Fractional Differential Calculus
Volume 7, Number 2 (2017), 385–393 doi:10.7153/fdc-2017-07-19

ON A STURM–LIOUVILLE TYPE DIFFERENTIAL INCLUSION OF FRACTIONAL ORDER

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(*Communicated by M. Fečkan*)

Abstract. The existence of solutions for a Sturm-Liouville type differential inclusion of fractional order is investigated. New results are obtained by using suitable fixed point theorems when the right hand side has convex or non convex values.

1. Introduction

This note is concerned with the following problem

$$D_C^q y(t) \in F(t, x(t)) \text{ a.e. } ([0,T]), \quad x(0) = x_0, \quad y(0) = y_0, \tag{1.1}$$

where $y(t) \equiv p(t)x'(t)$, $F(.,.) : [0,T] \times \mathbf{R} \to \mathscr{P}(\mathbf{R})$ is a set-valued map, $p(.) : [0,T] \to (0,\infty)$ is a continuous function, $x_0, y_0 \in \mathbf{R}$ and D_C^q denotes Caputo's fractional derivative of order $q \in (0,1)$.

In the theory of ordinary differential equations it is wellknown that any linear real second-order differential equation may be written in the self adjoint form

$$-(r(t)x')' + q(t)x = 0. \tag{1.2}$$

Equation (1.2) together with boundary conditions of the form $a_1x(0) - a_2x'(0) = 0$, $b_1x(T) - b_2x'(T) = 0$ is called the Sturm-Liouville problem. For a complete disscusion on Sturm-Liouville problems we refer, for example, to [13]]. This is the reason why differential inclusions of the form $(r(t)x')' \in F(t,x)$ are usually called Sturm-Liouville type differential inclusions, even if the boundary value problems associated are not as at the original Sturm-Liouville problem.

In the last years one may see a strong development of the theory of differential equations and inclusions of fractional order ([6, 9, 10, 11, 15] etc.). The main reason is that fractional differential equations are very useful tools in order to model many physical phenomena. In the fractional calculus there are several fractional derivatives. From them, the fractional derivative introduced by Caputo in [2], allows to use Cauchy conditions which have physical meanings.

Mathematics subject classification (2010): 34A60.
Keywords and phrases: Differential inclusion, measurable selection, fractional derivative.

The aim of our paper is to consider the extension of the Sturm-Liouville problem to the fractional framework, given by problem (1.1), and to present several existence results for problem (1.1). Our results are essentially based on a nonlinear alternative of Leray-Schauder type, on Bressan-Colombo selection theorem for lower semicontinuous set-valued maps with decomposable values and on Covitz and Nadler set-valued contraction principle. The methods used are known in the theory of differential inclusions, however their exposition in the framework of problem (1.1) is new. We mention also that in [8], namely Theorem 2.4, it is provided a sufficient condition under which any nonoscilatory solution of problem (1.1), with $F(.,.)$ single-valued, is bounded.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main results.

2. Preliminaries

In this section we sum up some basic facts that we are going to use later.

Let (X,d) be a metric space with the corresponding norm $|.|$ and denote $I=[0,T]$. Denote by $\mathscr{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I, by $\mathscr{P}(X)$ the family of all nonempty subsets of X and by $\mathscr{B}(X)$ the family of all Borel subsets of X. If $A\subset I$ then $\chi_A(.):I\to\{0,1\}$ denotes the characteristic function of A. For any subset $A\subset X$ we denote by $\overline{A}$ the closure of A.

Recall that the Pompeiu-Hausdorff distance of the closed subsets $A,B\subset X$ is defined by $d_H(A,B)=\max\{d^*(A,B),d^*(B,A)\}$, where $d^*(A,B)=\sup\{d(a,B);a\in A\}$ and $d(x,B)=\inf_{y\in B}d(x,y)$.

As usual, we denote by $C(I,X)$ the Banach space of all continuous functions $x(.):I\to X$ endowed with the norm $|x(.)|_C=\sup_{t\in I}|x(t)|$, by $AC(I,X)$ the Banach space of all absolutely continuous functions $x(.):I\to X$ and by $L^p(I,X)$ the Banach space of all (Bochner) p-integrable functions $x(.):I\to X$; in particular, $L^1(I,X)$ is the Banach space of all (Bochner) integrable functions $x(.):I\to X$ endowed with the norm $|x(.)|_1=\int_I|x(t)|\mathrm{d}t$.

A subset $D\subset L^1(I,X)$ is said to be *decomposable* if for any $u(\cdot),v(\cdot)\in D$ and any subset $A\in\mathscr{L}(I)$ one has $u\chi_A+v\chi_B\in D$, where $B=I\backslash A$.

Consider $T:X\to\mathscr{P}(X)$ a set-valued map. A point $x\in X$ is called a fixed point for $T(.)$ if $x\in T(x)$. $T(.)$ is said to be bounded on bounded sets if $T(B):=\cup_{x\in B}T(x)$ is a bounded subset of X for all bounded sets B in X. $T(.)$ is said to be compact if $T(B)$ is relatively compact for any bounded sets B in X. $T(.)$ is said to be totally compact if $\overline{T(X)}$ is a compact subset of X. $T(.)$ is said to be upper semicontinuous if for any $x_0\in X$, $T(x_0)$ is a nonempty closed subset of X and if for each open set D of X containing $T(x_0)$ there exists an open neighborhood V_0 of x_0 such that $T(V_0)\subset D$. Let E a Banach space, $Y\subset E$ a nonempty closed subset and $T(.):Y\to\mathscr{P}(E)$ a multifunction with nonempty closed values. $T(.)$ is said to be lower semicontinuous if for any open subset $D\subset E$, the set $\{y\in Y;T(y)\cap D\neq\emptyset\}$ is open. $T(.)$ is called completely continuous if it is upper semicontinuous and totally compact on X.

It is well known that a compact set-valued map $T(.)$ with nonempty compact values is upper semicontinuous if and only if $T(.)$ has a closed graph.

We recall the following nonlinear alternative of Leray-Schauder type proved in [14] and its consequences.

THEOREM 2.1. *Let D and $\overline{D}$ be the open and closed subsets in a normed linear space X such that $0 \in D$ and let $T : \overline{D} \to \mathscr{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*

i) the inclusion $x \in T(x)$ has a solution, or

ii) there exists $x \in \partial D$ (the boundary of D) such that $\lambda x \in T(x)$ for some $\lambda > 1$.

COROLLARY 2.1. *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $T : \overline{B_r(0)} \to \mathscr{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*

i) the inclusion $x \in T(x)$ has a solution, or

ii) there exists $x \in X$ with $|x| = r$ and $\lambda x \in T(x)$ for some $\lambda > 1$.

COROLLARY 2.2. *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $T : \overline{B_r(0)} \to X$ be a completely continuous single valued map with compact convex values. Then either*

i) the equation $x = T(x)$ has a solution, or

ii) there exists $x \in X$ with $|x| = r$ and $x = \lambda T(x)$ for some $\lambda < 1$.

If $F(.,.) : I \times X \to \mathscr{P}(X)$ is a set-valued map with compact values we define $S_F : C(I,X) \to \mathscr{P}(L^1(I,X))$ by $S_F(x) := \{f \in L^1(I,X);\ f(t) \in F(t,x(t))$ a.e. $(I)\}$. We say that $F(.,.)$ is of *lower semicontinuous type* if $S_F(.)$ is lower semicontinuous with nonempty closed and decomposable values. The next result is proved in [1].

THEOREM 2.2. *Let S be a separable metric space and $G(.) : S \to \mathscr{P}(L^1(I,X))$ be a lower semicontinuous set-valued map with closed decomposable values.*

Then $G(.)$ has a continuous selection (i.e., there exists a continuous mapping $g(.) : S \to L^1(I,X)$ such that $g(s) \in G(s)\ \ \forall s \in S$).

A set-valued map $G : I \to \mathscr{P}(X)$ with nonempty compact convex values is said to be *measurable* if for any $x \in X$ the function $t \to d(x, G(t))$ is measurable.

A set-valued map $F(.,.) : I \times X \to \mathscr{P}(X)$ is said to be *Carathéodory* if $t \to F(t,x)$ is measurable for any $x \in X$ and $x \to F(t,x)$ is upper semicontinuous for almost all $t \in I$. Moreover, $F(.,.)$ is said to be L^1*-Carathéodory* if for any $l > 0$ there exists $h_l(.) \in L^1(I,\mathbf{R})$ such that $\sup\{|v|; v \in F(t,x)\} \leqslant h_l(t)$ a.e. (I), $\forall x \in \overline{B_l(0)}$. The following theorem is proved in [12].

THEOREM 2.3. *Let X be a Banach space, let $F(.,.) : I \times X \to \mathscr{P}(X)$ be a L^1-Carathéodory set-valued map with $S_F(x) \neq \emptyset$ for all $x(.) \in C(I,X)$ and let $\Gamma : L^1(I,X) \to C(I,X)$ be a linear continuous mapping.*

Then the set-valued map $\Gamma \circ S_F : C(I,X) \to \mathscr{P}(C(I,X))$ defined by

$$(\Gamma \circ S_F)(x) = \Gamma(S_F(x))$$

has compact convex values and has a closed graph in $C(I,X) \times C(I,X)$.

Note that if $dimX < \infty$, and $F(.,.)$ is as in Theorem 2.3, then $S_F(x) \neq \emptyset$ for any $x(.) \in C(I,X)$ (e.g., [12]). For the next definitions we refer, for example, to [11].

DEFINITION 2.1. a) The fractional integral of order $p > 0$ of a Lebesgue integrable function $f : (0,\infty) \to \mathbf{R}$ is defined by

$$I^p f(t) = \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} f(s)ds,$$

provided the right-hand side is pointwise defined on $(0,\infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}dt$.

b) The Riemann-Liouville fractional derivative of order $p > 0$ of a continuous function $f : (0,\infty) \to \mathbf{R}$ is defined by

$$D^p f(t) = \frac{1}{\Gamma(n-p)} \frac{d^n}{dt^n} \int_0^t (t-s)^{-p+n-1} f(s)ds,$$

where $n = [p]+1$, provided the right-hand side is pointwise defined on $(0,\infty)$.

c) Caputo's fractional derivative of order $p \in (0,1)$ of a function $f : [0,\infty) \to \mathbf{R}$ is defined by

$$D_C^p f(t) = D^p[f(t) - f(0)].$$

In what follows $q \in (0,1)$ and $\gamma \in (0,q)$.

A mapping $x(.) \in AC(I,\mathbf{R})$ is called a *solution* of problem (1.1) if there exists a function $f(.) \in L^{\frac{1}{\gamma}}(I,\mathbf{R})$ such that

$$f(t) \in F(t,x(t)) \ \text{ a.e. } (I), \tag{2.1}$$

$$x(t) = x_0 + y_0 \int_0^t \frac{1}{p(s)} ds + \int_0^t \frac{1}{p(s)} \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u)du \right) ds \quad \forall t \in I. \tag{2.2}$$

This definition of the solution is justified by the fact that (see Lemmas 2.8 and 2.9 in [16]) if $f(.) \in L^{\frac{1}{\gamma}}(I,\mathbf{R})$ and $y(.) : I \to \mathbf{R}$ is such that

$$y(t) = y_0 + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds \tag{2.3}$$

then $D_C^q y(t) = f(t)$ a. e. (I) and $y(0) = y_0$. Since $p(t)x'(t) \equiv y(t)$, integrating by parts in (2.3) we obtain (2.2).

We note that $x(.)$ in (2.2) may be written as

$$x(t) = x_0 + y_0 \int_0^t \frac{1}{p(s)} ds + \frac{1}{\Gamma(q)} \int_0^t \left(\int_u^t \frac{(s-u)^{q-1}}{p(s)} ds \right) f(u)du.$$

Since $p(.) : [0,T] \to (0,\infty)$ is continuous, we denote $M := \sup_{t\in I} \frac{1}{p(t)}$. We put also $a(t) = x_0 + y_0 \int_0^t \frac{1}{p(s)} ds$, $m_1 = |x_0| + |y_0|MT$, $K(t,u) = \frac{1}{\Gamma(q)} \int_u^t \frac{(s-u)^{q-1}}{p(s)} ds$, $M_1 = \frac{T^q}{Mq\Gamma(q)}$. Note, that $|K(t,u)| \leqslant \frac{1}{M\Gamma(q)} \int_u^t (s-u)^{q-1} ds \leqslant \frac{1}{Mq\Gamma(q)} (t-u)^q \leqslant M_1 \ \ \forall t,u \in I$.

3. The main results

We are able now to present the existence results for problem (1.1). We consider first the case when $F(.,.)$ is convex valued.

HYPOTHESIS 1. i) $F(.,.) : I \times \mathbf{R} \to \mathscr{P}(\mathbf{R})$ has nonempty compact convex values and is Carathéodory.

ii) There exist $\varphi(.) \in L^{\frac{1}{\gamma}}(I,\mathbf{R})$ with $\varphi(t) > 0$ a.e. (I) and there exists a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$\sup\{|v|; \quad v \in F(t,x)\} \leqslant \varphi(t)\psi(|x|) \text{ a.e. } (I), \quad \forall x \in \mathbf{R}.$$

THEOREM 3.1. *Assume that Hypothesis 1 is satisfied and there exists $r > 0$ such that*

$$r > m_1 + M_1|\varphi|_1\psi(r). \tag{3.1}$$

Then problem (1.1) has at least one solution $x(.)$ such that $|x(.)|_C < r$.

Proof. Let $X = C(I,\mathbf{R})$ and consider $r > 0$ as in (3.1). It is obvious that the existence of solutions to problem (1.1) reduces to the existence of the solutions of the integral inclusion

$$x(t) \in a(t) + \int_0^t K(t,s)F(s,x(s))ds, \quad t \in I. \tag{3.2}$$

Consider the set-valued map $T : \overline{B_r(0)} \to \mathscr{P}(C(I,\mathbf{R}))$ defined by

$$T(x) := \{v(.) \in C(I,\mathbf{R});\ v(t) = a(t) + \int_0^t K(t,s)f(s)ds,\ f \in S_F(x)\}. \tag{3.3}$$

We show that $T(.)$ satisfies the hypotheses of Corollary 2.1. First, we show that $T(x) \subset C(I,\mathbf{R})$ is convex for any $x \in C(I,\mathbf{R})$. If $v_1, v_2 \in T(x)$ then there exist $f_1, f_2 \in S_F(x)$ such that for any $t \in I$ one has $v_i(t) = a(t) + \int_0^t K(t,s)f_i(s)ds$, $i = 1,2$.

Let $0 \leqslant \alpha \leqslant 1$. Then for any $t \in I$ we have $(\alpha v_1 + (1-\alpha)v_2)(t) = a(t) + \int_0^t K(t,s)[\alpha f_1(s) + (1-\alpha)f_2(s)]ds$. The values of $F(.,.)$ are convex, thus $S_F(x)$ is a convex set and hence $\alpha f_1 + (1-\alpha)f_2 \in T(x)$.

Secondly, we show that $T(.)$ is bounded on bounded sets of $C(I,\mathbf{R})$. Let $B \subset C(I,\mathbf{R})$ be a bounded set. Then there exist $m > 0$ such that $|x|_C \leqslant m \ \forall x \in B$. If $v \in T(x)$ there exists $f \in S_F(x)$ such that $v(t) = a(t) + \int_0^t K(t,s)f(s)ds$. One may write for any $t \in I$

$$|v(t)| \leqslant m_1 + \int_0^t |K(t,s)|.|f(s)|ds \leqslant m_1 + \int_0^t |K(t,s)|\varphi(s)\psi(|x(t)|)ds$$

and therefore $|v|_C \leqslant m_1 + M_1|\varphi|_1\psi(m) \ \forall v \in T(x)$, i.e., $T(B)$ is bounded.

We show next that $T(.)$ maps bounded sets into equi-continuous sets. Let $B \subset C(I,\mathbf{R})$ be a bounded set as before and $v \in T(x)$ for some $x \in B$. There exists $f \in S_F(x)$ such that $v(t) = a(t) + \int_0^t K(t,s)f(s)ds$. Then for any $t,\tau \in I$ we have

$$|v(t)-v(\tau)| \leqslant |a(t)-a(\tau)|+|\int_0^t K(t,s)f(s)ds-\int_0^t K(\tau,s)f(s)ds|+|\int_\tau^t K(\tau,s)f(s)ds|$$
$$\leqslant |a(t)-a(\tau)|+M_1\int_\tau^t \varphi(s)\psi(m)ds+\int_0^t |K(t,s)-K(\tau,s)|\varphi(s)\psi(m)ds.$$

It follows that $|v(t)-v(\tau)| \to 0$ as $\tau \to t$. Therefore, $T(B)$ is an equi-continuous set in $C(I,\mathbf{R})$. We apply now Arzela-Ascoli's theorem we deduce that $T(.)$ is completely continuous on $C(I,\mathbf{R})$.

In the next step of the proof we prove that $T(.)$ has a closed graph. Let $x_n \in C(I,\mathbf{R})$ be a sequence such that $x_n \to x^*$ and $v_n \in T(x_n)$ $\forall n \in \mathbf{N}$ such that $v_n \to v^*$. We prove that $v^* \in T(x^*)$. Since $v_n \in T(x_n)$, there exists $f_n \in S_F(x_n)$ such that $v_n(t) = a(t) + \int_0^t K(t,s)f_n(s)ds$. Define $\Gamma : L^1(I,\mathbf{R}) \to C(I,\mathbf{R})$ by $(\Gamma(f))(t) := \int_0^t K(t,s)f(s)ds$. One has $\max_{t\in I}|v_n(t)-a(t)-(v^*(t)-a(t))| = |v_n(.)-v^*(.)|_C \to 0$ as $n \to \infty$. We apply Theorem 2.3 to find that $\Gamma \circ S_F$ has closed graph and from the definition of Γ we get $v_n \in \Gamma \circ S_F(x_n)$. Since $x_n \to x^*$, $v_n \to v^*$ it follows the existence of $f^* \in S_F(x^*)$ such that $v^*(t) = a(t) + \int_0^t K(t,s)f^*(s)ds$.

Therefore, $T(.)$ is upper semicontinuous and compact on $\overline{B_r(0)}$. We apply Corollary 2.1 to deduce that either i) the inclusion $x \in T(x)$ has a solution in $\overline{B_r(0)}$, or ii) there exists $x \in X$ with $|x|_C = r$ and $\lambda x \in T(x)$ for some $\lambda > 1$.

Assume that ii) is true. With the same arguments as in the second step of our proof we get $r = |x(.)|_C \leqslant m_1 + M_1|\varphi|_1\psi(r)$ which contradicts (3.1). Hence only i) is valid and theorem is proved. □

We consider now the case when $F(.,.)$ is not necessarily convex valued. Our first existence result in this case is based on the Leray-Schauder alternative for single valued maps and on Bressan Colombo selection theorem.

HYPOTHESIS 2. i) $F(.,.) : I \times \mathbf{R} \to \mathscr{P}(\mathbf{R})$ has compact values, $F(.,.)$ is $\mathscr{L}(I)\otimes\mathscr{B}(\mathbf{R})$ measurable and $x \to F(t,x)$ is lower semicontinuous for almost all $t \in I$.

ii) There exist $\varphi(.) \in L^{\frac{1}{\gamma}}(I,\mathbf{R})$ with $\varphi(t) > 0$ a.e. (I) and there exists a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$\sup\{|v|;\quad v \in F(t,x)\} \leqslant \varphi(t)\psi(|x|) \text{ a.e. } (I), \quad \forall x \in \mathbf{R}.$$

THEOREM 3.2. *Assume that Hypothesis 2 is satisfied and there exists $r > 0$ such that condition (3.1) is satisfied.*

Then problem (1.1) has at least one solution on I.

Proof. We note first that if Hypothesis 2 is satisfied then $F(.,.)$ is of lower semicontinuous type (e.g., [7]). Therefore, we apply Theorem 2.2 with $S = C(I,\mathbf{R})$ and $G(.) = S_F(.)$ to deduce that there exists a continuous mapping $f(.) : C(I,\mathbf{R}) \to L^1(I,\mathbf{R})$ such that $f(x) \in S_F(x)$ $\forall x \in C(I,\mathbf{R})$.

We consider the corresponding problem

$$x(t)=a(t)+\int_0^t K(t,s)f(x(s))ds, \quad t\in I \tag{3.4}$$

in the space $X=C(I,\mathbf{R})$. It is clear that if $x(.)\in C(I,\mathbf{R})$ is a solution of the problem (3.4) then $x(.)$ is a solution to problem (1.1). Let $r>0$ that satisfies condition (3.1) and define the map $T:\overline{B_r(0)}\to C(I,\mathbf{R})$ by $(T(x))(t):=a(t)+\int_0^t K(t,s)f(x(s))ds$. Obviously, the integral equation (3.4) is equivalent with the operator equation

$$x(t)=(T(x))(t), \quad t\in I. \tag{3.5}$$

It remains to show that $T(.)$ satisfies the hypotheses of Corollary 2.2. We show that $T(.)$ is continuous on $\overline{B_r(0)}$. From Hypotheses 2 ii) we have $|f(x(t))|\leqslant\varphi(t)\psi(|x(t)|)$ a.e. (I) for all $x(.)\in C(I,\mathbf{R})$. Let $x_n,x\in\overline{B_r(0)}$ such that $x_n\to x$. Then $|f(x_n(t))|\leqslant\varphi(t)\psi(r)$ a.e. (I). From Lebesgue's dominated convergence theorem and the continuity of $f(.)$ we obtain, for all $t\in I$, $\lim_{n\to\infty}\int_0^t K(t,s)f(x_n(s))ds=\int_0^t K(t,s)f(x(s))ds$ i.e., $T(.)$ is continuous on $\overline{B_r(0)}$.

Repeating the arguments in the proof of Theorem 3.1 with corresponding modifications it follows that $T(.)$ is compact on $\overline{B_r(0)}$. We apply Corollary 2.2 and we find that either i) the equation $x=T(x)$ has a solution in $\overline{B_r(0)}$, or ii) there exists $x\in X$ with $|x|_C=r$ and $x=\lambda T(x)$ for some $\lambda<1$.

As in the proof of Theorem 3.1 if the statement ii) holds true, then we obtain a contradiction to (3.1). Thus only the statement i) is true and problem (1.1) has a solution $x(.)\in C(I,\mathbf{R})$ with $|x(.)|_C<r$. □

In order to obtain an existence result for problem (1.1) by using the set-valued contraction principle we introduce the following hypothesis on F.

HYPOTHESIS 3. i) $F:I\times\mathbf{R}\to\mathscr{P}(\mathbf{R})$ has nonempty compact values is is integrably bounded and for every $x\in\mathbf{R}$, $F(.,x)$ is measurable.

ii) There exists $L\in L^{\frac{1}{\gamma}}(I,\mathbf{R}_+)$ such that for almost all $t\in I$,

$$d_H(F(t,x_1),F(t,x_2))\leqslant L(t)|x_1-x_2| \quad \forall\ x_1,x_2\in\mathbf{R}.$$

iii) There exists $l\in L^1(I,\mathbf{R}_+)$ such that for almost all $t\in I$, $d(0,F(t,0))\leqslant l(t)$.

THEOREM 3.3. *Assume that Hypothesis 3. is satisfied and $M_1|L|_1<1$. Then problem (1.1) has a solution.*

Proof. We transform problem (1.1) into a fixed point problem. Consider the set-valued map $T:C(I,\mathbf{R})\to\mathscr{P}(C(I,\mathbf{R}))$ defined by $T(x):=\{v(.)\in C(I,\mathbf{R});\ v(t)=a(t)+\int_0^t K(t,s)f(s)ds,\ f\in S_F(x)\}$.

Since the set-valued map $t\to F(t,x(t))$ is measurable with the measurable selection theorem it admits a measurable selection $f:I\to\mathbf{R}$. Moreover, since F is integrably bounded, $f\in L^1(I,\mathbf{R})$. Therefore, $S_F(x)\neq\emptyset$.

It is clear that the fixed points of T are solutions of problem (1.1). We shall prove that T fulfills the assumptions of Covitz Nadler contraction principle.

First, we note that since $S_F(x) \neq \emptyset$, $T(x) \neq \emptyset$ for any $x \in C(I,\mathbf{R})$.

Secondly, we prove that $T(x)$ is closed for any $x \in C(I,\mathbf{R})$. Let $\{x_n\}_{n\geqslant 0} \in T(x)$ such that $x_n \to x^*$ in $C(I,\mathbf{R})$. Then $x^* \in C(I,\mathbf{R})$ and there exists $f_n \in S_F(x_n)$ such that $x_n(t) = a(t) + \int_0^t K(t,s)f_n(s)ds$, $t \in I$. Since F has compact values and Hypothesis 3 is satisfied we may pass to a subsequence (if necessary) to get that f_n converges to $f \in L^1(I,\mathbf{R})$ in $L^1(I,\mathbf{R})$. In particular, $f \in S_F(x)$ and for any $t \in I$ we have $x_n(t) \to x^*(t) = a(t) + \int_0^t K(t,s)f(s)ds$, i.e., $x^* \in T(x)$ and $T(x)$ is closed.

Finally, we show that T is a contraction on $C(I,\mathbf{R})$. Let $x_1, x_2 \in C(I,\mathbf{R})$ and $v_1 \in T(x_1)$. Then there exist $f_1 \in S_F(x_1)$ such that $v_1(t) = a(t) + \int_0^t K(t,s)f_1(s)ds$, $t \in I$. Consider the set-valued map

$$H(t) := F(t,x_2(t)) \cap \{x \in \mathbf{R};\ |f_1(t) - x| \leqslant L(t)|x_1(t) - x_2(t)|\}, \quad t \in I.$$

From Hypothesis 3 one has

$$d_H(F(t,x_1(t)),F(t,x_2(t))) \leqslant L(t)|x_1(t) - x_2(t)|, \quad t \in I.$$

hence H has nonempty closed values. Moreover, since H is measurable, there exists f_2 a measurable selection of H. It follows that $f_2 \in S_F(x_2)$ and for any $t \in I$, $|f_1(t) - f_2(t)| \leqslant L(t)|x_1(t) - x_2(t)|$. Define

$$v_2(t) = a(t) + \int_0^t K(t,s)f_2(s)ds, \quad t \in I$$

We have

$$\begin{aligned} |v_1(t) - v_2(t)| &\leqslant \int_0^t |K(t,s)||f_1(s) - f_2(s)|ds \\ &\leqslant M_1 \int_0^t L(s)|x_1(s) - x_2(s)|ds \\ &\leqslant M_1|L|_1|x_1 - x_2|_C. \end{aligned}$$

So, $|v_1 - v_2|_C \leqslant M_1|L|_1|x_1 - x_2|_C$. From an analogous reasoning by interchanging the roles of x_1 and x_2 it follows

$$d_H(T(x_1),T(x_2)) \leqslant M_1|L|_1|x_1 - x_2|_C.$$

Therefore, T admits a fixed point which is a solution to problem (1.1). □

Acknowledgement. The author wishes to thank an anonymous referee for his helpful comments which improved the paper.

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(Received January 18, 2017)

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