t=T$ in $\mathbf{C}_t$.

Given a function $\chi:(0,1)\times[-\tau,T]\to\mathbb{R}$ such that for each $t\in[-\tau,T]$, $\chi(.,t):[-\tau,T]\to\mathbf{H}$, we may identify it with the function $\chi:[-\tau,T]\to\mathbf{H}$ given by $\chi(t)(x)=\chi(x,t)$. For a given Lipschitz continuous function $g:(0,1)\times[a,b]\times\mathbb{R}\times\mathbf{C}_0\to\mathbb{R}$ and χ as above with the additional property that $\chi\in\mathbf{C}_T$, we identify it with a function $g:[a,b]\times\mathbf{H}\times\mathbf{C}_0\to\mathbf{H}$ by $g(t,\chi(t),\chi_t)(x)=g(x,t,\chi(x,t),\chi_t)$. For $\chi\in\mathbf{H}$, we define $|\chi|\in\mathbf{H}$ by $|\chi|(x)=|\chi(x)|$, $x\in(0,1)$. Let $\mathbf{D}=\{\chi\in\mathbf{C}_0:\chi(\theta)\in\mathbf{H},\ \theta\in[-\tau,0]\}$.

We put the following assumptions, in order to prove our results.

(A1) $f(t,u,\phi)\in\mathbf{H}$ for $(t,u,\phi)\in[0,T]\times\mathbf{H}\times\mathbf{D}$ and satisfies the local Lipschitz condition

$$\|f(t_1,u_1,\phi_1)-f(t_2,u_2,\phi_2)\|_{B_2^1}\leqslant L_f(r_1,r_2)[|t_1-t_2|+\|u_1-u_2\|_{B_2^1}+\|\phi_1-\phi_2\|_{\mathbf{C}_0}]$$

for all $t_i\in[0,T]$, $u_i\in\mathbf{V}$ with $\|u_i\|_{B_2^1}\leqslant r_1$, and $\phi_i\in\mathbf{D}$ with $\|\phi_i\|_{B_2^1}\leqslant r_2$, $i=1,2$, $L_f(r_1,r_2)$ is a non-negative non-decreasing function of r_1 and r_2.

(A2) $\Phi:[-\tau,0]\to\mathbf{H}^2$. Further, we also assume that $\Phi\in C^{0,1}([-\tau,0];\mathbf{H})$ with the uniform Lipchitz constant L_Φ.

(A3) $\frac{d\Phi(0,x)}{dx}=0$, $\int_0^1\Phi(0,x)dx=0$. We further assume that there exist positive constants M and N such that $N\|u\|^2\geqslant(\partial_t^{1-\alpha}u,u)\geqslant M\|u\|^2$.

DEFINITION 2.1. By a weak solution of of our problem, we mean a function $u:[-\tau,T]\to\mathbf{H}$; such that

(i) $u=\Phi$ on $[-\tau,0]$;

(ii) $u\in L^\infty([0,T];\mathbf{V})\cap C^{0,1}([-\tau,T];B_2^1(0,1))$;

(iii) u has ($a.e$ in $[0,T]$) a strong derivative $\frac{du}{dt}\in L^\infty([0,T];B_2^1(0,1))$;

(iv) For all $\phi\in\mathbf{V}$ and a.e. $t\in[0,T]$, the identity

$$\left(\frac{du(t)}{dt},\phi\right)_{B_2^1}+(\partial_t^{1-\alpha}u(t),\phi)=(f(t,u(t),u_t),\phi)_{B_2^1}, \tag{2.1}$$

is satisfied.

THEOREM 2.1. *Under the assumptions* $(A1)$–$(A3)$, *the problem (1.2) has a unique weak solution on* $[-\tau,T_0]$ *for some* $0<T_0\leqslant T$. *Moreover, the solution u can be extended uniquely either on the whole interval* $[-\tau,T]$ *or there exist a maximal interval* $[0,T_{\max})$, $0<T_{\max}<T$, *of existence such that* $\lim_{t\to T_{max}-}\|u(t)\|=+\infty$.

The proof of the above theorem is given in the later part of the paper after obtaining necessary estimates.

3. Discretization scheme and a priori estimates

This section focuses on existence and uniqueness of a weak solution. In order to apply the method of line we take the following steps. We first choose a suitable $h_n = \frac{T_0}{n}$ where $0 < T_0 \leqslant T$. Next, we set $u_0^n = \Phi(0)$ for all $n \in \mathbb{N}$ and then define each of $\{u_j^n\}_{j=1}^n$ as the unique solution of the time- discretized problems.

$$\delta u_j^n - \partial_{t_j}^{1-\alpha}\Big(\frac{d^2 u_j^n}{dx^2}\Big) = f_j^n, \quad x \in (0,1), \tag{3.1}$$

$$\frac{du_j^n}{dx}(0) = 0, \tag{3.2}$$

$$\int_0^1 u_j^n(x)dx = 0, \tag{3.3}$$

where the sequence $\delta u_j^n = \frac{u_j^n - u_{j-1}^n}{h_n}$, and the discretized function is $f_j^n = f(t_j^n, u_{j-1}^n, \tilde{u}_{j-1}^n)$. Here $\partial_{t_j}^{1-\alpha} u = \partial_t^{1-\alpha} u|_{t=t_j}$. The initial point is define by $\tilde{u}_0^n(t) = \Phi(t)$ for $t \in [-\tau, 0]$, $\tilde{u}_0^n(t) = \Phi(0)$ for $t \in [0, T_0]$ and for $2 \leqslant j \leqslant n$,

$$\tilde{u}_j^n(\theta) = \begin{cases} \Phi(t_j^n + \theta), & \theta \leqslant -t_j^n, \\ u_{i-1}^n + (\theta - t_{j+1-i}^n)\delta u_i^n, & \theta \geqslant -t_j^n,\ \theta \in [-t_{j+1-i}^n, -t_{j-i}^n],\ 1 \leqslant i \leqslant j. \end{cases} \tag{3.4}$$

Established in [14] Lemma 3.1, the existence of unique $u_j^n \in \mathbf{H^2}$ satisfying (3.1)–(3.3) is similarly ensured. Here, first we prove the estimates for u_j^n and difference quotients $\{(u_j^n - u_{j-1}^n)/h_n\}$ using our assumptions $(A1)-(A3)$. Then moving to next step, we introduce sequences $\{U^n\}$ of polygonal functions from $U^n : [-\tau, T_0] \to H^2(0,1) \cap V$ defined as follows

$$U^n(t) = \begin{cases} \Phi(t), & t \in [-\tau, 0], \\ u_{j-1}^n + (t - t_{j-1})\delta u_j^n, & t \in [t_{j-1}^n, t_j^n]. \end{cases} \tag{3.5}$$

We establish the convergence of $\{U^n\}$ to a unique solution u of our problem in the space $C([-\tau, T_0], B_2^1(0,1))$ as $n \to \infty$. We some time suppress the superscript n, throughout, C will represent a generic constant independent of j, h_n and n for the notational convenience and to make computation easy.

THEOREM 3.1. *Under the assumptions* $(A1)-(A3)$, *there exists a positive constant* C, *independent of* j, h *and* n *such that.*

$$\|u_j\| \leqslant C, \tag{3.6}$$

$$\|\delta u_j\|_{B_2^1} \leqslant C, \tag{3.7}$$

$n \geqslant 1$ *and* $j = 1, \dots n$.

Proof. Let us first fix $R>0$ and then choose $0<T_0\leqslant T$ such that $T_0M\leqslant R$. Here the constants are defined by

$$M=[L_f(R,\tilde{R})(T+\tilde{R})+\|\partial_{t_0}^{1-\alpha}\tfrac{d^2u_0}{dx^2}\|_{B_2^1}+\|f(0,u_0,\Phi)\|_{B_2^1}], \tag{3.8}$$

$$\tilde{R}=4(\|\Phi\|_{\mathbf{C}_0}+R)+\|u_0\|_{B_2^1}. \tag{3.9}$$

For any $\phi\in\mathbf{V}$, by taking the inner product in $B_2^1(0,1)$ of (3.1), we obtain

$$(\delta u_j,\phi)_{B_2^1}-\left(\partial_{t_j}^{1-\alpha}\left(\frac{d^2u_j^n}{dx^2}\right),\phi\right)_{B_2^1}=(f_j,\phi)_{B_2^1}. \tag{3.10}$$

By doing integration by parts and using (3.2), we get

$$\begin{aligned}\left(\partial_{t_j}^{1-\alpha}\frac{d^2u_j}{dx^2},\phi\right)_{B_2^1}&=\int_0^1\partial_{t_j}^{1-\alpha}\frac{du_j}{dx}(x)\mathfrak{I}_x\phi dx\\&=\partial_{t_j}^{1-\alpha}u_j(x)\mathfrak{I}_x\phi|_{x=0}^{x=1}-\int_0^1\partial_{t_j}^{1-\alpha}u_j\phi dx.\end{aligned}$$

From the above observation, we obtain $(\partial_{t_j}^{1-\alpha}\frac{d^2u_j}{dx^2},\phi)_{B_2^1}=-(\partial_{t_j}^{1-\alpha}u_j,\phi)$, (3.10) becomes

$$(\delta u_j,\phi)_{B_2^1}+(\partial_{t_j}^{1-\alpha}u_j,\phi)=(f_j,\phi)_{B_2^1}. \tag{3.11}$$

For $j=1$ in (3.11), and taking $\phi=u_1-u_0$, we get

$$(u_1-u_0,u_1-u_0)_{B_2^1}+h_n(\partial_{t_1}^{1-\alpha}(u_1),u_1-u_0)=h_n\big(f_1,u_1-u_0\big)_{B_2^1},$$

which further implies

$$(u_1-u_0,u_1-u_0)_{B_2^1}+h_n(\partial_{t_1}^{1-\alpha}(u_1-u_0),u_1-u_0)=h_n\left(f_1+\partial_{t_1}^{1-\alpha}\frac{d^2u_0}{dx^2},u_1-u_0\right)_{B_2^1},$$

An important computation is the second term in the inner product. By assumption $(A3)$ and using the above relations, we have

$$\|u_1-u_0\|_{B_2^1}\leqslant h_n\left[\left\|f(t_1,u_0,\tilde{u_0})+\partial_{t_1}^{1-\alpha}\frac{d^2u_0}{dx^2}\right\|_{B_2^1}\right] \tag{3.12}$$

Now, we obtain

$$\begin{aligned}\|f(t_1,u_0,\tilde{u_0})\|_{B_2^1}&\leqslant\|f(t_1,u_0,\tilde{u_0})-f(0,u_0,\phi)\|_{B_2^1}+\|f(0,u_0,\phi)\|_{B_2^1}\\&\leqslant L_f(R,\tilde{R})[T+R+\tilde{R}]+\|f(0,u_0,\phi)\|_{B_2^1}.\end{aligned} \tag{3.13}$$

Thus

$$\|u_1-u_0\|_{B_2^1}\leqslant R.$$

From equation (3.11), we get

$$(u_j-u_0,\phi)_{B_2^1}+h_n(\partial_{t_j}^{1-\alpha}(u_j-u_0),\phi)=(u_{j-1}-u_0,\phi)_{B_2^1}+h_n\Big(f_j+\partial_{t_j}^{1-\alpha}\frac{d^2u_0}{dx^2},\phi\Big)_{B_2^1}. \tag{3.14}$$

Since $t_0\leqslant t_1<t_2<\cdots<t_N\leqslant T$, it is not difficult to see the following inequality

$$\int_0^{t_0}(t_0-s)^{\alpha-1}ds\leqslant\int_0^{t_{j-1}}(t_{j-1}-s)^{\alpha-1}ds\leqslant\int_0^{t_j}(t_j-s)^{\alpha-1}ds\leqslant\frac{T^\alpha}{\alpha}.$$

Now putting $\phi=u_j-u_0$, we obtain

$$\|u_j-u_0\|_{B_2^1}\leqslant\|u_{j-1}-u_0\|_{B_2^1}+h_n\Big\|f_j+\partial_{t_j}^{1-\alpha}\frac{d^2u_0}{dx^2}\Big\|_{B_2^1}. \tag{3.15}$$

Repeating the above inequality, we obtain

$$\|u_j-u_0\|_{B_2^1}\leqslant jh_n\Big\|f_j+\partial_{t_j}^{1-\alpha}\frac{d^2u_0}{dx^2}\Big\|_{B_2^1}. \tag{3.16}$$

Now we use mathematical induction to establish our result. Let us assume that $\|u_i-u_0\|_{B_2^1}\leqslant R$ is true for $i=1,2,\ldots,j-1$, now we need to show that $\|u_j-u_0\|_{B_2^1}\leqslant R$. We have,

$$\|f_j\|_{B_2^1}\leqslant\|f(t_j,u_{j-1},\tilde{u}_{j-1})-f(0,u_0,\Phi)\|_{B_2^1}+\|f(0,u_0,\Phi)\|_{B_2^1}.$$

Using our first assumption (A1), we obtain

$$\|f_j\|_{B_2^1}\leqslant L_f(R,\tilde{R})[T+R+\tilde{R}]+\|f(0,u_0,\Phi)\|_{B_2^1}. \tag{3.17}$$

Using the above estimate i.e. estimate (3.17) in (3.16), we conclude that $\|u_j-u_0\|\leqslant R$. In the further calculations, for notational convenience, we denote by $L_f:=L_f(R,\tilde{R})$.

Further for $j=1$ in (3.11) and $(\partial_{t_j}^{1-\alpha}\Phi(0),\phi)=-(\partial_{t_j}^{1-\alpha}\frac{d^2\Phi(0)}{dx^2},\phi)_{B_2^1}$. We get

$$(\delta u_1,\phi)_{B_2^1}+h_n(\partial_{t_1}^{1-\alpha}\delta u_1,\phi)=\Big(f_1+\partial_{t_1}^{1-\alpha}\frac{d^2\Phi(0)}{dx^2},\phi\Big)_{B_2^1}.$$

Verifying this equality with the function $\phi=\delta u_1=\frac{u_1-\Phi(0)}{h_n}\in V$ and using assumption $(A3)$, we obtain

$$\begin{aligned}\|\delta u_1\|_{B_2^1}^2+h_nM\|\delta u_1\|^2&\leqslant[\|f_1\|_{B_2^1}+\Big\|\partial_{t_1}^{1-\alpha}\frac{d^2\Phi(0)}{dx^2}\Big\|_{B_2^1}]\|\delta u_1\|_{B_2^1}\\&\leqslant[\|f_1\|_{B_2^1}+N\|\Phi(0)\|]\|\delta u_1\|_{B_2^1}\end{aligned} \tag{3.18}$$

Hence

$$\|\delta u_1\|_{B_2^1}\leqslant C+D=C. \tag{3.19}$$

Now let $2 \leqslant j \leqslant n$. Subtracting equation (3.11) for $j-1$ from equation (3.11) for j and putting the function $\phi = \delta u_j$, we obtain

$$(\delta u_j - \delta u_{j-1}, \delta u_j)_{B_2^1} + (\partial_{t_j}^{1-\alpha} u_j - \partial_{t_{j-1}}^{1-\alpha} u_{j-1}, \delta u_j) = (f_j - f_{j-1}, \delta u_j)_{B_2^1},$$

which is equivalent to

$$\|\delta u_j\|_{B_2^1}^2 + \frac{M^*}{h_n}\|u_j - u_{j-1}\|^2 \leqslant (\|f_j - f_{j-1}\|_{B_2^1} + \|\delta u_{j-1}\|_{B_2^1})\|\delta u_j\|_{B_2^1},$$

for some constant M^*, using assumption $(A3)$. Using assumptions, we can verify the following

$$\begin{aligned}
&(\partial_{t_j}^{1-\alpha} u_j - \partial_{t_{j-1}}^{1-\alpha} u_{j-1}, u_j - u_{j-1}) \\
&= (\partial_{t_j}^{1-\alpha} u_j, u_j) - (\partial_{t_j}^{1-\alpha} u_j, u_{j-1}) - (\partial_{t_{j-1}}^{1-\alpha} u_{j-1}, u_j) + (\partial_{t_{j-1}}^{1-\alpha} u_{j-1}, u_{j-1}) \\
&\geqslant M\|u_j\|^2 - \|\partial_{t_j}^{1-\alpha} u_j\|\|u_{j-1}\| - \|\partial_{t_{j-1}}^{1-\alpha} u_{j-1}\|\|u_j\| + M\|u_{j-1}\|^2 \\
&\geqslant M\|u_j\|^2 - N\|u_j\|\|u_{j-1}\| - N\|u_{j-1}\|\|u_j\| + M\|u_{j-1}\|^2 \\
&= M\|u_j\|^2 - 2N\|u_j\|\|u_{j-1}\| + M\|u_{j-1}\|^2 \\
&\geqslant M\|u_j\|^2 - N(\|u_j\|^2 + \|u_{j-1}\|^2) + M\|u_{j-1}\|^2 \\
&\geqslant (M-N)\|u_j\|^2 + \|u_{j-1}\|^2 \\
&\geqslant M^*\|u_j - u_{j-1}\|^2.
\end{aligned} \tag{3.20}$$

Finally we get he following estimate

$$\|\delta u_j\|_{B_2^1} \leqslant \|f_j - f_{j-1}\|_{B_2^1} + \|\delta u_{j-1}\|_{B_2^1}.$$

Using assumption $(A1)$, for $j \geqslant 2$, we get

$$\begin{aligned}
\|f_j - f_{j-1}\|_{B_2^1} &= \|f(t_j, u_{j-1}, \tilde{u}_{j-1}) - f(t_{j-1}, u_{j-2}, \tilde{u}_{j-2})\|_{B_2^1} \\
&\leqslant L_f[|t_j - t_{j-1}| + h_n\|\delta u_{j-1}\|_{B_2^1} + h_n \max_{1\leqslant i\leqslant j-1} \|\delta u_i\|] \\
&\leqslant Ch_n[1 + \|\delta u_{j-1}\|_{B_2^1} + \max_{1\leqslant i\leqslant j-1} \|\delta u_i\|].
\end{aligned}$$

Thus

$$\begin{aligned}
\max_{1\leqslant k\leqslant j} \|\delta u_k\|_{B_2^1} &\leqslant (1 + Ch_n) \max_{1\leqslant k\leqslant j-1} \|\delta u_k\|_{B_2^1} + Ch_n \\
&\leqslant (1 + Ch_n)[1 + \max_{1\leqslant k\leqslant j-1} \|\delta u_k\|_{B_2^1}].
\end{aligned}$$

Using iteration, we get the following

$$\max_{1\leqslant k\leqslant j} \|\delta u_j\|_{B_2^1} \leqslant (1 + Ch_n)^j C \leqslant Ce^{CT} \leqslant C. \tag{3.21}$$

Thus, we obtain

$$\|\delta u_j\|_{B_2^1} \leqslant C.$$

In order to obtain the first estimate, we choose $\phi = u_j$ in (3.11), $j = 1,2,\ldots,n$, to obtain

$$\frac{1}{h_n}\|u_j\|^2_{B^1_2} + M\|u_j\|^2 \leqslant \Big(\|f_j\|_{B^1_2} + \frac{1}{h_n}\|u_{j-1}\|_{B^1_2}\Big)\|u_j\|_{B^1_2},$$

which implies

$$\|u_j\|_{B^1_2} \leqslant h_n\|f_j\|_{B^1_2} + \|u_{j-1}\|_{B^1_2}. \tag{3.22}$$

Again using assumption $(A1)$, we have for all $j \geqslant 1$,

$$\begin{aligned}\|f_j\|_{B^1_2} &\leqslant \|f(t_j,u_{j-1},\tilde{u}_{j-1}) - f(0,\Phi(0),\Phi)\|_{B^1_2} + \|f(0,\Phi(0),\Phi)\|_{B^1_2}\\ &\leqslant L_f(jh_n + \|u_{j-1} - \Phi(0)\|_{B^1_2} + \|\tilde{u}_{j-1} - \Phi\|_{B^1_2}) + \|f(0,\Phi(0),\Phi)\|_{B^1_2}\\ &\leqslant L_f(jh_n + \|u_{j-1}\|_{B^1_2} + \|\Phi(0)\|_{B^1_2} + \|\tilde{u}_{j-1}\|_{B^1_2} + \|\Phi\|_{B^1_2}) + C\\ &\leqslant L_f(jh_n + \|u_{j-1}\|_{B^1_2} + \max_{1\leqslant i\leqslant j-1}\|u_i\|_{B^1_2} + C) + C.\end{aligned} \tag{3.23}$$

Using the above estimate in (3.22), we get

$$\begin{aligned}\max_{1\leqslant i\leqslant j}\|u_k\|_{B^1_2} &\leqslant (1+Ch_n)\max_{1\leqslant k\leqslant j-1}\|u_k\|_{B^1_2} + Ch_n + L_f jh_n^2\\ &\leqslant (1+Ch_n)\max_{1\leqslant k\leqslant j-1}\|u_k\|_{B^1_2} + Ch_n.\end{aligned}$$

Using iteration in the above inequality, we obtain

$$\max_{1\leqslant k\leqslant j}\|u_k\|_{B^1_2} \leqslant (1+Ch_n)^{j-1}[\|u_1\|_{B^1_2} + Ch_n(j-1)]. \tag{3.24}$$

Further replacing $(1+Ch_n)^{j-1} \leqslant e^{CT}$ and $Ch_n(j-1) \leqslant CT$, we obtain

$$\max_{1\leqslant k\leqslant j}\|u_k\|_{B^1_2} \leqslant C. \tag{3.25}$$

Thus we get

$$\|u_k\|_{B^1_2} \leqslant C.$$

Now choosing $\phi = u_j - u_{j-1}$ in (3.11), we have

$$h_n\|\delta u_j\|^2_{B^1_2} + (\partial^{1-\alpha}_{t_j}u_j, u_j - u_{j-1}) = (f_j, u_j - u_{j-1})_{B^1_2},$$

which again implies

$$(\partial^{1-\alpha}_{t_j}u_j, u_j - u_{j-1}) \leqslant (f_j, u_j - u_{j-1})_{B^1_2}.$$

The above equation can be rewritten as

$$(\partial^{1-\alpha}_{t_j}u_j, u_j) - (\partial^{1-\alpha}_{t_j}u_j, u_{j-1}) \leqslant (f_j, u_j - u_{j-1})_{B^1_2},$$

which implies

$$\begin{aligned}(\partial^{1-\alpha}_{t_j}u_j, u_j) - \|\partial^{1-\alpha}_{t_j}u_j\|\|u_{j-1}\| &\leqslant 2h_n\|f_j\|_{B^1_2}\|\delta u_j\|_{B^1_2}\\ &\leqslant Ch_n(1+\|u_{j-1}\|_{B^1_2})\|u_{j-1}\|_{B^1_2}\\ &\leqslant Ch_n.\end{aligned} \tag{3.26}$$

Using assumption $\|\partial_{t_j}^{1-\alpha} u_j\| \leqslant M\|u_j\|$ for some $M,$ we have

$$(\partial_{t_j}^{1-\alpha} u_j, u_j) - M\|u_j\|\|u_{j-1}\| \leqslant Ch_n.$$

Using Archimedean property, we get

$$N^*N\|u_j\|^2 - M\|u_j\|\|u_{j-1}\| \leqslant Ch_n.$$

Hence, we obtain $\|u_j\| \leqslant C.$ □

DEFINITION 3.1. The Rothe's sequence $\{U^n\}$ is defined by (3.4). Furthermore, we define another sequences $\{X^n\}$ of step functions from $[-h_n, T]$ into $\mathbf{H}^2 \cap \mathbf{V}$ given by

$$X^n(t) = \begin{cases} \Phi(0), & t \in [-h_n, 0], \\ u_j, & t \in (t_{j-1}, t_j]. \end{cases}$$

REMARK 3.1. It is easy to note that the function U^n is Lipschitz continuous on $[0, T_0]$ by Theorem 3.1. The sequences $\{U^n\}$ and $\{X^n\}$ are bounded in $C([0,T]; B_2^1(0,1))$ uniformly in $n \in \mathbb{N}$ and $t \in [0,T]$, which implies

$$\|U^n\| \leqslant C, \qquad \|X^n\| \leqslant C, \qquad \left\|\frac{dU^n}{dt}(t)\right\|_{B_2^1} \leqslant C,$$

$$\|U^n(t) - U^n(s)\| \leqslant C|t-s|,$$

$$\|X^n(t) - U^n(t)\|_{B_2^1} \leqslant \frac{C}{n}, \quad \text{and} \quad \|X^n(t) - X^n(t-h_n)\|_{B_2^1} \leqslant \frac{C}{n}.$$

For notational convenience, let

$$f^n(t) = f(t_j^n, u_{j-1}^n, \tilde{u}_{j-1}^n),$$

$t \in (t_{j-1}^n, t_j^n], 1 \leqslant j \leqslant n.$ Using above notation, the inner product (3.8) may be rewritten as

$$\left(\frac{dU^n}{dt}(t), \phi\right)_{B_2^1} + (\partial_t^{1-\alpha} X^n(t), \phi) = (f^n(t), \phi)_{B_2^1}, \tag{3.27}$$

for all $\phi \in \mathbf{V}$ and a.e. $t \in (0,T]$.

LEMMA 3.2. *There exists $u \in C([0,T]; B_2^1(0,1))$ such that $U^n(t) \to u(t)$ uniformaly on I. Furthermore $u(t)$ is Lipschitz continuous on $[0,T]$.*

Proof. From inner product (3.27) for a.e. $t \in (0,T]$, we obtain

$$\left(\frac{d}{dt}(U^n(t) - U^k(t)), U^n(t) - U^k(t)\right)_{B_2^1} + (\partial_t^{1-\alpha} X^n(t) - \partial_t^{1-\alpha} X^k(t), U^n(t) - U^k(t))$$
$$= (f^n(t) - f^k(t), U^n(t) - U^k(t))_{B_2^1}.$$

From the above equality, we obtain the following

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|U^n(t)-U^k(t)\|^2_{B^1_2}+\|\partial_t^{1-\alpha}X^n(t)-\partial_t^{1-\alpha}X^k(t)\|^2\\&=(\partial_t^{1-\alpha}X^n(t)-\partial_t^{1-\alpha}X^k(t),\partial_t^{1-\alpha}X^n(t)-\partial_t^{1-\alpha}X^k(t)-U^n(t)+U^k(t))\\&\quad+(f^n(t)-f^k(t),U^n(t)-U^k(t))_{B^1_2}.\end{aligned}\tag{3.28}$$

Using equation (3.23) and estimate $\|f^n(t)\|_{B^1_2}\leqslant C,$ the following identity

$$(\partial_t^{1-\alpha}X^n(t),\phi)=\Big(f^n(t,u^n,\tilde{u}^n)-\frac{dU^n}{dt},\phi\Big)_{B^1_2}$$

implies

$$|(\partial_t^{1-\alpha}X^n(t),\phi)|\leqslant\Big\|\Big[\|f^n\|_{B^1_2}+\Big\|\frac{dU^n}{dt}\Big\|_{B^1_2}\Big]\Big\|\phi\|_{B^1_2}.$$

Hence, we obtain

$$|(\partial_t^{1-\alpha}X^n(t),\phi)|\leqslant C\|\phi\|_{B^1_2}.\tag{3.29}$$

Now using the above relation (3.29), we obtain

$$\begin{aligned}&(\partial_t^{1-\alpha}X^n(t)-\partial_t^{1-\alpha}X^k(t),\partial_t^{1-\alpha}X^n(t)-\partial_t^{1-\alpha}X^k(t)-U^n(t)+U^k(t))\\&\leqslant 2C(\|\partial_t^{1-\alpha}X^n(t)-U^n(t)\|_{B^1_2}+\|\partial_t^{1-\alpha}X^k(t)-U^k(t)\|_{B^1_2})\leqslant 4C\Big(\frac{1}{n}+\frac{1}{k}\Big).\end{aligned}\tag{3.30}$$

Using the following inequality $\alpha\beta\leqslant\frac{1}{2}\alpha^2+\frac{1}{2}\beta^2,\ \alpha,\beta\in\mathbb{R}>0,$ we may write

$$\begin{aligned}&(f^n(t)-f^k(t),U^n(t)-U^k(t))_{B^1_2}\\&\leqslant\|f^n(t)-f^k(t)\|_{B^1_2}\|U^n(t)-U^k(t)\|_{B^1_2}\\&\leqslant\frac{1}{2}\|f^n(t)-f^k(t)\|^2_{B^1_2}+\frac{1}{2}\|U^n(t)-U^k(t)\|^2_{B^1_2}.\end{aligned}\tag{3.31}$$

Now by assumption (A1), we get

$$\begin{aligned}\|f^n(t)-f^k(t)\|_{B^1_2}&=\|f(t_j,u^n_{j-1},\tilde{u}^n_{j-1})-f(t_l,u^k_{l-1},\tilde{u}^k_{l-1})\|_{B^1_2}\\&\leqslant L_f[|t^n_j-t^k_l|+\|u^n_{j-1}-u^k_{l-1}\|_{B^1_2}+\|\tilde{u}^n_{j-1}-\tilde{u}^k_{l-1}\|_{\mathbf{C}_0}]\\&\leqslant\delta_{nk}(t)+L_f\|U^n-U^k\|_{\mathbf{C}_t},\end{aligned}$$

where

$$\begin{aligned}\delta_{nk}(t)=L_f[&|t^n_j-t^k_l|+\|X^n(t-h_n)-U^n(t)\|_{B^1_2}+\|X^k(t-h_k)-U^k(t)\|_{B^1_2}\\&+\|\tilde{u}^n_{j-1}-U^n_t\|_{\mathbf{C}_0}+\|\tilde{u}^k_{l-1}-U^k_t\|_{\mathbf{C}_0}]\end{aligned}$$

for $t\in(t^n_{j-1},t^n_j]$ and $t\in(t^k_{l-1},t^k_l],\ 1\leqslant j\leqslant n,1\leqslant l\leqslant k.$ It is now easy to see that $\delta_{nk}(t)\to 0$ uniformly on $[0,T]$ as $n,k\to\infty$. Moreover

$$\|f^n(t)-f^k(t)\|^2_{B^1_2}\leqslant\delta^1_{nk}(t)+L_f^2(\|U^n-U^k\|_{\mathbf{C}_t})^2.$$

Thus the equation (3.31) becomes

$$(f^n(t)-f^k(t),U^n(t)-U^k(t))_{B_2^1} \leqslant \delta_{nk}^1+\frac{1}{2}L_f^2(\|U^n-U^k\|_{\mathbf{C}_t})^2+\frac{1}{2}\|U^n(t)-U^k(t)\|_{B_2^1}^2,\ \forall t\in(0,T], \tag{3.32}$$

where δ_{nk}^1 is a sequence of numbers converging to zero as $n,k\to\infty$. Now combining relations (3.30), (3.31) and (3.32), the relation (3.28) becomes

$$\frac{d}{dt}\|U^n(t)-U^k(t)\|_{B_2^1}^2+2\|\partial_t^{1-\alpha}X^n(t)-\partial_t^{1-\alpha}X^k(t)\|^2 \leqslant 2C\Big(\frac{1}{n}+\frac{1}{k}\Big)+\delta_{nk}^1+(1+L_f^2)\|U^n-U^k\|_{\mathbf{C}_t}^2,\ \forall t\in(0,T].$$

Now by integrating the above over $(0,s)$, $0<s\leqslant t\leqslant T$, and using the fact that $U^n=\Phi$ on $[-\tau,0]$ for all n, we obtain

$$\|U^n(s)-U^k(s)\|_{B_2^1}^2\leqslant 2CT\Big(\frac{1}{n}+\frac{1}{k}\Big)+CT\delta_{nk}^1+C\int_0^t\|U^n-U^k\|_{\mathbf{C}_s}^2ds.$$

By taking the supremum over $(0,t)$, we have

$$(\|U^n-U^k\|)_{\mathbf{C}_t}^2\leqslant 2CT\Big(\frac{1}{n}+\frac{1}{k}\Big)+CT\delta_{nk}^1+C\int_0^t\|U^n-U^k\|_{\mathbf{C}_s}^2ds.$$

Where C is a positive constant independent of j,h and n. Using Gronwall's inequality, we conclude that there exists a function $u\in C([-\tau,T];B_2^1(0,1))$ such that $U^n\to u$ in this space. Moreover by the Remark, we can deduce that u is Lipschitz continuous on $[-\tau,T]$. Hence this completes the proof of the lemma. $\square$

By the Remark 3.1 and Lemma 3.2 we have the following remark on the weak convergence (denoted by $\rightharpoonup$) U^n and its strong derivative to the function u and its strong derivative, respectively.

REMARK 3.2. (i) $u\in L^\infty([-\tau,T];\mathbf{V})\cap C^{0,1}([-\tau,T];B_2^1(0,1))$;
(ii) u is strongly differentiable $a.e$ in $[0,T]$ and $\frac{du}{dt}\in L^\infty([0,T];B_2^1(0,1))$;
(iii) $U^n(t)$ and $X^n(t)\rightharpoonup u(t)$ in V for all $t\in[-\tau,T]$;
(iv) $\frac{dU^n(t)}{dt}\rightharpoonup\frac{du}{dt}$ in $L^\infty([0,T];B_2^1(0,1))$.

Now we give proof of our first theorem.

Proof of Theorem 2.1. In this proof, first we show the existence on the interval $[-\tau,T]$. By integrating the relation (3.27) over $(0,t)\subset[0,T]$ and using the fact that $U^n(0)=\Phi(0)$, we obtain

$$(U^n(t)-\Phi(0),\phi)_{B_2^1}+\int_0^t(\partial_t^{1-\alpha}X^n(s),\phi)ds=\int_0^t(f^n,\phi)_{B_2^1}ds. \tag{3.33}$$

We know that $U^n(t) \rightharpoonup u(t)$ in $\mathbf{V}$ for all $t \in [0,T_0]$ and $\forall \phi \in \mathbf{V}$. Also the linear functional $v \to (v,\phi)_{B^1_2}$ is bounded on $\mathbf{V}$. Hence, we obtain

$$(U^n(t),\phi)_{B^1_2} \to (u(t),\phi)_{B^1_2}, \quad \forall t \in [0,T_0]. \tag{3.34}$$

Now using the Lipschitz continuity of f and by Remark 3.1, we obtain the following estimate

$$f^n(s,u^n(s),\tilde{u}^n(s)) \to f(s,u(s),\tilde{u}(s)), \quad \text{in} \quad B^1_2(0,1), \tag{3.35}$$

as $n \to \infty$. One can see that from (3.27) and (3.29) the functions $|(f^n,\phi)_{B^1_2}|$ and $|(X^n,\phi)|$ are uniformly bounded. Hence we can use bounded convergence theorem, which along with (3.27) implies,

$$(u(t)-\Phi(0),\phi)_{B^1_2} + \int_0^t (\partial_t^{1-\alpha} u(s),\phi)ds = \int_0^t (f(s,u(s),\tilde{u}(s)),\phi)_{B^1_2} ds,$$

as $n \to \infty$ and for all $\phi \in \mathbf{V}$ and $t \in [0,T_0]$. Differentiating the above identity we get the desired relation. Now the next step is to prove the uniqueness.The method is standard, let u_1 and u_2 be two such solutions of the model. Now consider the difference $U(t) = u_1(t) - u_2(t)$, which gives

$$\left(\frac{dU(t)}{dt},U(t)\right)_{B^1_2} + (\partial_t^{1-\alpha}U(t),U(t)) = (f(t,u_1(t),\tilde{u}_1(t)) - f(t,u_2(t),\tilde{u}_2(t)),U(t))_{B^1_2},$$

Using assumption $(A3)$, we can rewrite the above equation

$$\left(\frac{dU(t)}{dt},U(t)\right)_{B^1_2} \leqslant (f(t,u_1(t),\tilde{u}_1(t)) - f(t,u_2(t),\tilde{u}_2(t)),U(t))_{B^1_2},$$

Which further implies

$$\begin{aligned} \frac{d}{dt}\|U(t)\|^2_{B^1_2} &\leqslant 2\|f(t,u_1(t),(u_1)_t) - f(t,u_2(t),(u_2)_t)\|\|U(t)\|_{B^1_2} \\ &\leqslant 2L_f[\|u_1(t)-u_2(t)\|_{B^1_2} + \|(u_1)_t)-(u_2)_t)\|_{B^1_2}]\|U(t)\|_{B^1_2} \\ &\leqslant 2L_f[\|U(t)\|_{B^1_2} + (\|U\|_t)_{B^1_2}]\|U(t)\|_{B^1_2} \\ &\leqslant C(\|U\|_t)^2_{B^1_2}, \end{aligned} \tag{3.36}$$

using the boundedness of U. Here $(\|U\|_t(s))^2_{B^1_2} = \|U(t+s)\|^2_{B^1_2}$ for $s \in [-\tau,0]$.

Now integrating the above relation over $(0,s)$ for $0,s \leqslant t \leqslant T$. Further using the fact that $U(0) \equiv 0$ on $[-\tau,0]$, we obtain

$$\|U(s)\|^2_{B^1_2} \leqslant C\int_0^s (\|U\|_t)^2_{B^1_2} dt.$$

Now by taking the supremum over $[0,t]$, we have

$$\sup_{s\in[0,t]} \|U(s)\|^2_{B^1_2} \leqslant C\int_0^t (\|U\|^2_z)_{B^1_2} dz.$$

Hence, we obtain

$$(\|U\|_t^2)_{B_2^1} \leqslant C\int_0^t (\|U\|_s^2)_{B_2^1} ds.$$

Finally using the generalized Gronwall's inequality gives $(\|U\|_t^2)_{B_2^1} \leqslant 0$ for each t. Hence, we obtain $U \equiv 0$ on the interval $[-\tau, T]$.

Further, we establish the unique continuation property of the solution u to either on whole interval $[-\tau, \infty)$ or to the maximal interval $[-\tau, T_{max})$ of existence where $0 < T_{max} < \infty$ and $\lim_{t\to T_{max}-} \|u(t)\| = \infty$. Let us assume that $T_{max} < \infty$ and $\|u(T_{max})\| < \infty$. Now consider the following problem

$$\begin{aligned}
\frac{\partial w}{\partial t} - \partial_t^{1-\alpha}\frac{\partial^2 w}{\partial x^2} &= \tilde{f}(x,t,w(t),w_t), \quad x \in (0,1),\ 0 < t \leqslant T - T_0,\\
w(x,t) &= \tilde{\Phi}(x,t), \quad x \in (0,1),\ t \in [-\tau - T_0, 0],\\
\frac{\partial w}{\partial x}(0,t) &= 0, \quad t \in [0, T - T_0],\\
\int_0^1 w(x,t)dx &= 0, \quad x \in (0,1),\ t \in [0, T - T_0],
\end{aligned} \tag{3.37}$$

where $\tilde{f}(x,t,w(t),w_t) = f(x,t+T_0,w(t),w_t),\ x \in (0,1),\ 0 < t \leqslant T - T_0,$

$$\tilde{\Phi}(t) = \begin{cases} \Phi(t+T_0), & t \in [-\tau - T_0, -T_0],\\ u(t+T_0), & t \in [-T_0, 0], \end{cases}$$

Since $\|\tilde{\Phi}(0)\| = \|u(T_0)\| < \infty$ and $\tilde{f}$ satisfies the assumption $(A1)$ on $[0, T - T_0]$, the existence of a unique $w(t) \in C([-\tau - T_0, T_1]; B_2^1(0,1)),\ 0 < T_1 \leqslant T - T_0,$ is evident from last result. Further we have Lipschitz continuity of w on $[0, T_1]$. The function w satisfies

$$\begin{aligned}
\frac{\partial w}{\partial x} - \partial_t^{1-\alpha}\frac{\partial^2 w}{\partial x^2} &= \tilde{f}(x,t,w(t),w_t), \quad x \in (0,1),\ 0 < t \leqslant T_1,\\
w(x,t) &= \tilde{\Phi}(x,t), \quad x \in (0,1),\ t \in [-\tau - T_0, 0],\\
\frac{\partial w}{\partial x}(0,t) &= 0, \quad t \in [0, T - T_0],\\
\int_0^1 w(x,t)dx &= 0, \quad t \in [0, T - T_0].
\end{aligned} \tag{3.38}$$

The function defined by

$$\overline{u}(t) = \begin{cases} u(t), & t \in [-\tau, T_0],\\ w(t - T_0), & t \in [T_0, T_0 + T_1], \end{cases}$$

is Lipschitz continuous on $[0, T_0 + T_1]$, $\overline{u}(t) \in C([0, T_0 + T_1], B_2^1(0,1))$ for $t \in [0, T_0 + T_1]$ and satisfies our main partial differential equation of fractional order on $[0, T_0 + T_1]$. Proceeding same way we may prove the existence on the whole interval $[-\tau, T]$ or there is the maximal interval $[-\tau, T_{max})$, $0 < T_{max} \leqslant T$, such that u is the weak solution on every subinterval $[-\tau, \tilde{T}], 0 < \tilde{T} < T_{max}$. If the limit $\lim_{t \to t_{max}} \|u(t)\| < \infty$, in this case we may continue the solution beyond T_{max}. This will contradict the definition of maximal interval of existence. Hence the proof of Theorem 2.1 is complete. □

REMARK 3.3. One can also define the sequence

$$\partial_{t_j}^{1-\alpha} u_j^n = \frac{1}{\Gamma(\alpha)} \frac{(Mu)_j^n - (Mu)_{j-1}^n}{h_n} = \frac{1}{\Gamma(\alpha)} \delta(Mu)_j^n,$$

where $(Mu)(t) = \int_0^t (t-s)^{\alpha-1} u(s) ds$. Here for convenience we denote $(Mu)_j^n(t) = (Mu)(t)|_{t=t_j}$. In this case we have a new discritized problem. Presently we are trying to find the estimate with this kind of formulation.

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