Fractional Differential Calculus
Volume 8, Number 1 (2018), 1–31
doi:10.7153/fdc-2018-08-01

# SHIFTED CHEBYSHEV SPECTRAL–COLLOCATION METHOD FOR SOLVING OPTIMAL CONTROL OF FRACTIONAL MULTI–STRAIN TUBERCULOSIS MODEL

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(*Communicated by M. Kirane*)

*Abstract.* In this paper, optimal control for a novel fractional multi-strain Tuberculosis model is presented. The proposed model is governed by a system of fractional differential equations, where the fractional derivative is defined in the Caputo sense. Modified parameters are introduced to account for the fractional order. Four controls variables are proposed to minimize the cost of interventions. Necessary and sufficient conditions that guarantee the existence and the uniqueness of the solution of the resulting systems are given. The optimality system is approximated by shifted Chebyshev polynomials which transformed the system of differential equations to a nonlinear system of algebraic equations with unknown coefficients. The convergence analysis and an upper bound of the error of the derived formula are given. Newton's iteration method is used to solve this system of nonlinear algebraic equations. The value of the objective functional which is obtained by the proposed method are compared with those obtained by the generalized Euler method. It is found that, Shifted Chebyshev spectral-collocation method is better than the generalized Euler method.

## 1. Introduction

It is well known that, the mathematical models are a quite important and efficient tool to describe and investigate several problems in natural sciences disciplines such as biology, physics, weather science and many other fields ([3], [12], [15], [16], [19], [43], [46], [44], [45]). Numerical simulations are sometimes the only way to solve these mathematical models or to derive the desired information out of it. The accuracy of these numerical solutions is a major factor to consider while deciding on which numerical method is to be used in solving a mathematical model.

Recently, the theory of fractional optimal control problems has been under development. Necessary optimality conditions have been developed for (FOCPs). For instance, in ([4]–[7]) necessary conditions of optimization for fractional optimal control problems FOCPs in the sense of Riemann-Liouville derivative have been achieved and solved the problem numerically using finite difference methods. In [28], the authors presented a numerical method for solving FOCPs in the Caputo sense is based on Chebyshev polynomials approximation and finite difference method. In [9] Baleanu

*Mathematics subject classification* (2010): 37N25, 49J15, 26A33, 74S25.
*Keywords and phrases*: Tuberculosis model, emerging multi-drug resistant, extensively drug-resistant, fractional order model, Caputo definition, optimal control problem, shifted Chebyshev spectral-collocation method.


Paper FDC-08-01

et al., used central difference scheme for solving FOCPs. In [10] Biswas and Sen presented a numerical technique for the solution of fractional optimal control problems defined both in terms of Riemann-Liouville and Caputo fractional derivatives. Agrawal et al. in [8] formulated FOCPs in multi-dimensions of the state and control variables. They used Riemann-Liouville fractional derivative with Grünwald-Letinkov approximation to get their numerical scheme. In [47], Tricaud and Chen discussed FOCPs and their solutions by means of rational approximation. Lotfi et al. in [29], considered FOCPs in terms of Caputo operators and solve it using the Legendre orthonormal polynomial basis. Jafari and Trajadodi in [30] have studied FOCPs using Bernstein polynomials.

The past three decades have seen rapid development in the fractional calculus (fractional differential equations) field ([21], [27], [37]). The applications of fractional calculus are becoming increasingly important in science and biology, ([43]–[44]), control theory of dynamical systems [42], magnetic plasma [22], physics [40] and the process can be successfully modeled by fractional differential equations (FDEs) ([38], [39]). For certain applications the use of fractional derivatives is justified since they provide a better model than integer order derivative models do since they provide a powerful instrument for incorporation of memory and hereditary properties of the systems as opposed to the integer order models, where such effects are neglected or difficult to incorporate. The memory effect is due to the fact that fractional derivatives are non-local as opposed to the local behavior of integer derivatives ([1], [23]–[25]).

Spectral methods have developed rapidly over the past four decades by a huge number of studies see for details, ([13], [17], [31], [32]). The principal advantage of spectral methods lies in their ability to achieve accurate results with substantially fewer degrees of freedom. In recent years, Chebyshev polynomials [11] which are families of orthogonal polynomials on the interval $[a,b]$ have become increasingly important in numerical analysis, from both theoretical and practical points of view. We refer here to the excellent book [31], for the reader who is interested in Chebyshev polynomials of all kinds.

Tuberculosis (TB) can be considered as one of the most important infectious diseases, it is the second largest cause of mortality by infectious diseases and is a challenging disease to control [26]. It is caused by various strains of Mycobacteria. Specifically, Mycobacterium tuberculosis. TB primarily affects the lungs, but it can also affect organs in the central nervous system, lymphatic system, and circulatory system among others.

Several papers considered modeling TB such as ([3], [14], [15], [34], [36], [48]). We consider in this work a multi-strain TB model of fractional order derivatives as extension the model of TB which given in [2]. This model includes several factors of spreading TB such as the fast infection, the exogenous reinfection and secondary infection along with the resistance factor [42]. The model incorporates three strains, drug sensitive emerging, multi-drug resistant $(MDR)$ and extensively drug-resistant $(XDR)$. Sweilam and AL-Mekhlafi introduced some numerical studies of this model in ([42]–[45]).

The aim of this paper is to study optimal control of fractional multi-strain TB model with modified parameters, this modified parameters are introduced to account for the fractional order [1]. Four controls represent the effort that prevents the failure of treatment in active TB infectious individuals. Shifted Chebyshev spectral method, is used to solve such optimality system. The state and control variables are expanded in shifted Chebyshev series with unknown coefficients, the resultant system of algebraic equations is solved using Newton iteration method. Two numerical methods are used to study the optimal control problem (OCP). The methods are the shifted Chebyshev spectral method (SCSM) and the generalized Euler method (GEM). Comparative studies are implemented.

This paper is organized as follows: In Section 2, a multi-strain TB model with control is given. In Section 3, formulation of the optimal control problem and the necessary optimality conditions for the multi-strain TB model are derived. In Section 4, Numerical methods for solving the optimal control problem are given, also, we derive an approximate formula for derivatives using Chebyshev series expansion, In section 5, we study the error analysis of the introduced approximate formula, moreover the numerical implementation of the proposed technique is given in section 6. The conclusions are given in section 7.

## 2. Multi-strain TB model with controls

In the following, the fraction multi-strain TB model is presented. The population of interest is divided into eight compartments depending on their epidemiological stages as follows: susceptible $(S)$; latently infected with drug sensitive TB $(L_s)$; latently infected with MDR TB $(L_m)$; latently infected with XDR TB $(L_x)$; sensitive drug TB infectious $(I_s)$; MDR TB infectious $(I_m)$; XDR TB infectious $(I_x)$; recovered $R$. All interpretation and meaning of this variables see Table 1. One of the main assumptions of this model is that, the total population, $N$; with $N = S(t)+L_s(t)+L_m(t)+L_x(t)+I_s(t)+I_m(t)+I_x(t)+R(t)$, is constant in time. In other words, we assume $b^\alpha = (dN)^\alpha$, where, $b^\alpha$ is birth rate and $d^\alpha$ is natural death rate also, we assume there is no disease-induced death rate, i.e., $\delta_s^\alpha = \delta_m^\alpha = \delta_x^\alpha$=0. Four control functions $u_1(\cdot),\ u_2(\cdot),\ u_3(\cdot)$ and $u_4(\cdot)$, and four real positive constants $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3$, and $\varepsilon_4$, will be added to the model. The control $u_1$ governs the latent individuals $L_s$ that is put under treatment. The controls $u_2,\ u_3$ and $u_4$ represents the effort in preventing the failure of treatment in active TB infectious individuals $I_s$, $I_m$, $I_x$, respectively, e.g., supervising the patients, helping them to take the TB medications regularly and to complete the TB treatment. The parameters $\varepsilon_i \in (0,1),\ i=1,2,3,4,$ measure the effectiveness of the controls $u_k,\ k=1,2,3,4,$ respectively, i.e., these parameters measure the efficacy of treatment interventions for active and persistent latent TB individuals. The new parameters of the model are described in Table 2, this modified parameters are introduced to account for the fractional order [1]. The new system is described by fractional order derivatives as follows:

$$D_t^\alpha S = b^\alpha - d^\alpha S - \beta_s^\alpha \frac{SI_s}{N} - \beta_m^\alpha \frac{SI_m}{N} - \beta_x^\alpha \frac{SI_x}{N}, \tag{1}$$

$$\begin{aligned} D_t^\alpha L_s = & \lambda_s^\alpha \beta_s^\alpha \frac{SI_s}{N} + \sigma_s^\alpha \lambda_s^\alpha \beta_s^\alpha \frac{RI_s}{N} + \gamma_s^\alpha I_s - \alpha_{ss}^\alpha \beta_s^\alpha \frac{L_s I_s}{N} - \alpha_{sm}^\alpha \beta_m^\alpha \frac{L_s I_m}{N} - \alpha_{sx}^\alpha \beta_x^\alpha \frac{L_s I_x}{N} \\ & - (d^\alpha + \varepsilon_s^\alpha + t_{1s}^\alpha + \varepsilon_1 u_1(t)) L_s, \end{aligned} \tag{2}$$

$$\begin{aligned} D_t^{\alpha} L_m =& \lambda_m^{\alpha} \beta_m^{\alpha} \frac{SI_m}{N} + \sigma_m^{\alpha} \lambda_m^{\alpha} \beta_m^{\alpha} \frac{RI_m}{N} + \gamma_m^{\alpha} I_m + \alpha_{sm}^{\alpha} \beta_m^{\alpha} \lambda_m^{\alpha} \frac{L_s I_m}{N} + (1 - P_1^{\alpha}) t_{1s}^{\alpha} L_s \\ & + \varepsilon_1 u_1(t) L_s + (1 - P_2^{\alpha}) t_{2s}^{\alpha} I_s + \varepsilon_2 u_2(t)) I_s - \alpha_{mm}^{\alpha} \beta_m^{\alpha} \frac{L_m I_m}{N} - \alpha_{mx}^{\alpha} \beta_x^{\alpha} \frac{L_m I_x}{N} \\ & - (d^{\alpha} + \varepsilon_m^{\alpha}) L_m, \end{aligned} \tag{3}$$

$$\begin{aligned} D_t^{\alpha} L_x =& \lambda_x^{\alpha} \beta_x^{\alpha} \frac{SI_x}{N} + \sigma_x^{\alpha} \lambda_x^{\alpha} \beta_x^{\alpha} \frac{RI_x}{N} + \gamma_x^{\alpha} I_x + \alpha_{sx}^{\alpha} \beta_x^{\alpha} \lambda_x^{\alpha} \frac{L_s I_x}{N} + \alpha_{mx}^{\alpha} \beta_x^{\alpha} \lambda_x^{\alpha} \frac{L_m I_x}{N} \\ & + (1 - P_3^{\alpha}) t_{2m}^{\alpha} I_m + \varepsilon_2 u_2(t)) I_s - \alpha_{xx}^{\alpha} \beta_x^{\alpha} \frac{L_x I_x}{N} - (d^{\alpha} + \varepsilon_x^{\alpha}) L_x, \end{aligned} \tag{4}$$

$$\begin{aligned} D_t^{\alpha} I_s =& \alpha_{ss}^{\alpha} \beta_s^{\alpha} \frac{L_s I_s}{N} + (1 - \lambda_s^{\alpha}) \beta_s^{\alpha} \left( \frac{SI_s}{N} + \sigma_s^{\alpha} \frac{RI_s}{N} \right) + \varepsilon_s^{\alpha} L_s \\ & - (d^{\alpha} + \delta_s^{\alpha} + t_{2s}^{\alpha} + \gamma_s^{\alpha} + \varepsilon_2 u_2(t)) I_s, \end{aligned} \tag{5}$$

$$\begin{aligned} D_t^{\alpha} I_m =& \alpha_{mm}^{\alpha} \beta_m^{\alpha} \frac{L_m I_m}{N} + (1 - \lambda_m^{\alpha}) \beta_m^{\alpha} \left( \frac{SI_m}{N} + \sigma_m^{\alpha} \frac{RI_m}{N} + \alpha_{sm}^{\alpha} \frac{L_s I_m}{N} \right) + \varepsilon_m^{\alpha} L_m \\ & - (d^{\alpha} + \delta_m^{\alpha} + t_{2m}^{\alpha} + \varepsilon_3 u_3(t) + \gamma_m^{\alpha}) I_m, \end{aligned} \tag{6}$$

$$\begin{aligned} D_t^{\alpha} I_x =& \alpha_{xx}^{\alpha} \beta_x^{\alpha} \frac{L_x I_x}{N} + (1 - \lambda_x^{\alpha}) \beta_x^{\alpha} \left( \frac{SI_x}{N} + \sigma_x^{\alpha} \frac{RI_x}{N} + \alpha_{sx}^{\alpha} \frac{L_s I_x}{N} + \alpha_{mx}^{\alpha} \frac{L_m I_x}{N} \right) + \varepsilon_x^{\alpha} L_x \\ & - (d^{\alpha} + \delta_x^{\alpha} + t_{2x}^{\alpha} + \gamma_x^{\alpha} + \varepsilon_4 u_4(t)) I_x, \end{aligned} \tag{7}$$

$$\begin{aligned} D_t^{\alpha} R =& P_1^{\alpha} t_{1s}^{\alpha} L_s + P_2^{\alpha} t_{2s}^{\alpha} I_s + P_3^{\alpha} t_{2m}^{\alpha} I_m + t_{2x}^{\alpha} I_x + \varepsilon_4 u_4(t) I_x - \sigma_s^{\alpha} \beta_s^{\alpha} \frac{RI_s}{N} - \sigma_m^{\alpha} \beta_m^{\alpha} \frac{RI_m}{N} \\ & - \sigma_x^{\alpha} \beta_x^{\alpha} \frac{RI_x}{N} - d^{\alpha} R. \end{aligned} \tag{8}$$

Table 1: *All variables in the system* (1)–(8) *and their definition.*

| Variable | Definition |
|---|---|
| $S(t)$ | The susceptible population, individuals who have never encountered TB. |
| $L_s(t)$ | The individuals infected with the drug sensitive TB strain but who are in a latent stage, $i \cdot e$, who are neither showing symptoms nor infecting others. |
| $L_m(t)$ | Individuals latently infected with MDR-TB. |
| $L_x(t)$ | Individuals latently infected with XDR-TB. |
| $I_s(t)$ | Individuals infected with the drug-sensitive TB strain who are infectious to others (and most likely, showing symptoms as well). |
| $I_m(t)$ | Those individuals who are infectious with the MDR-TB strain. |
| $I_x(t)$ | Individuals who infectious with the XDR-TB strain. |
| $R(t)$ | Those individuals for whom treatment was successful. |
| $N(t)$ | The total population . $N = S + L_s + L_m + L_x + I_s + I_m + I_x + R$. |

Also all parameters and their interpretation as follows:

Table 2: *All adapted parameters in the system* (1)–(8) *and their interpretation.*

| Parameter | Interpretation |
|---|---|
| $b^{\alpha}$ | birth/recruitment rate |
| $d^{\alpha}$ | per capita natural death rate |
| | Disease dynamics |
| $\beta_r^{\alpha}$ | transmission coefficient for strain $r$ |
| $\lambda_r^{\alpha}$ | proportion of newly infected individuals developing $LTBI$ with strain $r$ |
| $1-\lambda_r^{\alpha}$ | proportion of newly infected individuals progressing to active TB with strain $r$ due to fast infection |
| $\varepsilon_r^{\alpha}$ | per capita rate of endogenous reactivation of $L_r$ |
| $\alpha_{r1}^{\alpha}, \alpha_{r2}^{\alpha}$ | proportion of exogenous reinfection of $L_{r1}$ due to contact with $I_{r2}$ |
| $\gamma_r^{\alpha}$ | per capita rate of natural recovery to the latent stage $L_r$ |
| $\delta_r^{\alpha}$ | per capita rate of death due to$TB$ of strain $r$ |
| | Treatment related |
| $t_{1s}^{\alpha}$ | per capita rate of treatment for $L_s$ |
| $t_{2r}^{\alpha}$ | per capita rate of treatment for $I_r$. Note that $t_{2x}$ is the rate of successful treatment of $I_x, r \in \{x, m, s\}$ |
| $1-\sigma_r^{\alpha}$ | efficiency of treatment in preventing infection with strain $r$ |
| $P_1^{\alpha}$ | probability of treatment success for $L_s$ |
| $1-P_1^{\alpha}$ | proportion of treated $L_s$ moved to $L_m$ due to incomplete treatment or lack of strict compliance in the use of drugs |
| $P_2^{\alpha}$ | probability of treatment success for $I_s$ |
| $1-P_2^{\alpha}$ | proportion of treated $I_s$ moved to $L_m$ due to incomplete treatment or lack of strict compliance in the use of drugs |
| $P_3^{\alpha}$ | probability of treatment success for $I_m$ |
| $1-P_3^{\alpha}$ | proportion of treated $I_m$ moved to $L_x$due to incomplete treatment or lack of strict compliance in the use of drugs |

Table 3: *All parameters in the system* (1)–(8).

| parameter | value |
|---|---|
| $b^{\alpha}$ | $(N^*d)^{\alpha}(\frac{1}{\text{year}})^{\alpha}$ |
| $d^{\alpha}$ | $(1/73.45)^{\alpha}(\frac{1}{\text{year}})^{\alpha}$ |
| $\beta_s^{\alpha} = \beta_m^{\alpha} = \beta_x^{\alpha}$ | $14^{\alpha}(\frac{1}{\text{year}})^{\alpha}$ |
| $\lambda_s^{\alpha} = \lambda_m^{\alpha} = \lambda_x^{\alpha}$ | $0.5^{\alpha}(\frac{1}{\text{year}})^{\alpha}$ |
| $\varepsilon_s^{\alpha} = \varepsilon_m^{\alpha} = \varepsilon_x^{\alpha}$ | $0.0002^{\alpha}(\frac{1}{\text{year}})^{\alpha}$ |
| $\alpha_{r1,r2}^{\alpha}$ | $0.05^{\alpha}(\frac{1}{\text{year}})^{\alpha}$ |

| | |
|---|---|
| $\gamma_s^\alpha = \gamma_m^\alpha = \gamma_x^\alpha$ | $0.00002^\alpha(\frac{1}{\text{year}})^\alpha$ |
| $t_{1s}^\alpha$ | $2^\alpha(\frac{1}{\text{year}})^\alpha$ |
| $t_{2r}^\alpha : r \in (s,m,x)$ | $t_{2s}^\alpha = 2^\alpha;\ t_{2m}^\alpha = t_{2x}^\alpha = (\frac{1}{\text{year}})^\alpha$ |
| $\sigma_r^\alpha$ | $0.25^\alpha(\frac{1}{\text{year}})^\alpha$ |
| $P_r^\alpha$ | $0.88^\alpha(\frac{1}{\text{year}})^\alpha$ |
| $\delta_r^\alpha$ | $0$ |

### 2.1. The basic reproduction number

The basic reproduction number ($R_0$) represents the expected average number of new TB infections produced by a single TB active infected individual when in contact with a completely susceptible population [50].

To derive a formula for $R_0$ using the next generation method, we follow the method of [50] and order the infected variables as

$$\Im := (L_s, L_m, L_x, I_s, I_m, I_x)'$$

The vector representing new infections into the infected classes $F$ is given by

$$F := \begin{pmatrix} \lambda_s^\alpha \beta_s^\alpha \frac{SI_s}{N} + \sigma_s^\alpha \lambda_s^\alpha \beta_s^\alpha \frac{RI_s}{N} \\ \lambda_m^\alpha \beta_m^\alpha \frac{SI_m}{N} + \sigma_m^\alpha \lambda_m^\alpha \beta_m^\alpha \frac{RI_m}{N} \\ \lambda_x^\alpha \beta_x^\alpha \frac{SI_x}{N} + \sigma_x^\alpha \lambda_x^\alpha \beta_x^\alpha \frac{RI_x}{N} \\ (1-\lambda_s^\alpha)\beta_s^\alpha(\frac{SI_s}{N} + \sigma_s^\alpha \frac{RI_s}{N}) \\ (1-\lambda_m^\alpha)\beta_m^\alpha(\frac{SI_m}{N} + \sigma_m^\alpha \frac{RI_m}{N}) \\ (1-\lambda_x^\alpha)\beta_x^\alpha(\frac{SI_x}{N} + \sigma_x^\alpha \frac{RI_x}{N}) \end{pmatrix},$$

The vector $V$ representing other flows within and out of the infected classes $\Im$ is given by

$$V := \begin{pmatrix} \alpha_{ss}^\alpha \beta_s^\alpha \frac{L_sI_s}{N} - \alpha_{sm}^\alpha \beta_m^\alpha \frac{L_sI_m}{N} - \alpha_{sx}^\alpha \beta_x^\alpha \frac{L_sI_x}{N} \\ +\gamma_s^\alpha I_s - (d^\alpha + \varepsilon_s^\alpha + t_{1s}^\alpha) L_s, \\ +\gamma_m^\alpha I_m + \alpha_{sm}^\alpha \beta_m^\alpha \lambda_m^\alpha \frac{L_sI_m}{N} + (1-P_1^\alpha) t_{1s}^\alpha L_s + (1-P_2^\alpha) t_{2s}^\alpha I_s, \\ -\alpha_{mm}^\alpha \beta_m^\alpha \frac{L_mI_m}{N} - \alpha_{mx}^\alpha \beta_x^\alpha \frac{L_mI_x}{N} - (d^\alpha + \varepsilon_m^\alpha) L_m, \\ +\gamma_x^\alpha I_x + \alpha_{sx}^\alpha \beta_x^\alpha \lambda_x^\alpha \frac{L_sI_x}{N} + \alpha_{mx}^\alpha \beta_x^\alpha \lambda_x^\alpha \frac{L_mI_x}{N} \\ +(1-P_3^\alpha) t_{2m}^\alpha I_m - \alpha_{xx}^\alpha \beta_x^\alpha \frac{L_xI_x}{N} - (d^\alpha + \varepsilon_x^\alpha) L_x, \\ \alpha_{ss}^\alpha \beta_s^\alpha \frac{L_sI_s}{N} + \varepsilon_s^\alpha L_s - (d^\alpha + \delta_s^\alpha + t_{2s}^\alpha + \gamma_s^\alpha) I_s, \\ \alpha_{mm}^\alpha \beta_m^\alpha \frac{L_mI_m}{N} + \alpha_{sm}^\alpha \frac{L_sI_m}{N}) + \varepsilon_m^\alpha L_m - (d^\alpha + \delta_m^\alpha + \gamma_m^\alpha + t_{2m}^\alpha) I_m, \\ \alpha_{xx}^\alpha \beta_x^\alpha \frac{L_xI_x}{N} + (1-\lambda_x^\alpha)\beta_x^\alpha + \alpha_{sx}^\alpha \frac{L_sI_x}{N} + \alpha_{mx}^\alpha \frac{L_mI_x}{N}) + \varepsilon_x^\alpha L_x \\ -(d^\alpha + \delta_x^\alpha + \gamma_x^\alpha + t_{2x}^\alpha) I_x, \end{pmatrix},$$

The matrix of new infections $F$ and the matrix of transfers between compartments $V$ are the Jacobian matrices obtained by differentiating $F$ and $V$ with respect to the infected variables $\Im$ and evaluating at the disease free equilibrium. They take the form:

$$F := \begin{pmatrix} 0 & A \\ 0 & B \end{pmatrix}, \qquad V := \begin{pmatrix} C & D \\ E & F_2 \end{pmatrix},$$

where,

$$A = \begin{pmatrix} \lambda_s^\alpha \beta_s^\alpha & 0 & 0 \\ 0 & \lambda_m^\alpha \beta_m^\alpha & 0 \\ 0 & 0 & \lambda_s^\alpha \beta_s^\alpha \end{pmatrix},$$

$$B = \begin{pmatrix} (1-\lambda_s^\alpha)\beta_s^\alpha & 0 & 0 \\ 0 & (1-\lambda_m^\alpha)\beta_m^\alpha & 0 \\ 0 & 0 & (1-\lambda_s^\alpha)\beta_s^\alpha \end{pmatrix},$$

$$C = \begin{pmatrix} (d^\alpha + \varepsilon_s^\alpha + t_{1s}^\alpha) & 0 & 0 \\ (-1+P_1^\alpha)t_{1s}^\alpha & (d^\alpha + \varepsilon_m^\alpha) & 0 \\ 0 & 0 & (d^\alpha + \varepsilon_x^\alpha) \end{pmatrix},$$

$$D = \begin{pmatrix} -\gamma_s^\alpha & 0 & 0 \\ (-1+P_2^\alpha)t_{2s}^\alpha & -\gamma_m^\alpha & 0 \\ 0 & (-1+P_3^\alpha)t_{2m}^\alpha & -\gamma_x^\alpha \end{pmatrix},$$

$$F_2 = \begin{pmatrix} (d^\alpha + \delta_s^\alpha + \gamma_s^\alpha + t_{2s}^\alpha) & 0 & 0 \\ 0 & (d^\alpha + \delta_m^\alpha + \gamma_m^\alpha + t_{2m}^\alpha) & 0 \\ 0 & 0 & (d^\alpha + \delta_x^\alpha + \gamma_x^\alpha + t_{2x}^\alpha) \end{pmatrix},$$

$$E = \begin{pmatrix} -\varepsilon_s^\alpha & 0 & 0 \\ 0 & -\varepsilon_m^\alpha & 0 \\ 0 & 0 & -\varepsilon_x^\alpha \end{pmatrix},$$

Then the basic reproduction number $R_0$ for the system (1)–(8) is the spectral radius of the next generation matrix and is given by

$$R_0 = \rho(FV^{-1}) = \max(R_{0s}, R_{0m}, R_{0x}), \tag{9}$$

where,

$$R_{0s} = \frac{\beta_s^\alpha(\varepsilon_s^\alpha + (1-\lambda_s^\alpha)(d^\alpha + t_{1s}^\alpha))}{(\varepsilon_s^\alpha + d^\alpha + t_{1s}^\alpha)(t_{2s}^\alpha + \delta_s^\alpha + d^\alpha) + \gamma_s^\alpha(t_{1s}^\alpha + d^\alpha)},$$

$$R_{0m} = \frac{\beta_m^\alpha(\varepsilon_m^\alpha + (1-\lambda_m^\alpha)d^\alpha)}{(\varepsilon_m^\alpha + d^\alpha)(t_{2m}^\alpha + \delta_m^\alpha + d^\alpha) + d^\alpha\gamma_m^\alpha},$$

$$R_{0x} = \frac{\beta_x^\alpha(\varepsilon_x^\alpha + (1-\lambda_x^\alpha)d^\alpha)}{(\varepsilon_x^\alpha + d^\alpha)(t_{2x}^\alpha + \delta_x^\alpha + d^\alpha) + d^\alpha\gamma_x^\alpha}$$

## 3. Formulation of the optimal control problem

Let us consider the state system (1)–(8), in $R^8$ with the set of admissible control functions:

$$\Omega=\{(u_1(\cdot),u_2(\cdot),u_3(\cdot),u_4(\cdot))\in(L^{\infty}(0,T))\mid 0\leqslant u_1(\cdot),u_2(\cdot),u_3(\cdot),u_4(\cdot)\leqslant 1,\forall t\in[0,T]\}.$$

The objective functional is given by [42] as follows:

$$J(u_k)=\int_0^T \eta(S,L_s,L_m,L_x,I_s,I_m,I_x,R,u_k,t)dt, \tag{10}$$

where $k=1,2,3,4,$ and

$$\begin{aligned}J(u_1(\cdot),u_2(\cdot),u_3(\cdot),u_4(\cdot))=\int_0^T [I_s(t)+I_m(t)+I_x(t)+L_s(t)+\frac{1}{2}B_1u_1^2(t)+\frac{1}{2}B_2u_2^2(t)\\ +\frac{1}{2}B_3u_3^2(t)+\frac{1}{2}B_4u_4^2(t)]dt,\end{aligned} \tag{11}$$

where the constants $B_1$, $B_2$, $B_3$ and $B_4$ are a measure of the relative cost of the interventions associated with the controls $u_1$, $u_2$, $u_3$ and $u_4$, respectively.

The main point in fraction optimal contol problems is to find the optimal controls $u_k(t)$, where $k=1,2,3,4,$ which minimizes the objective function (11), subject to the following state system:

$$ {}_a^cD_t^{\alpha}S=\xi_1(S,L_s,L_m,L_x,I_s,I_m,I_x,R,u_k,t), \tag{12}$$
$$ {}_a^cD_t^{\alpha}L_s=\xi_2(S,L_s,L_m,L_x,I_s,I_m,I_x,R,u_k,t), \tag{13}$$
$$ {}_a^cD_t^{\alpha}L_m=\xi_3(S,L_s,L_m,L_x,I_s,I_m,I_x,R,u_k,t), \tag{14}$$
$$ {}_a^cD_t^{\alpha}L_x=\xi_4(S,L_s,L_m,L_x,I_s,I_m,I_x,R,u_k,t), \tag{15}$$
$$ {}_a^cD_t^{\alpha}I_s=\xi_5(S,L_s,L_m,L_x,I_s,I_m,I_x,R,u_k,t), \tag{16}$$
$$ {}_a^cD_t^{\alpha}I_m=\xi_6(S,L_s,L_m,L_x,I_s,I_m,I_x,R,u_k,t), \tag{17}$$
$$ {}_a^cD_t^{\alpha}I_x=\xi_7(S,L_s,L_m,L_x,I_s,I_m,I_x,R,u_k,t), \tag{18}$$
$$ {}_a^cD_t^{\alpha}R=\xi_8(S,L_s,L_m,L_x,I_s,I_m,I_x,R,u_k,t), \tag{19}$$

and satisfying the inital conditions:

$$\begin{gathered}S(0)=S_0,\quad L_s(0)=L_{s0},\quad L_m(0)=L_{m0},\quad L_x(0)=L_{x0},\\ I_s(0)=I_{s0},\quad I_m(0)=I_{m0},\quad I_x(0)=I_{x0},\quad R(0)=R_0.\end{gathered}$$

The following expression defines a modified objective function:

$$\begin{aligned}\tilde{J}=\int_0^T &[H(S,L_s,L_m,L_x,I_s,I_m,I_x,R,u_k,t)\\ &-\sum_{i=1}^{8}\lambda_i\xi_i(S,L_s,L_m,L_x,I_s,I_m,I_x,R,u_k,t)]dt,\end{aligned} \tag{20}$$

where $H(S,L_s,L_m,L_x,I_s,I_m,I_x,R,u_k,t)$ is the following Hamiltonian

$$H(S,L_s,L_m,L_x,I_s,I_m,I_x,R,u_k,\lambda_i,t) = \eta(S,L_s,L_m,L_x,I_s,I_m,I_x,R,u_k,t) + \sum_{i=1}^{8} \lambda_i \xi_i(S,L_s,L_m,L_x,I_s,I_m,I_x,R,u_k,t). \quad (21)$$

From (20) and (21), we can derive ([7]–[8]):

$$\begin{aligned}
{}_t^cD_{t_f}^{\alpha}\lambda_1 &= \frac{\partial H}{\partial S}, \quad {}_t^cD_{t_f}^{\alpha}\lambda_2 = \frac{\partial H}{\partial L_s},\\
{}_t^cD_{t_f}^{\alpha}\lambda_3 &= \frac{\partial H}{\partial L_m}, \quad {}_t^cD_{t_f}^{\alpha}\lambda_4 = \frac{\partial H}{\partial L_x},\\
{}_t^cD_{t_f}^{\alpha}\lambda_5 &= \frac{\partial H}{\partial I_s}, \quad {}_t^cD_{t_f}^{\alpha}\lambda_6 = \frac{\partial H}{\partial I_m},\\
{}_t^cD_{t_f}^{\alpha}\lambda_7 &= \frac{\partial H}{\partial I_x}, \quad {}_t^cD_{t_f}^{\alpha}\lambda_8 = \frac{\partial H}{\partial R},
\end{aligned} \quad (22)$$

$$0 = \frac{\partial H}{\partial u_k}, \quad (23)$$

$$\begin{aligned}
{}_0^cD_t^{\alpha}S &= \frac{\partial H}{\partial \lambda_1}, \quad {}_0^cD_t^{\alpha}L_s = \frac{\partial H}{\partial \lambda_2},\\
{}_0^cD_t^{\alpha}L_m &= \frac{\partial H}{\partial \lambda_3}, \quad {}_0^cD_t^{\alpha}L_x = \frac{\partial H}{\partial \lambda_4},\\
{}_0^cD_t^{\alpha}I_s &= \frac{\partial H}{\partial \lambda_5}, \quad {}_0^cD_t^{\alpha}I_m = \frac{\partial H}{\partial \lambda_6},\\
{}_0^cD_t^{\alpha}I_x &= \frac{\partial H}{\partial \lambda_7}, \quad {}_0^cD_t^{\alpha}R = \frac{\partial H}{\partial \lambda_8},
\end{aligned} \quad (24)$$

and it is also required that:

$$\lambda_i(b) = 0, \quad \text{where} \quad i = 1,2,3,\ldots,8. \quad (25)$$

Eqs. (23) and (25) describe the necessary conditions in terms of a Hamiltonian for the optimal control problem defined above.

THEOREM 3.1. *If $u_1^*(\cdot), u_2^*(\cdot),\ u_3^*(\cdot)$, and $u_4^*(\cdot)$ are optimal controls with corresponding staste $S^*(\cdot),\ L_s^*(\cdot),\ L_m^*(\cdot),\ L_x^*(\cdot),\ I_s^*(\cdot),\ I_m^*(\cdot),\ I_x^*(\cdot)$, and $R^*(\cdot)$ then there exist adjoint variables $\lambda_1^*,\ \lambda_2^*,\ \lambda_3^*,\ \lambda_4^*,\ \lambda_5^*,\ \lambda_6^*,\ \lambda_7^*$, and $\lambda_8^*$ satisfy the following:*

*(i) adjoint equations:*

$$\begin{aligned}
{}_t^cD_{t_f}^{\alpha}\lambda_1^* = \frac{\partial H}{\partial S} = &-(\lambda_1^*(t)\Big(\frac{\beta_s^{\alpha}}{N}I_s^*(t) + \frac{\beta_m^{\alpha}}{N}I_m^*(t) + \frac{\beta_x^{\alpha}}{N}I_x^*(t) + d^{\alpha}\Big) - \lambda_2^*(t)\frac{\lambda_s^{\alpha}\beta_s^{\alpha}}{N}I_s^*(t)\\
&- \lambda_3^*(t)\frac{\lambda_m^{\alpha}\beta_m^{\alpha}}{N}I_m^*(t) - \lambda_4^*(t)\frac{\lambda_x^{\alpha}\beta_x^{\alpha}}{N}I_s^*(t) - \lambda_5^*(t)\frac{(1-\lambda_s^{\alpha})\beta_s^{\alpha}}{N}I_s^*(t)\\
&- \lambda_6^*(t)\frac{(1-\lambda_m^{\alpha})\beta_m^{\alpha}}{N}I_m^*(t) - \lambda_7^*(t)\frac{(1-\lambda_x^{\alpha})\beta_x^{\alpha}}{N}I_x^*(t)),
\end{aligned} \quad (26)$$

$$
\begin{aligned}
{}_t^cD_{t_f}^{\alpha}\lambda_2^* = \frac{\partial H}{\partial L_s} =& -\Big(-1+\lambda_2^*(t)\Big(\frac{\beta_s^{\alpha}\alpha_{ss}^{\alpha}I_s^*(t)}{N}+\frac{\beta_m^{\alpha}\alpha_{sm}^{\alpha}I_m^*(t)}{N}+\frac{\beta_x^{\alpha}\alpha_{sx}^{\alpha}I_x^*(t)}{N}\Big)\\
&+\lambda_2^*(t)(d^{\alpha}+\varepsilon_s^{\alpha}+t_{1s}^{\alpha}+\varepsilon_1u_1^*(t))-\lambda_3^*(t)\frac{\lambda_m^{\alpha}\beta_m^{\alpha}\alpha_{sm}^{\alpha}}{N}I_m^*(t)\\
&-(t_{1s}^{\alpha}+\varepsilon_1u_1^*(t)-P_1^{\alpha}t_{1s}^{\alpha})\lambda_3^*(t)-\lambda_4^*(t)\frac{\lambda_x^{\alpha}\beta_x^{\alpha}\alpha_{sx}^{\alpha}}{N}I_s^*(t)\\
&-\lambda_5^*(t)\frac{\beta_s^{\alpha}\alpha_{ss}^{\alpha}}{N}I_s^*(t)-\lambda_5^*(t)\varepsilon_s^{\alpha}-\lambda_6^*(1-\lambda_m^{\alpha})\frac{\beta_m^{\alpha}\alpha_{sm}^{\alpha}}{N}I_m^*(t)\\
&-\lambda_7^*(1-\lambda_x^{\alpha})\frac{\beta_x^{\alpha}\alpha_{sx}^{\alpha}}{N}I_x^*(t)-P_1^{\alpha}t_{1s}^{\alpha}\lambda_8^*(t)\Big),
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
{}_t^cD_{t_f}^{\alpha}\lambda_3^* = \frac{\partial H}{\partial L_m} =& -\Big(\lambda_3^*(t)\Big(\alpha_{mm}^{\alpha}\beta_m^{\alpha}\frac{I_m^*(t)}{N}+\alpha_{mx}^{\alpha}\beta_x^{\alpha}\frac{I_x^*(t)}{N}+d^{\alpha}+\varepsilon_m^{\alpha}\Big)\\
&-\lambda_4^*(t)\alpha_{mx}^{\alpha}\beta_x^{\alpha}\lambda_x^{\alpha}\frac{I_x^*(t)}{N}-\lambda_6^*(t)\Big(\alpha_{mm}^{\alpha}\beta_m^{\alpha}\frac{I_m^*(t)}{N}+\varepsilon_m^{\alpha}\Big)\\
&-\lambda_7^*(t)(1-\lambda_x^{\alpha})\alpha_{mx}^{\alpha}\beta_x^{\alpha}\frac{I_x^*(t)}{N}\Big),
\end{aligned} \tag{28}
$$

$$
{}_t^cD_{t_f}^{\alpha}\lambda_4^* = \frac{\partial H}{\partial L_x} = -\Big(\lambda_4^*(t)\Big(\alpha_{xx}^{\alpha}\beta_x^{\alpha}\frac{I_x^*(t)}{N}+d^{\alpha}+\varepsilon_x^{\alpha}\Big)-\lambda_7^*(t)\Big(\alpha_{xx}^{\alpha}\beta_x^{\alpha}\frac{I_x^*(t)}{N}-\varepsilon_x^{\alpha}\Big), \tag{29}
$$

$$
\begin{aligned}
{}_t^cD_{t_f}^{\alpha}\lambda_5^* = \frac{\partial H}{\partial I_s} =& -1+\lambda_1^*(t)\beta_s^{\alpha}\frac{S^*(t)}{N}+\lambda_2^*(t)\Big(\alpha_{ss}^{\alpha}\beta_s^{\alpha}\frac{L_s^*(t)}{N}-\lambda_s^{\alpha}\beta_s^{\alpha}\frac{S^*(t)}{N}\\
&-\lambda_s^{\alpha}\sigma_s^{\alpha}\beta_s^{\alpha}\frac{R^*(t)}{N}-\gamma_s^{\alpha}\Big)+\lambda_5^*(t)\Big(\Big(d^{\alpha}+\delta_s^{\alpha}+t_{2s}^{\alpha}+\varepsilon_2u_2^*(t)+\gamma_s^{\alpha}\Big)\\
&-(1-\lambda_s^{\alpha})\beta_s^{\alpha}\frac{S^*(t)}{N}-\sigma_s^{\alpha}(1-\lambda_s^{\alpha})\beta_s^{\alpha}\frac{R^*(t)}{N}+\frac{\beta_s^{\alpha}\alpha_{ss}^{\alpha}}{N}L_s^*(t)\Big)\\
&-\lambda_3^*(t)(t_{2s}^{\alpha}+\varepsilon_2u_2^*(t)-P_2^{\alpha}t_{2s}^{\alpha})+\lambda_8^*(t)\Big(\sigma_s^{\alpha}\beta_s^{\alpha}\frac{R^*(t)}{N}-P_2^{\alpha}t_{2s}^{\alpha}\Big)\Big),
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
{}_t^cD_{t_f}^{\alpha}\lambda_6 = \frac{\partial H}{\partial I_m} =& -\Big(-1+\lambda_1^*(t)\beta_m^{\alpha}\frac{S^*(t)}{N}+\lambda_2^*(t)\Big(\alpha_{sm}^{\alpha}\beta_m^{\alpha}\frac{L_s^*(t)}{N}-\lambda_3^*(t)\Big(\lambda_m^{\alpha}\beta_m^{\alpha}\frac{S^*(t)}{N}\\
&+\lambda_m^{\alpha}\sigma_m^{\alpha}\beta_m^{\alpha}\frac{R^*(t)}{N}+\frac{\lambda_m^{\alpha}\beta_m^{\alpha}\alpha_{sm}^{\alpha}}{N}L_s^*(t)-\frac{\beta_m^{\alpha}\alpha_{mm}^{\alpha}}{N}L_m^*(t)+\gamma_m^{\alpha}\Big)\\
&-\lambda_4^*(t)t_{2m}^{\alpha}-\lambda_4^*(t)\varepsilon_3u_3^*(t)+\lambda_4^*(t)P_3^{\alpha}t_{2m}^{\alpha}\\
&-\lambda_6^*(t)\Big(\frac{\beta_m^{\alpha}\alpha_{mm}^{\alpha}}{N}L_m^*(t)+(1-\lambda_m^{\alpha})\beta_m^{\alpha}\frac{S^*(t)}{N}+\sigma_m^{\alpha}(1-\lambda_m^{\alpha})\beta_m^{\alpha}\frac{R^*(t)}{N}\\
&+\alpha_{sm}^{\alpha}(1-\lambda_m^{\alpha})\beta_m^{\alpha}\frac{L_s^*(t)}{N}-(d^{\alpha}+\delta_m^{\alpha}+t_{2m}^{\alpha}+\varepsilon_3u_3^*(t)+\gamma_m^{\alpha})\Big)\\
&-\lambda_8^*(t)\Big(P_3^{\alpha}t_{2m}^{\alpha}-\sigma_m^{\alpha}\beta_m^{\alpha}\frac{R^*}{N}\Big)\Big)\Big),
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
{}_t^cD_{t_f}^{\alpha}\lambda_7^* = \frac{\partial H}{\partial I_x} =& -\Big(-1+\lambda_1^*(t)\beta_x^{\alpha}\frac{S^*(t)}{N}+\lambda_2^*(t)\alpha_{sx}^{\alpha}\beta_x^{\alpha}\frac{L_s^*(t)}{N}+\lambda_3^*(t)\alpha_{mx}^{\alpha}\beta_x^{\alpha}\frac{L_m^*(t)}{N}\\
&-\lambda_4^*(t)\Big(\lambda_x^{\alpha}\sigma_x^{\alpha}\beta_x^{\alpha}\frac{R^*(t)}{N}+\lambda_x^{\alpha}\beta_x^{\alpha}\frac{S^*(t)}{N}+\lambda_x^{\alpha}\alpha_{sx}^{\alpha}\beta_x^{\alpha}\frac{L_s^*(t)}{N}
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda_x^\alpha \alpha_{mx}^\alpha \beta_x^\alpha \frac{L_m^*(t)}{N} + \gamma_x^\alpha - \alpha_{xx}^\alpha \beta_x^\alpha \frac{L_x^*(t)}{N}\Big) \\
&-\lambda_7^*(t)\Big(\alpha_{xx}^\alpha \beta_x^\alpha \frac{L_x^*(t)}{N} + (1-\lambda_x^\alpha)\beta_x^\alpha \frac{S^*}{N} + (1-\lambda_x^\alpha)\beta_x^\alpha \alpha_{sx}^\alpha \frac{L_s^*}{N} \\
&+(1-\lambda_x^\alpha)\beta_x^\alpha \sigma_x^\alpha \frac{R^*}{N} + (1-\lambda_x^\alpha)\beta_x^\alpha \alpha_{mx}^\alpha \frac{L_m^*}{N} \\
&+(d^\alpha + \delta_x^\alpha + t_{2x}^\alpha + \varepsilon_4 u_4^* + \gamma_x^\alpha)\Big) - \lambda_8^*(t)\Big(t_{2x}^\alpha + \varepsilon_4 u_4^* - \sigma_x^\alpha \beta_x^\alpha \frac{R}{N}\Big)\Big),
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
{}_t^cD_{t_f}^\alpha \lambda_8^* = \frac{\partial H}{\partial R} = -\Big(&-\lambda_2^*(t)\beta_s^\alpha \lambda_s^\alpha \sigma_s^\alpha \frac{I_s^*(t)}{N} - \lambda_3^*(t)\beta_m^\alpha \lambda_m^\alpha \sigma_m^\alpha \frac{I_m^*(t)}{N} \\
&-\lambda_4^*(t)\beta_x^\alpha \lambda_x^\alpha \sigma_x^\alpha \frac{I_x^*(t)}{N} - \lambda_5^*(t)(1-\lambda_s^\alpha)\beta_s^\alpha \sigma_s^\alpha \frac{I_s^*(t)}{N} \\
&-\lambda_6^*(t)(1-\lambda_m^\alpha)\beta_m^\alpha \sigma_m^\alpha \frac{I_m^*(t)}{N} - \lambda_7^*(t)(1-\lambda_x^\alpha)\beta_x^\alpha \sigma_x^\alpha \frac{I_x^*(t)}{N} \\
&+\lambda_8^*(t)\Big(\sigma_s^\alpha \beta_s^\alpha \frac{I_s^*}{N} + \sigma_m^\alpha \beta_m^\alpha \frac{I_m^*}{N} + \sigma_x^\alpha \beta_x^\alpha \frac{I_x^*}{N} + d^\alpha\Big)\Big),
\end{aligned}
\tag{33}
$$

*(ii) with transversality conditions* $\lambda_i^*(T) = 0,\ i = 1,\ldots,8.$
*(iii) optimality conditions:*

$$
\begin{aligned}
&H(S^*(t), L_s^*(t), L_m^*(t), L_x^*(t), I_s^*(t), I_m^*(t), I_x^*(t), R^*(t), \lambda^*(t), u_k^*(t)) \\
&= \min_{0 \leqslant u_k \leqslant 1} H(S^*(t), L_s^*(t), L_m^*(t), L_x^*(t), I_s^*(t), I_m^*(t), I_x^*(t), R^*(t), \lambda^*(t), u_k(t)),
\end{aligned}
\tag{34}
$$

$$
u_1^*(t) = \min\Big\{\max\Big\{0, \frac{\varepsilon_1 L_s^*(\lambda_2^*(t) - \lambda_3^*(t))}{W_1}\Big\}, 1\Big\}, \tag{35}
$$

$$
u_2^*(t) = \min\Big\{\max\Big\{0, \frac{\varepsilon_2 I_s^*(\lambda_5^*(t) - \lambda_3^*(t))}{W_2}\Big\}, 1\Big\}, \tag{36}
$$

$$
u_3^*(t) = \min\Big\{\max\Big\{0, \frac{\varepsilon_3 I_m^*(\lambda_6^*(t) - \lambda_4^*(t))}{W_3}\Big\}, 1\Big\}, \tag{37}
$$

$$
u_4^*(t) = \min\Big\{\max\Big\{0, \frac{\varepsilon_4 I_x^*(\lambda_7^*(t) - \lambda_8^*(t))}{W_4}\Big\}, 1\Big\}, \tag{38}
$$

*where the stationarity condition is* $\frac{\partial H}{\partial u_k} = 0,\ k = 1,2,3,4.$

*Proof.* Using the conditions (22), we can get equations (26)–(33), where the Hamiltonian H is given by:

$$
\begin{aligned}
H =& H(S, L_s, L_m, L_x, I_s, I_m, I_x, R, \lambda, u_k) \\
=& I_s + I_m + I_x + L_s + u_1^2 \frac{B_1}{2} + u_2^2 \frac{B_2}{2} + u_3^2 \frac{B_3}{2} + u_4^2 \frac{B_4}{2} \\
&+\lambda_1\Big(b^\alpha - d^\alpha S - \beta_s^\alpha \frac{SI_s}{N} - \beta_m^\alpha \frac{SI_m}{N} - \beta_x^\alpha \frac{SI_x}{N}\Big) \\
&+\lambda_2\Big(\lambda_s^\alpha \beta_s^\alpha \frac{SI_s}{N} + \sigma_s^\alpha \lambda_s^\alpha \beta_s^\alpha \frac{RI_s}{N} - \alpha_{ss}^\alpha \beta_s^\alpha \frac{L_s I_s}{N} - \alpha_{sm}^\alpha \beta_m^\alpha \frac{L_s I_m}{N} - \alpha_{sx}^\alpha \beta_x^\alpha \frac{L_s I_x}{N} + \gamma_s^\alpha I_s
\end{aligned}
$$

$$
\begin{aligned}
& - (d^\alpha + \varepsilon_s^\alpha + t_{1s}^\alpha + \varepsilon_1 u_1(t))L_s\Big) \\
& + \lambda_3\Big(\lambda_m^\alpha \beta_m^\alpha \frac{SI_m}{N} + \sigma_m^\alpha \lambda_m^\alpha \beta_m^\alpha \frac{RI_m}{N} + \alpha_{sm}^\alpha \beta_m^\alpha \lambda_m^\alpha \frac{L_s I_m}{N} - \alpha_{mm}^\alpha \beta_m^\alpha \frac{L_m I_m}{N} - \alpha_{mx}^\alpha \beta_x^\alpha \frac{L_m I_x}{N} \\
& + \gamma_m^\alpha I_m - (d^\alpha + \varepsilon_m^\alpha)L_m + t_{1s}^\alpha L_s + \varepsilon_1 u_1(t) L_s - P_1^\alpha t_{1s}^\alpha L_s + t_{2s}^\alpha I_s + \varepsilon_2 u_2(t) I_s - P_2^\alpha t_{2s}^\alpha I_s\Big) \\
& + \lambda_4\Big(\lambda_x^\alpha \beta_x^\alpha \frac{SI_x}{N} + \sigma_x^\alpha \lambda_x^\alpha \beta_x^\alpha \frac{RI_x}{N} + \alpha_{sx}^\alpha \beta_x^\alpha \lambda_x^\alpha \frac{L_s I_x}{N} + \alpha_{mx}^\alpha \beta_x^\alpha \lambda_x^\alpha \frac{L_m I_x}{N} - \alpha_{xx}^\alpha \beta_x^\alpha \frac{L_x I_x}{N} \\
& - (d^\alpha + \varepsilon_x^\alpha)L_x + \gamma_x^\alpha I_x + t_{2m}^\alpha I_m + \varepsilon_3 u_3(t) I_m - P_3^\alpha t_{2m}^\alpha I_m\Big) \\
& + \lambda_5(\alpha_{ss}^\alpha \beta_s^\alpha \frac{L_s I_s}{N} + (1-\lambda_s^\alpha)\beta_s^\alpha \Big(\frac{SI_s}{N} + \sigma_s^\alpha \frac{RI_s}{N}\Big) + \varepsilon_s^\alpha L_s \\
& - (d^\alpha + \delta_s^\alpha + t_{2s}^\alpha + \gamma_s^\alpha + \varepsilon_2 u_2(t))I_s) \\
& + \lambda_6\Big(\alpha_{mm}^\alpha \beta_m^\alpha \frac{L_m I_m}{N} + (1-\lambda_m^\alpha)\beta_m^\alpha \Big(\frac{SI_m}{N} + \sigma_m^\alpha \frac{RI_m}{N} + \alpha_{sm}^\alpha \frac{L_s I_m}{N}\Big) + \varepsilon_m^\alpha L_m \\
& - (d^\alpha + \delta_m^\alpha + t_{2m}^\alpha + \varepsilon_3 u_3(t) + \gamma_m^\alpha)I_m\Big) \\
& + \lambda_7\Big(\alpha_{xx}^\alpha \beta_x^\alpha \frac{L_x I_x}{N} + (1-\lambda_x^\alpha)\beta_x^\alpha \Big(\frac{SI_x}{N} + \sigma_x^\alpha \frac{RI_x}{N} + \alpha_{sx}^\alpha \frac{L_s I_x}{N} + \alpha_{mx}^\alpha \frac{L_m I_x}{N}\Big) + \varepsilon_x^\alpha L_x \\
& - (d^\alpha + \delta_x^\alpha + t_{2x}^\alpha + \gamma_x^\alpha + \varepsilon_4 u_4(t))I_x\Big) \\
& + \lambda_8\Big(P_1^\alpha t_{1s}^\alpha L_s + P_2^\alpha t_{2s}^\alpha I_s + P_3^\alpha t_{2m}^\alpha I_m + t_{2x}^\alpha I_x + \varepsilon_4 u_4(t) I_x - \sigma_s^\alpha \beta_s^\alpha \frac{RI_s}{N} \\
& - \sigma_m^\alpha \beta_m^\alpha \frac{RI_m}{N} - \sigma_x^\alpha \beta_x^\alpha \frac{RI_x}{N} - d^\alpha R\Big), \qquad (39)
\end{aligned}
$$

where $\lambda_i,\ i = 1,2,3,\ldots,8$ are the Lagrange multipliers. It is known as a co-state or adjoint variables.

Moreover, the transversality conditions $\lambda_i(T) = 0,\ i = 1,\ldots,8,$ hold and the optimal controls (35)–(38) can be claimed from the minimization condition (34). □

## 4. Numerical methods for solving FOCP

### 4.1. Generalized Euler method

Generalized Euler method (GEM), is a generalization of the classical Euler's method, for more details see [33]. The headlines of this method is given as follows, let us consider (12)–(19): Let $[0,a]$ be the interval over which we want to find the solution of the problem (12)–(19). The interval $[0,a]$ will be subdivided into $K$ subintervals $[t_j, t_{j+1}]$ of equal width $h = \frac{a}{K}$ by using the nodes $t_j = jh$, for $j = 0,1,2,\ldots K$. The general formula for GEM when $t_{j+1} = t_j + h$ is

$$
\begin{aligned}
S(t_{j+1}) = S(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)} \xi_1(& S(t_j), L_s(t_j), L_m(t_j), L_x(t_j), I_s(t_j), I_m(t_j), I_x(t_j), R(t_j), \\
& u_k(t_j), t_j),
\end{aligned}
$$

$$
\begin{aligned}
L_s(t_{j+1}) &= L_s(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)}\xi_2(S(t_j),L_s(t_j),L_m(t_j),L_x(t_j),I_s(t_j),I_m(t_j),I_x(t_j),R(t_j),\\
&\quad u_k(t_j),t_j),\\
L_m(t_{j+1}) &= L_m(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)}\xi_3(S(t_j),L_s(t_j),L_m(t_j),L_x(t_j),I_s(t_j),I_m(t_j),I_x(t_j),R(t_j),\\
&\quad u_k(t_j),t_j),\\
L_x(t_{j+1}) &= L_x(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)}\xi_4(S(t_j),L_s(t_j),L_m(t_j),L_x(t_j),I_s(t_j),I_m(t_j),I_x(t_j),R(t_j),\\
&\quad u_k(t_j),t_j),\\
I_s(t_{j+1}) &= I_s(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)}\xi_5(S(t_j),L_s(t_j),L_m(t_j),L_x(t_j),I_s(t_j),I_m(t_j),I_x(t_j),R(t_j),\\
&\quad u_k(t_j),t_j),\\
I_m(t_{j+1}) &= I_m(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)}\xi_6(S(t_j),L_s(t_j),L_m(t_j),L_x(t_j),I_s(t_j),I_m(t_j),I_x(t_j),R(t_j),\\
&\quad u_k(t_j),t_j),\\
I_x(t_{j+1}) &= I_x(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)}\xi_7(S(t_j),L_s(t_j),L_m(t_j),L_x(t_j),I_s(t_j),I_m(t_j),I_x(t_j),R(t_j),\\
&\quad u_k(t_j),t_j),\\
R(t_{j+1}) &= R(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)}\xi_8(S(t_j),L_s(t_j),L_m(t_j),L_x(t_j),I_s(t_j),I_m(t_j),I_x(t_j),R(t_j),\\
&\quad u_k(t_j),t_j),
\end{aligned}
$$

for $j = 0,1,\ldots,K-1$. It is clear that if $\alpha = 1$, then the GEM [9], reduces to the classical Euler's method.

### 4.2. Shifted Chebyshev spectral method

It is well known Chebyshev polynomials of the first kind are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formula [11].

$$T_{k+1}(z) = 2zT_k(z) - T_{k-1}(z), \quad k = 1,2,3,\ldots,$$

where $T_0(z) = 1$, and $T_1(z) = z$. In order to use these polynomials on the interval $[0,L]$ we define the so called shifted Chebyshev polynomials of the first kind by introducing the change of variable $z = \frac{2t}{L} - 1$. Let the shifted Chebyshev polynomials $T_k(\frac{2t}{L} - 1)$ be denoted by

$$T^*_{k+1}(z) = 2\left(\frac{2t}{L} - 1\right)T^*_k(z) - T^*_{k-1}(z), \quad k = 1,2,3,\ldots,$$

where $T^*_0(z) = 1$ and $T^*_1(z) = \frac{2t}{L} - 1$. The analytic form of the shifted Chebyshev polynomials $T^*_k(t)$ of degree $k$ is given by

$$T^*_k(t) = k\sum_{i=0}^{k}(-1)^{k-i}2^{2i}\frac{k+i-1!}{(k-i)!(2i!)L^i}t^i. \tag{40}$$

Note that $T_k^*(0) = (-1)^k$ and $T_k^*(1) = 1$. The function $u(t)$, which is a square integrable function in $[0,L]$, may be expressed in terms of shifted Chebyshev polynomials as

$$u(t) = \sum_{i=0}^{\infty} c_i T_i^*(t), \tag{41}$$

where the coefficients $c_i$ are given by:

$$c_i = \frac{2}{\pi h_i} \int_0^L u(t) T_i^*(t) dt, \quad h_0 = 2, \quad h_i = 1, \quad i = 0, 1, \ldots. \tag{42}$$

In practice, only the first $(m+1)$-terms of shifted Chebyshev polynomials are considered. Then we have

$$u_m(t) = \sum_{i=0}^{m} c_i T_i^*(t). \tag{43}$$

$$D^{\alpha} u_m(t)) = \sum_{i=0}^{m} c_i D^{\alpha}(T_i^*(t)). \tag{44}$$

Therefore, for $i = 1, 2, \ldots, m$,

$$\begin{aligned} D^{\alpha}(T_i^*(t)) &= i \sum_{k=\lceil \alpha \rceil}^{i} (-1)^{i-k} 2^{2k} \frac{(k+i-1!)}{(i-k)!(2k!)L^i} D^{\alpha} t^k, \\ &= i \sum_{k=\lceil \alpha \rceil}^{i} (-1)^{i-k} 2^{2k} \frac{(k+i-1)!\Gamma(k+1)}{(i-k)!(2k!)\Gamma(k+1-\alpha)L^i} t^{k-\alpha}. \end{aligned} \tag{45}$$

Then

$$D^{\alpha} u_m(t)) = \sum_{i=0}^{m} c_i \Theta_{i,k}, \tag{46}$$

where $\Theta_{i,k}$ is given by

$$\Theta_{i,k} = i \sum_{k=\lceil \alpha \rceil}^{i} (-1)^{i-k} 2^{2k} \frac{(k+i-1)!\Gamma(K+1)}{(i-k)!(2k!)\Gamma(K+1-\alpha)L^i} t^{k-\alpha}. \tag{47}$$

## 5. Error analysis

In this section, special attention is given to study the convergence analysis and evaluate an upper bound of the error of the proposed formula.

THEOREM 5.1. (Chebyshev truncation theorem) [41] *The error in approximating $u(t)$ by the sum of its first m terms is bounded by the sum of the absolute values of all the neglected coefficients. If*

$$u_m(t) = \sum_{i=0}^{m} c_i T_i^*(t). \tag{48}$$

*then*

$$E_T(m) \equiv |u_m(t) - u(t)| \leqslant \sum_{i=m+1}^{\infty} c_i T_i^*(t), \tag{49}$$

*for all* $u(t)$, *all* $m$, *and all* $t \in [-1,1]$.

THEOREM 5.2. *The error* $E_T(m)$ *in approximating* $D^\alpha u(t)$ *and* $D^\alpha u_m(t)$ *is bounded by*

$$E_T(m) \leqslant \sum_{i=m+1}^{\infty} c_i \sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{k-\lceil\alpha\rceil} \Theta_{i,k} t^{k-1}, \tag{50}$$

*Proof.* A combination of Eqs. (41), (43) and (44)

$$\begin{aligned} |E_T(m)| &= |D^\alpha u_m(t) - D^\alpha u(t)| \\ &= |\sum_{i=m+1}^{\infty} c_i \sum_{k=\lceil\alpha\rceil}^{m} \sum_{j=0}^{k-\lceil\alpha\rceil} \Theta_{i,k} t^{k-1} T_j^*|, \end{aligned}$$

but $|T_j^*| \leqslant 1$ so, we can obtain

$$E_T(m) \leqslant |\sum_{i=m+1}^{\infty} c_i \sum_{i=\lceil\alpha\rceil}^{m} \sum_{j=0}^{k-\lceil\alpha\rceil} \Theta_{i,k} t^{k-1}|,$$

and subtracting the truncated series from the infinite series, bounding each term in the difference, and summing the bounds completes the proof of the theorem. □

## 5.1. Discretizations and numerical results

Consider the systems given in Eqs. (1)–(8) and (26)–(33). In order to use SCSM, we first approximate $S(t)$, $L_s(t)$, $L_m(t)$, $L_x(t)$, $I_s(t)$, $I_m(t)$, $I_x(t)$, $R(t)$, $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$, $\lambda_4(t)$, $\lambda_5(t)$, $\lambda_6(t)$, $\lambda_7(t)$ and $\lambda_8(t)$ as follows:

$$S(t) = \sum_{i=0}^{m} a_i T_i^*(t), \quad L_s(t) = \sum_{i=0}^{m} b_i T_i^*(t), \tag{51}$$

$$L_m(t) = \sum_{i=0}^{m} c_i T_i^*(t), \quad L_x(t) = \sum_{i=0}^{m} d_i T_i^*(t), \tag{52}$$

$$I_s(t) = \sum_{i=0}^{m} e_i T_i^*(t), \quad I_m(t) = \sum_{i=0}^{m} f_i T_i^*(t), \tag{53}$$

$$I_x(t) = \sum_{i=0}^{m} g_i T_i^*(t), \quad R(t) = \sum_{i=0}^{m} h_i T_i^*(t), \tag{54}$$

$$\lambda_1 = \sum_{i=0}^{m} k_i T_i^*(t), \quad \lambda_2 = \sum_{i=0}^{m} r_i T_i^*(t), \tag{55}$$

$$\lambda_3=\sum_{i=0}^{m}u_iT_i^*(t),\;\; \lambda_4=\sum_{i=0}^{m}v_iT_i^*(t), \tag{56}$$

$$\lambda_5=\sum_{i=0}^{m}w_iT_i^*(t),\;\; \lambda_6=\sum_{i=0}^{m}x_iT_i^*(t), \tag{57}$$

$$\lambda_7=\sum_{i=0}^{m}y_iT_i^*(t),\;\; \lambda_8=\sum_{i=0}^{m}z_iT_i^*(t). \tag{58}$$

Now we collocate the soluation at $m+1-\lceil\alpha\rceil$ points $t_p$, $(p=0,1,\ldots,m+1-\lceil\alpha\rceil)$ as:

$$\begin{aligned}
\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i}a_i\Theta_{i,k}t_p^{k-1}=&b^\alpha-d^\alpha\sum_{i=0}^{m}a_iT_i^*(t_p)-\beta_s^\alpha\frac{\sum_{i=0}^{m}a_iT_i^*(t_p)\sum_{i=0}^{m}e_iT_i^*(t_p)}{N}\\
&-\beta_m^\alpha\frac{\sum_{i=0}^{m}a_iT_i^*(t_p)}{N}\sum_{i=0}^{m}f_iT_i^*(t_p)\\
&-\beta_x^\alpha\frac{\sum_{i=0}^{m}a_iT_i^*(t_p)\sum_{i=0}^{m}g_iT_i^*(t_p)}{N},
\end{aligned}\tag{59}$$

$$\begin{aligned}
\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i}b_i\Theta_{i,k}t_p^{k-1}=&\lambda_s^\alpha\beta_s^\alpha\frac{\sum_{i=0}^{m}a_iT_i^*(t_p)\sum_{i=0}^{m}e_iT_i^*(t_p)}{N}\\
&+\sigma_s^\alpha\lambda_s^\alpha\beta_s^\alpha\frac{\sum_{i=0}^{m}h_iT_i^*(t_p)\sum_{i=0}^{m}e_iT_i^*(t_p)}{N}\\
&-\alpha_{ss}^\alpha\beta_s^\alpha\frac{\sum_{i=0}^{m}b_iT_i^*(t_p)\sum_{i=0}^{m}e_iT_i^*(t_p)}{N}\\
&-\alpha_{sm}^\alpha\beta_m^\alpha\frac{\sum_{i=0}^{m}b_iT_i^*(t_p)\sum_{i=0}^{m}f_iT_i^*(t_p)}{N}\\
&-\alpha_{sx}^\alpha\beta_x^\alpha\frac{\sum_{i=0}^{m}b_iT_i^*(t_p)\sum_{i=0}^{m}g_iT_i^*(t_p)}{N}+\gamma_s^\alpha\sum_{i=0}^{m}e_iT_i^*(t_p)\\
&-(d^\alpha+\varepsilon_s^\alpha+t_{1s}^\alpha+\varepsilon_1u_1(t))\sum_{i=0}^{m}b_iT_i^*(t_p),
\end{aligned}\tag{60}$$

$$\begin{aligned}
\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i}c_i\Theta_{i,k}t_p^{k-1}=&\lambda_m^\alpha\beta_m^\alpha\frac{\sum_{i=0}^{m}a_iT_i^*(t_p)\sum_{i=0}^{m}f_iT_i^*(t_p)}{N}\\
&+\sigma_m^\alpha\lambda_m^\alpha\beta_m^\alpha\frac{\sum_{i=0}^{m}h_iT_i^*(t_p)\sum_{i=0}^{m}f_iT_i^*(t)}{N}\\
&+\alpha_{sm}^\alpha\beta_m^\alpha\lambda_m^\alpha\frac{\sum_{i=0}^{m}b_iT_i^*(t)\sum_{i=0}^{m}f_iT_i^*(t)}{N}\\
&-\alpha_{mm}^\alpha\beta_m^\alpha\frac{\sum_{i=0}^{m}c_iT_i^*(t)\sum_{i=0}^{m}f_iT_i^*(t)}{N}\\
&-\alpha_{mx}^\alpha\beta_x^\alpha\frac{\sum_{i=0}^{m}c_iT_i^*(t)\sum_{i=0}^{m}g_iT_i^*(t)}{N}+\gamma_m^\alpha\sum_{i=0}^{m}f_iT_i^*(t_p)
\end{aligned}$$

$$
\begin{aligned}
&-(d^\alpha+\varepsilon_m^\alpha)\sum_{i=0}^{m} c_i T_i^*(t_p)+t_{1s}^\alpha\sum_{i=0}^{m} b_i T_i^*(t_p)\\
&+\varepsilon_1 u_1(t)\sum_{i=0}^{m} b_i T^*(t_p)-P_1^\alpha t_{1s}^\alpha\sum_{i=0}^{m} b_i T^*(t_p)+t_{2s}^\alpha\sum_{i=0}^{m} e_i T^*(t_p)\\
&+\varepsilon_2 u_2(t)\sum_{i=0}^{m} e_i T^*(t_p)-P_2^\alpha t_{2s}^\alpha\sum_{i=0}^{m} e_i T^*(t_p),
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} d_i\Theta_{i,k}t_p^{k-1}=&\lambda_x^\alpha\beta_x^\alpha\frac{\sum_{i=0}^m a_i T^*(t_p)\sum_{i=0}^m g_i T^*(t_p)}{N}\\
&+\sigma_x^\alpha\lambda_x^\alpha\beta_x^\alpha\frac{\sum_{i=0}^m h_i T^*(t_p)\sum_{i=0}^m g_i T^*(t_p)}{N}\\
&+\alpha_{sx}^\alpha\beta_x^\alpha\lambda_x^\alpha\frac{\sum_{i=0}^m b_i T^*(t_p)\sum_{i=0}^m g_i T^*(t_p)}{N}\\
&+\alpha_{mx}^\alpha\beta_x^\alpha\lambda_x^\alpha\frac{\sum_{i=0}^m c_i T^*(t_p)\sum_{i=0}^m g_i T^*(t_p)}{N}\\
&-\alpha_{xx}^\alpha\beta_x^\alpha\frac{\sum_{i=0}^m d_i T^*(t_p)\sum_{i=0}^m g_i T^*(t_p)}{N}-(d^\alpha+\varepsilon_x^\alpha)\sum_{i=0}^{m} d_i T^*(t_p)\\
&+\gamma_x^\alpha\sum_{i=0}^{m} g_i T^*(t_p)+t_{2m}^\alpha\sum_{i=0}^{m} f_i T^*(t_p)\\
&+\varepsilon_3 u_3(t)\sum_{i=0}^{m} f_i T^*(t_p)-P_3^\alpha t_{2m}^\alpha\sum_{i=0}^{m} f_i T^*(t_p),
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} e_i\Theta_{i,k}t_p^{k-1}=&\alpha_{ss}^\alpha\beta_s^\alpha\frac{\sum_{i=0}^m b_i T^*(t_p)\sum_{i=0}^m e_i T^*(t_p)}{N}\\
&+(1-\lambda_s^\alpha)\beta_s^\alpha\Bigg(\frac{\sum_{i=0}^m a_i T^*(t_p)\sum_{i=0}^m e_i T^*(t_p)}{N}\\
&+\sigma_s^\alpha\frac{\sum_{i=0}^m h_i T^*(t_p)\sum_{i=0}^m e_i T^*(t_p)}{N}\Bigg)+\varepsilon_s^\alpha\sum_{i=0}^{m} b_i T^*(t_p)\\
&-(d^\alpha+\delta_s^\alpha+t_{2s}^\alpha+\gamma_s^\alpha+\varepsilon_2 u_2(t))\sum_{i=0}^{m} e_i T^*(t_p),
\end{aligned}
\tag{63}
$$

$$
\begin{aligned}
\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} f_i\Theta_{i,k}t_p^{k-1}=&\alpha_{mm}^\alpha\beta_m^\alpha\frac{\sum_{i=0}^m c_i T^*(t_p)\sum_{i=0}^m f_i T^*(t)}{N}\\
&+(1-\lambda_m^\alpha)\beta_m^\alpha\Bigg(\frac{\sum_{i=0}^m a_i T^*(t_p)\sum_{i=0}^m f_i T^*(t_p)}{N}\\
&+\sigma_m^\alpha\frac{\sum_{i=0}^m h_i T^*(t_p)\sum_{i=0}^m f_i T^*(t_p)}{N}\\
&+\alpha_{sm}^\alpha\frac{\sum_{i=0}^m b_i T^*(t_p)\sum_{i=0}^m f_i T^*(t_p)}{N}\Bigg)+\varepsilon_m^\alpha\sum_{i=0}^{m} c_i T^*(t_p)
\end{aligned}
$$

$$
\begin{aligned}
&-(d^{\alpha}+\delta_m^{\alpha}+t_{2m}^{\alpha}+\varepsilon_3 u_3(t)+\gamma_m^{\alpha})\sum_{i=0}^{m} f_i T^*(t_p), \qquad (64)\\
\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} g_i\Theta_{i,k}t_p^{k-1} =&\alpha_{xx}^{\alpha}\beta_x^{\alpha}\frac{\sum_{i=0}^{m} d_iT^*(t_p)\sum_{i=0}^{m} g_iT^*(t_p)}{N}\\
&+(1-\lambda_x^{\alpha})\beta_x^{\alpha}\Big(\frac{\sum_{i=0}^{m} a_iT^*(t_p)\sum_{i=0}^{m} g_iT^*(t_p)}{N}\\
&+\sigma_x^{\alpha}\frac{\sum_{i=0}^{m} h_iT^*(t_p)\sum_{i=0}^{m} g_iT^*(t_p)}{N}\\
&+\alpha_{sx}^{\alpha}\frac{\sum_{i=0}^{m} b_iT^*(t_p)\sum_{i=0}^{m} g_iT^*(t_p)}{N}\\
&+\alpha_{mx}^{\alpha}\frac{\sum_{i=0}^{m} c_iT^*(t_p)\sum_{i=0}^{m} g_iT^*(t_p)}{N}\Big)+\varepsilon_x^{\alpha}\sum_{i=0}^{m} d_iT^*(t_p)\\
&-(d^{\alpha}+\delta_x^{\alpha}+t_{2x}^{\alpha}+\gamma_x^{\alpha}+\varepsilon_4 u_4(t))\sum_{i=0}^{m} g_iT^*(t_p), \qquad (65)\\
\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} h_i\Theta_{i,k}t_p^{k-1} =&P_1^{\alpha}t_{1s}^{\alpha}\sum_{i=0}^{m} b_iT^*(t_p)+P_2^{\alpha}t_{2s}^{\alpha}\sum_{i=0}^{m} e_iT^*(t_p)+P_3^{\alpha}t_{2m}^{\alpha}\sum_{i=0}^{m} f_iT^*(t_p)\\
&+t_{2x}^{\alpha}\sum_{i=0}^{m} g_iT^*(t_p)+\varepsilon_4 u_4(t)\sum_{i=0}^{m} g_iT^*(t_p)\\
&-\sigma_s^{\alpha}\beta_s^{\alpha}\frac{\sum_{i=0}^{m} h_iT_i^*(t_p)\sum_{i=0}^{m} e_iT^*(t_p)}{N}\\
&-\sigma_m^{\alpha}\beta_m^{\alpha}\frac{\sum_{i=0}^{m} h_iT^*(t_p)\sum_{i=0}^{m} f_iT^*(t_p)}{N}\\
&-\sigma_x^{\alpha}\beta_x^{\alpha}\frac{\sum_{i=0}^{m} h_iT^*(t_p)\sum_{i=0}^{m} g_iT^*(t_p)}{N}-d^{\alpha}\sum_{i=0}^{m} h_iT^*(t_p). \qquad (66)\\
\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} k_i\Theta_{i,k}t_p^{k-1} =&-\Big(\sum_{i=0}^{m} k_iT^*(t_p)\Big(\frac{\beta_s^{\alpha}}{N}\sum_{i=0}^{m} e_iT^*(t_p)+\frac{\beta_m^{\alpha}}{N}\sum_{i=0}^{m} f_iT^*(t_p)\\
&+\frac{\beta_x^{\alpha}}{N}\sum_{i=0}^{m} g_iT^*(t_p)+d^{\alpha}\Big)-\sum_{i=0}^{m} r_iT^*(t_p)\frac{\lambda_s^{\alpha}\beta_s^{\alpha}}{N}\sum_{i=0}^{m} e_iT^*(t_p)\\
&-\sum_{i=0}^{m} u_iT^*(t_p)\frac{\lambda_m^{\alpha}\beta_m^{\alpha}}{N}\sum_{i=0}^{m} f_iT^*(t_p)-\sum_{i=0}^{m} v_iT^*(t_p)\frac{\lambda_x^{\alpha}\beta_x^{\alpha}}{N}\sum_{i=0}^{m} e_iT^*(t_p)\\
&-\sum_{i=0}^{m} w_iT^*(t_p)\frac{(1-\lambda_s^{\alpha})\beta_s^{\alpha}}{N}\sum_{i=0}^{m} e_iT^*(t_p)\\
&-\sum_{i=0}^{m} x_iT^*(t_p)\frac{(1-\lambda_m^{\alpha})\beta_m^{\alpha}}{N}\sum_{i=0}^{m} f_iT^*(t_p)\\
&-\sum_{i=0}^{m} y_iT^*(t_p)\frac{(1-\lambda_x^{\alpha})\beta_x^{\alpha}}{N}\sum_{i=0}^{m} g_iT^*(t_p)\Big), \qquad (67)
\end{aligned}
$$

$$\begin{aligned}
\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} r_i\Theta_{i,k}t_p^{k-1} =& -\Big(-1+\sum_{i=0}^{m} r_iT^*(t_p)\Big(\frac{\beta_s^\alpha\alpha_{ss}^\alpha\sum_{i=0}^{m}e_iT^*(t_p)}{N} \\
&+\frac{\beta_m^\alpha\alpha_{sm}^\alpha\sum_{i=0}^{m}f_iT^*(t_p)}{N}+\frac{\beta_x^\alpha\alpha_{sx}^\alpha}{N}\Big)\Big) \\
&\times\sum_{i=0}^{m} g_iT^*\Big(t_p+\sum_{i=0}^{m} r_iT^*(t_p)(d^\alpha+\varepsilon_s^\alpha+t_{1s}^\alpha+\varepsilon_1 u_1^*(t)) \\
&-\sum_{i=0}^{m} u_iT^*(t_p)\frac{\lambda_m^\alpha\beta_m^\alpha\alpha_{sm}^\alpha}{N}\sum_{i=0}^{m} f_iT^*(t_p) \\
&-(t_{1s}^\alpha+\varepsilon_1u_1^*(t)-P_1t_{1s})\sum_{i=0}^{m} u_iT^*(t_p) \\
&-\sum_{i=0}^{m} v_iT^*(t_p)\frac{\lambda_x^\alpha\beta_x^\alpha\alpha_{sx}^\alpha}{N}\sum_{i=0}^{m} e_iT^*(t_p) \\
&-\sum_{i=0}^{m} w_iT^*(t_p)\frac{\beta_s^\alpha\alpha_{ss}^\alpha}{N}\sum_{i=0}^{m} e_iT^*(t_p)-\sum_{i=0}^{m} w_iT^*(t_p)\varepsilon_s^\alpha \\
&-\sum_{i=0}^{m} x_iT^*(t_p)(1-\lambda_m^\alpha)\frac{\beta_m^\alpha\alpha_{sm}^\alpha}{N}\sum_{i=0}^{m} f_iT^*(t_p) \\
&-\sum_{i=0}^{m} y_iT^*(t_p)(1-\lambda_x^\alpha)\frac{\beta_x^\alpha\alpha_{sx}^\alpha}{N}\sum_{i=0}^{m} g_iT^*(t_p)-P_1^\alpha t_{1s}^\alpha\sum_{i=0}^{m} z_iT^*(t_p)\Big),
\end{aligned} \tag{68}$$

$$\begin{aligned}
\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} u_i\Theta_{i,k}t_p^{k-1} =& -(\sum_{i=0}^{m} u_iT^*(t_p)\Big(\alpha_{mm}^\alpha\beta_m^\alpha\frac{\sum_{i=0}^{m}f_iT^*(t_p)}{N}+\alpha_{mx}^\alpha\beta_x^\alpha\frac{\sum_{i=0}^{m}g_iT^*(t_p)}{N} \\
&+d^\alpha+\varepsilon_m^\alpha\Big)-\sum_{i=0}^{m} v_iT^*(t_p)\alpha_{mx}^\alpha\beta_x^\alpha\lambda_x^\alpha\frac{\sum_{i=0}^{m}g_iT^*(t_p)}{N} \\
&-\sum_{i=0}^{m} x_iT^*(t_p)(\alpha_{mm}^\alpha\beta_m^\alpha\frac{\sum_{i=0}^{m}f_iT^*(t_p)}{N}+\varepsilon_m^\alpha) \\
&-\sum_{i=0}^{m} y_iT^*(t_p)(1-\lambda_x^\alpha)\alpha_{mx}^\alpha\beta_x^\alpha\frac{\sum_{i=0}^{m}g_iT^*(t_p)}{N}),
\end{aligned} \tag{69}$$

$$\begin{aligned}
\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} v_i\Theta_{i,k}t_p^{k-1} =& -\Big(\sum_{i=0}^{m} v_iT^*(t_p)(\alpha_{xx}^\alpha\beta_x^\alpha\frac{\sum_{i=0}^{m}g_iT^*(t_p)}{N}+d^\alpha+\varepsilon_x^\alpha) \\
&-\sum_{i=0}^{m} y_iT^*(t_p)\Big(\alpha_{xx}^\alpha\beta_x^\alpha\frac{\sum_{i=0}^{m}g_iT^*(t_p)}{N}-\varepsilon_x^\alpha\Big)\Big),
\end{aligned} \tag{70}$$

$$\begin{aligned}
\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} w_i\Theta_{i,k}t_p^{k-1} =& -(-1+\sum_{i=0}^{m} k_iT^*(t_p)\beta_s^\alpha\frac{\sum_{i=0}^{m}a_iT^*(t_p)}{N} \\
&+\sum_{i=0}^{m} r_iT^*(t_p)(\alpha_{ss}^\alpha\beta_s^\alpha\frac{\sum_{i=0}^{m}b_iT^*(t_p)}{N}
\end{aligned}$$

$$
\begin{aligned}
&- \lambda_s^\alpha \beta_s^\alpha \frac{\sum_{i=0}^m a_i T^*(t_p)}{N} - \lambda_s^\alpha \sigma_s^\alpha \beta_s^\alpha \frac{\sum_{i=0}^m h_i T^*(t_p)}{N} - \gamma_s^\alpha) \\
&+ \sum_{i=0}^m w_i T^*(t_p)\Big((d^\alpha + \delta_s^\alpha + t_{2s}^\alpha + \varepsilon_2 u_2^*(t) + \gamma_s^\alpha) \\
&- (1-\lambda_s^\alpha)\beta_s^\alpha \frac{\sum_{i=0}^m a_i T^*(t_p)}{N} - \sigma_s^\alpha (1-\lambda_s^\alpha)\beta_s^\alpha \frac{\sum_{i=0}^m h_i T^*(t_p)}{N} \\
&+ \frac{\beta_s^\alpha \alpha_{ss}^\alpha}{N} \sum_{i=0}^m b_i T^*(t_p)\Big) - \sum_{i=0}^m u_i T^*(t_p)(t_{2s} + \varepsilon_2 u_2^*(t) - P_2^\alpha t_{2s}^\alpha) \\
&+ \sum_{i=0}^m z_i T^*(t_p)(\sigma_s^\alpha \beta_s^\alpha \frac{\sum_{i=0}^m h_i T^*(t_p)}{N} - P_2^\alpha t_{2s}^\alpha)), \qquad (71)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=\lceil\alpha\rceil}^m \sum_{k=\lceil\alpha\rceil}^i x_i \Theta_{i,k} t_p^{k-1} = &- \Big(-1 + \sum_{i=0}^m k_i T^*(t_p)\beta_m^\alpha \frac{\sum_{i=0}^m a_i T^*(t_p)}{N} \\
&+ \sum_{i=0}^m r_i T^*(t_p)\Big(\alpha_{sm}^\alpha \beta_m^\alpha \frac{\sum_{i=0}^m b_i T^*(t_p)}{N} \\
&- \sum_{i=0}^m u_i T^*(t_p)(\lambda_m^\alpha \beta_m^\alpha \frac{\sum_{i=0}^m a_i T^*(t_p)}{N} + \lambda_m^\alpha \sigma_m^\alpha \beta_m^\alpha \frac{\sum_{i=0}^m h_i T^*(t_p)}{N} \\
&+ \frac{\lambda_m^\alpha \beta_m^\alpha \alpha_{sm}^\alpha}{N} \sum_{i=0}^m b_i T^*(t_p) - \frac{\beta_m^\alpha \alpha_{mm}^\alpha}{N} \sum_{i=0}^m c_i T^*(t_p) + \gamma_m^\alpha) \\
&- \sum_{i=0}^m v_i T^*(t_p) t_{2m} - \sum_{i=0}^m v_i T^*(t_p) \varepsilon_3 u_3^*(t) + \sum_{i=0}^m v_i T^*(t_p) P_3^\alpha t_{2m}^\alpha \\
&- \sum_{i=0}^m x_i T^*(t_p)\Big(\frac{\beta_m^\alpha \alpha_{mm}^\alpha}{N} \sum_{i=0}^m c_i T^*(t_p) + (1-\lambda_m^\alpha)\beta_m^\alpha \frac{\sum_{i=0}^m a_i T^*(t_p)}{N} \\
&+ \sigma_m^\alpha (1-\lambda_m^\alpha)\beta_m^\alpha \frac{\sum_{i=0}^m hiT^*(t_p)}{N} + \alpha_{sm}^\alpha (1-\lambda_m^\alpha)\beta_m^\alpha \frac{\sum_{i=0}^m biT^*(t_p)}{N} \\
&- (d^\alpha + \delta_m^\alpha + t_{2m}^\alpha + \varepsilon_3 u_3^*(t) + \gamma_m^\alpha)\Big) \\
&- \sum_{i=0}^m z_i T^*(t_p)\Big(P_3^\alpha t_{2m}^\alpha - \sigma_m^\alpha \beta_m^\alpha \frac{\sum_{i=0}^m h_i T^*(t_p)}{N}\Big)\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=\lceil\alpha\rceil}^m \sum_{k=\lceil\alpha\rceil}^i y_i \Theta_{i,k} t_p^{k-1} = &- \Big(-1 + \sum_{i=0}^m k_i T^*(t_p)\beta_x^\alpha \frac{\sum_{i=0}^m a_i T^*(t_p)}{N} \\
&+ \sum_{i=0}^m r_i T^*(t_p)\alpha_{sx}^\alpha \beta_x^\alpha \frac{\sum_{i=0}^m b_i T^*(t_p)}{N} \\
&+ \sum_{i=0}^m u_i T^*(t_p)\alpha_{mx}^\alpha \beta_x^\alpha \frac{\sum_{i=0}^m c_i T^*(t_p)}{N} \\
&- \sum_{i=0}^m v_i T^*(t_p)\Big(\lambda_x^\alpha \sigma_x^\alpha \beta_x^\alpha \frac{\sum_{i=0}^m h_i T^*(t_p)}{N} + \lambda_x^\alpha \beta_x^\alpha \frac{\sum_{i=0}^m a_i T^*(t_p)}{N}
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda_x^\alpha \alpha_{sx}^\alpha \beta_x^\alpha \frac{\sum_{i=0}^m b_i T^*(t_p)}{N} + \lambda_x^\alpha \alpha_{mx}^\alpha \beta_x^\alpha \frac{\sum_{i=0}^m c_i T^*(t_p)}{N} \\
&+\gamma_x^\alpha - \alpha_{xx}^\alpha \beta_x^\alpha \frac{\sum_{i=0}^m g_i T^*(t_p)}{N}\Big) - \sum_{i=0}^m y_i T^*(t_p)\Big(\alpha_{xx}^\alpha \beta_x^\alpha \frac{\sum_{i=0}^m d_i T^*(t_p)}{N} \\
&+(1-\lambda_x^\alpha)\beta_x^\alpha \frac{\sum_{i=0}^m a_i T^*(t_p)}{N} + (1-\lambda_x^\alpha)\beta_x^\alpha \alpha_{sx}^\alpha \frac{\sum_{i=0}^m b_i T^*(t_p)}{N} \\
&+(1-\lambda_x^\alpha)\beta_x^\alpha \sigma_x^\alpha \frac{\sum_{i=0}^m h_i T^*(t_p)}{N} \\
&+(1-\lambda_x^\alpha)\beta_x^\alpha \alpha_{mx}^\alpha \frac{\sum_{i=0}^m c_i T^*(t_p)}{N} + (d^\alpha + \delta_x^\alpha + t_{2x}^\alpha + \varepsilon_4 u_4^* + \gamma_x^\alpha)\Big) \\
&- \sum_{i=0}^m z_i T^*(t_p)\Big(t_{2x}^\alpha + \varepsilon_4 u_4^* - \sigma_x^\alpha \beta_x^\alpha \frac{\sum_{i=0}^m h_i T^*(t_p)}{N}\Big)\Big),
\end{aligned} \tag{72}
$$

$$
\begin{aligned}
\sum_{i=\lceil\alpha\rceil}^m \sum_{k=\lceil\alpha\rceil}^i z_i \Theta_{i,k} t_p^{k-1} = &-\Big(-\sum_{i=0}^m r_i T^*(t_p)\beta_s^\alpha \lambda_s^\alpha \sigma_s^\alpha \frac{\sum_{i=0}^m e_i T^*(t_p)}{N} \\
&- \sum_{i=0}^m u_i T^*(t_p)\beta_m^\alpha \lambda_m^\alpha \sigma_m^\alpha \frac{\sum_{i=0}^m f_i T^*(t_p)}{N} \\
&- \sum_{i=0}^m v_i T^*(t_p)\beta_x^\alpha \lambda_x^\alpha \sigma_x^\alpha \frac{\sum_{i=0}^m g_i T^*(t_p)}{N} \\
&- \sum_{i=0}^m w_i T^*(t_p)(1-\lambda_s^\alpha)\beta_s^\alpha \sigma_s^\alpha \frac{\sum_{i=0}^m e_i T^*(t_p)}{N} \\
&- \sum_{i=0}^m x_i T^*(t_p)(1-\lambda_m^\alpha)\beta_m^\alpha \sigma_m^\alpha \frac{\sum_{i=0}^m f_i T^*(t_p)}{N} \\
&- \sum_{i=0}^m y_i T^*(t_p)(1-\lambda_x^\alpha)\beta_x^\alpha \sigma_x^\alpha \frac{\sum_{i=0}^m g_i T^*(t_p)}{N} \\
&+ \sum_{i=0}^m z_i T^*(t_p)\Big(\sigma_s^\alpha \beta_s^\alpha \frac{\sum_{i=0}^m e_i T^*(t_p)}{N} + \sigma_m^\alpha \beta_m^\alpha \frac{\sum_{i=0}^m f_i T^*(t_p)}{N} \\
&+ \sigma_x^\alpha \beta_x^\alpha \frac{\sum_{i=0}^m g_i T^*(t_p)}{N} + d^\alpha\Big)\Big).
\end{aligned} \tag{73}
$$

In the following we will use the roots of shifted Chebyshev polynomials $T_i^*(t)$ as suitable collocation points. By substituting the initial conditions and the transeverslty conditions in Eqs. (51)–(58), we can obtain sixteen equations as follows:

$$
\sum_{i=0}^m (-1)^i a_i = S_0, \quad \sum_{i=0}^m (-1)^i b_i = L_{s0}, \quad \sum_{i=0}^m (-1)^i c_i = L_{m0}, \tag{74}
$$

$$
\sum_{i=0}^m (-1)^i d_i = L_{x0}, \quad \sum_{i=0}^m (-1)^i e_i = I_{s0}, \quad \sum_{i=0}^m (-1)^i f_i = I_{m0}, \tag{75}
$$

$$
\sum_{i=0}^m (-1)^i g_i = I_{x0}, \quad \sum_{i=0}^m (-1)^i h_i = R_0, \tag{76}
$$

$$\sum_{i=0}^{m}(-1)^i k_i=\lambda_1(t_f)=0,\quad \sum_{i=0}^{m}(-1)^i r_i=\lambda_2(t_f)=0,\quad \sum_{i=0}^{m}(-1)^i u_i=\lambda_3(t_f)=0, \tag{77}$$

$$\sum_{i=0}^{m}(-1)^i v_i=\lambda_4(t_f)=0,\quad \sum_{i=0}^{m}(-1)^i w_i=\lambda_5(t_f)=0,\quad \sum_{i=0}^{m}(-1)^i x_i=\lambda_6(t_f)=0, \tag{78}$$

$$\sum_{i=0}^{m}(-1)^i y_i=\lambda_7(t_f)=0,\quad \sum_{i=0}^{m}(-1)^i z_i=\lambda_8(t_f)=0. \tag{79}$$

Equations (59)–(73), together with the equations (74)–(79), give $(16m+16)$ of non-linear algebraic equations where $m$ is the degree of shifted Chebyshev polynomials, this algebraic equations can be solved using the Newton's iteration method for the unknowns $a_i$, $b_i$, $c_i$, $d_i$, $e_i$, $f_i$, $g_i$, $h_i$, $k_i$, $r_i$, $u_i$, $v_i$, $w_i$, $x_i$, $y_i$, and $z_i$, $i=0,1,\ldots,m$.

## 6. Numerical experiment

The purpose of this section is to show that, SCSM designed in this paper provides good approximations for the optimality system (1)–(8) and (26)–(33). The approximate solutions of the proposed system are given in Figures (1–8) by using SCSM and GEM. Also, using the initial condition $(S(0),L_s(0),L_m(0),L_x(0),I_s(0),I_m(0),I_x(0),R(0)) = \left(\frac{76}{120}N, \frac{20}{120}N, \frac{5}{120}N, \frac{2}{120}N, \frac{8}{120}N, \frac{4}{120}N, \frac{2}{120}N, \frac{3}{120}N\right)$, $m=8$ and the parameters in Table 3. Fig. 1, shows that $S(t)+L_s(t)+L_m(t)+L_x(t)+I_s(t)+I_m(t)+I_x(t)+R(t)/N$ is constant in time using SCSM for the controlled case when $0\leqslant u_k\leqslant 1$, where $k=1,2,3,4$ compared with the uncontrolled case i.e., $u_1=u_2=u_3=u_4=0$ with $\alpha=1$. Regarding the obtained results in Fig. 2, the effect of the controller on this model is

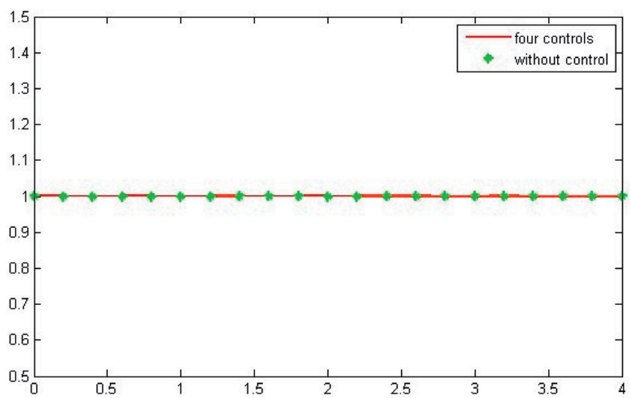


Figure 1: *Plot of* $S(t)+L_s(t)+L_m(t)+L_x(t)+I_s(t)+I_m(t)+I_x(t)+R(t)/N$ *versus* $t$ *in years by using SCSM for the controlled case when* $0\leqslant u_1,u_2,u_3,u_4\leqslant 1$, *compared with the uncontrolled case when* $u_1=u_2=u_3=u_4=0$.

reliable mainly for the two state variables $I_x(t)$ and $I_m(t)$. The effect of the controllers on other variables is not quite effective. Since Fig. 2, shows the numerical simulations of the model (1)–(8) for the controlled case using SCSM when $0 \leqslant u_k \leqslant 1$, compared with the uncontrolled case when $u_1 = u_2 = u_3 = u_4 = 0$. We note that the numbers $I_m$ and $I_x$ are larger in uncontrolled case compared with the controlled case. The number of $R(t), S(t)$ is larger in controlled case compared with the uncontrolled case. Also, from Table4, the value of objective functional is larger in uncontrolled case compared with the value of objective functional in controlled case. Fig. 3, shows the control variables $u_k^*$ in a time units of years by using SCSM.

Table 4: *Comparisons between the obtained result by using SCSM in controlled case and uncontrolled case, i.e., when* $u_1 = u_2 = u_3 = u_4 = 0$ *and* $T = 4$.

| | $J(u_1,u_2,u_3,u_4)$ | $L_s(4)+I_s(4)+I_m(4)+I_x(4)$ |
|---|---|---|
| With control | 16796.1772 | 1196 |
| Without control | 25783 | 3422 |

Table 5: *The values of objective functional by using SCSM with* $T = 4$ *and differieren values of* $\alpha$.

| $\alpha$ | $J(u_1,u_2,u_3,u_4)$ | $L_s(4)+I_s(4)+I_m(4)+I_x(4)$ |
|---|---|---|
| 0.90 | 16796.1772 | 1196 |
| 0.80 | 14685.8392 | 1882 |
| 0.7 | 15009.0953 | 1968 |
| 0.6 | 16944.1784 | 2055 |

Table 6: *Comparisons between GEM and SCSM where* $T = 4$ *and differieren values of* $\alpha$.

| $\alpha$ | Methods | $J(u_1,u_2,u_3,u_4)$ |
|---|---|---|
| 1 | GEM | 20938 |
| | SCSM | 16796.1772 |
| 0.98 | GEM | 19850 |
| | SCSM | 18132.8658 |
| 0.95 | GEM | 18277 |
| | SCSM | 17940.3829 |

Fig. 4, shows that $S(t)+L_s(t)+L_m(t)+L_x(t)+I_s(t)+I_m(t)+I_x(t)+R(t)/N$ is constant in time using GEM for the controlled case when $0 \leqslant u_k \leqslant 1$. Fig. 5, shows the control variables $u_k^*$ by using GEM at $\alpha = 1$. Fig. 6, shows the numerical simulations of the model (1)–(8) and (26)–(33) for the controlled case using GEM at different values of $\alpha$. Fig. (6–8), show that, how the fractional model is a generalization of the integer order model. In Table 5, the numerical value of sum the state variables $L_s$, $I_s$, $I_m$, and $I_x$ at $T = 4$ and the objective functional, obtained by SCSM with different values of $\alpha$. In Table 6, the values of the objective functional which are obtained by SCSM is compared with the results which obtained by GEM . From the numerical solutions, it is found that, the results which obtained by SCSM is better than GEM. All results were obtained by using MATLAB (R2013a). on a computer machine with intel(R) core $i3-3110M$ @ $2.40GHz$ and $4GB$ RAM.

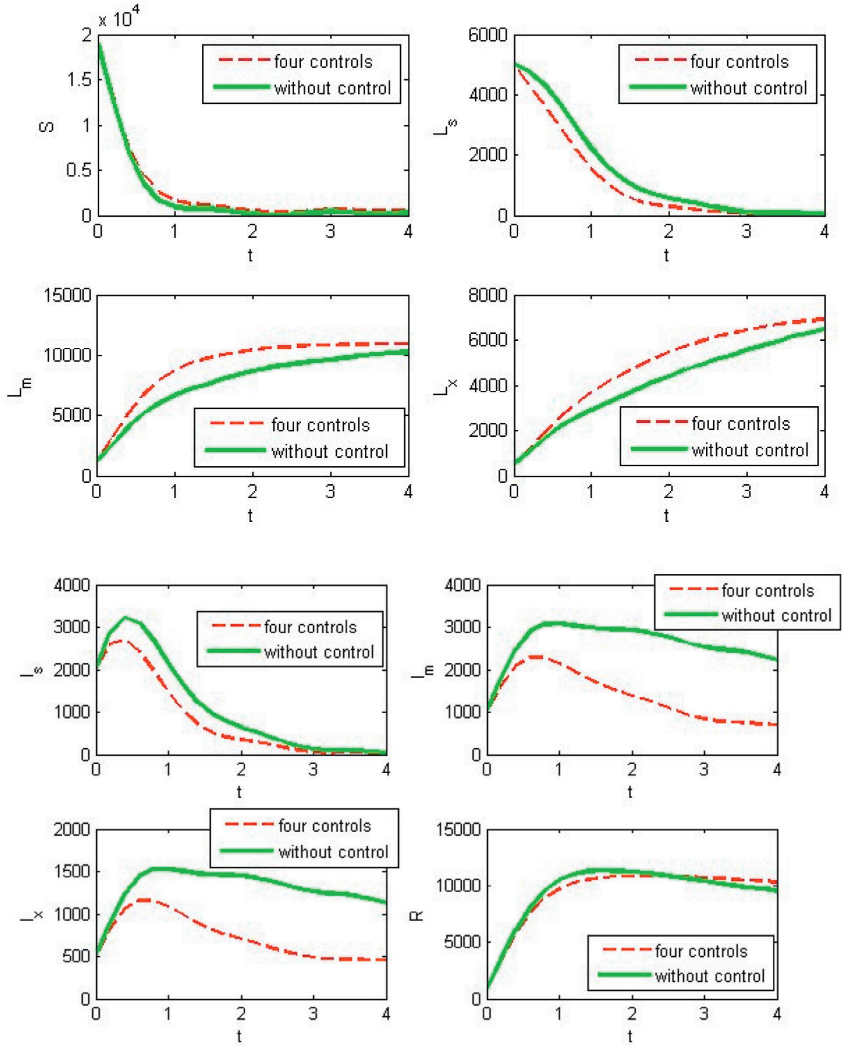


Figure 2: *The numerical simulations of the model (1)–(8) for the controlled case when $0 \leqslant u_1, u_2, u_3, u_4 \leqslant 1$, compared with the uncontrolled case when $u_1 = u_2 = u_3 = u_4 = 0$ by using SCSM .*

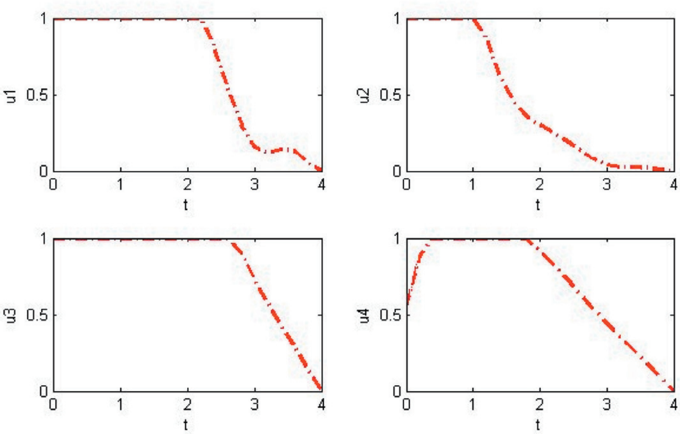


Figure 3: *The optimal control* $u_1^*, u_2^*, u_3^*, u_4^*$ *in a time units of years by using SCSM.*

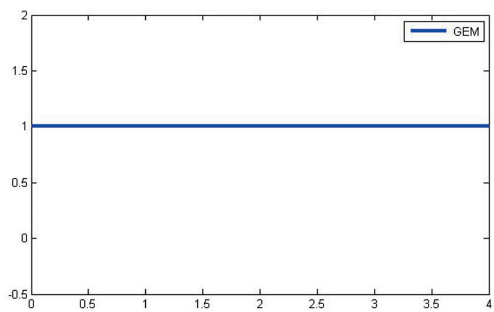


Figure 4: *Plot of* $S(t)+L_s(t)+L_m(t)+L_x(t)+I_s(t)+I_m(t)+I_x(t)+R(t)/N$ *versus* $t$ *in years by GEM.*

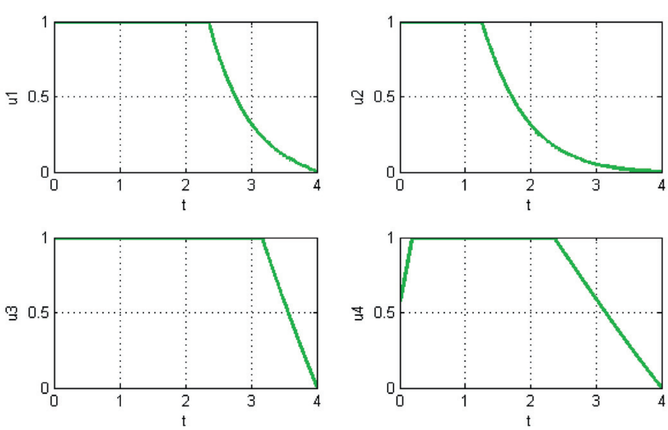


Figure 5: *The optimal control* $u_1^*, u_2^*, u_3^*, u_4^*$ *in a time units of years by using GEM.*

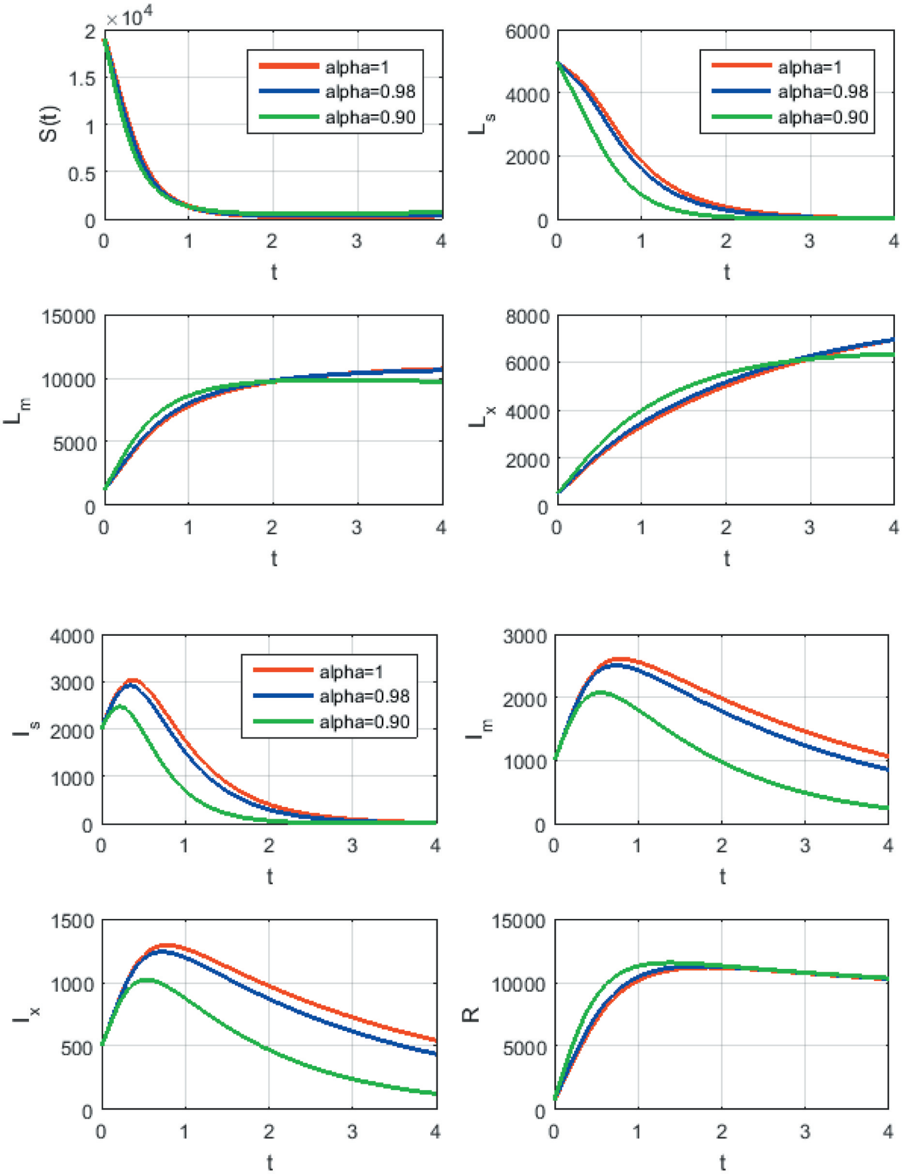


Figure 6: *The numerical simulations for the controlled case using GEM with different* $\alpha$.

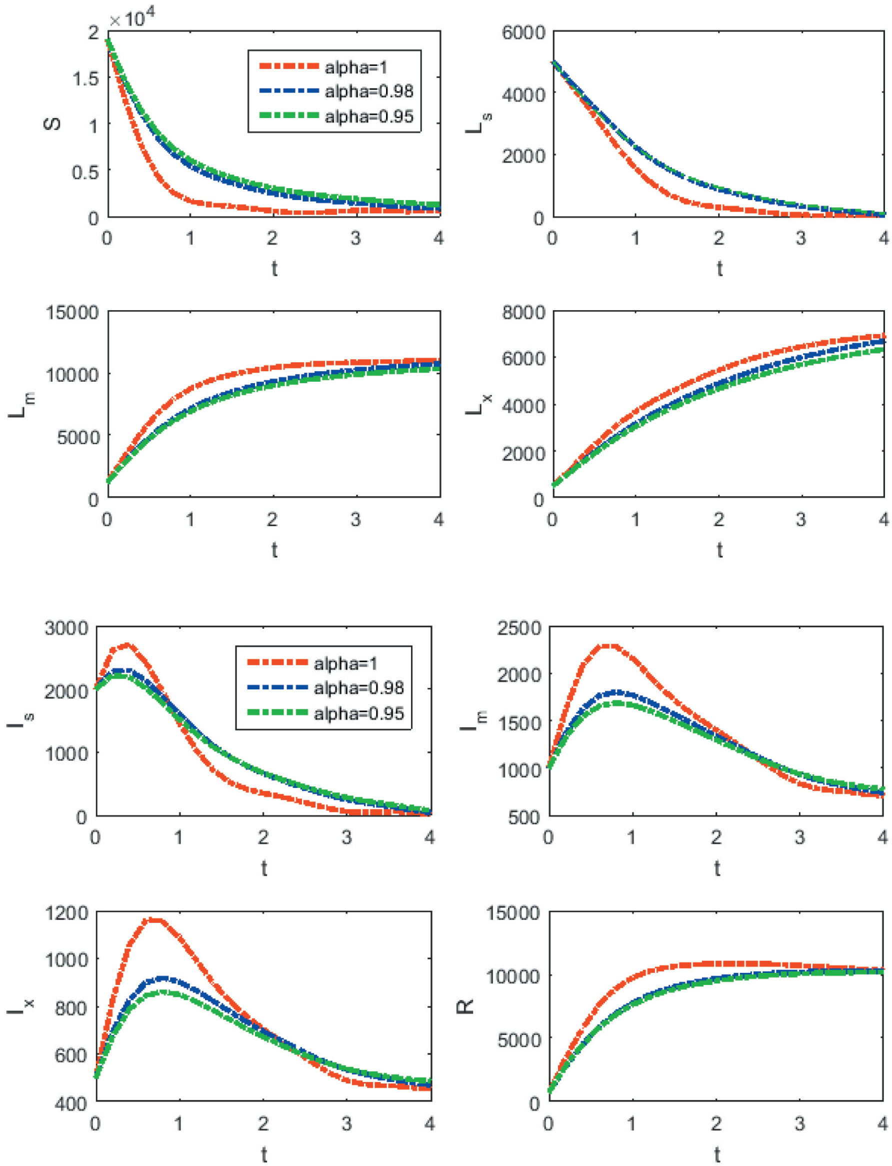


Figure 7: *The numerical simulations for the controlled case with different values of* $\alpha$ *using SCSM.*

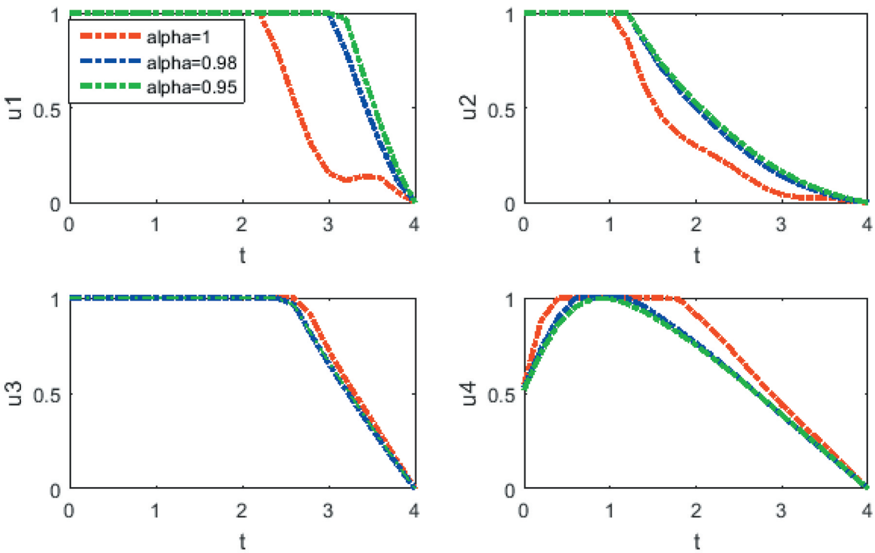


Figure 8: *The optimal control* $u_1^*, u_2^*, u_3^*, u_4^*$ *in a time units of years by using SCSM.*

## 7. Conclusions

In this paper, numerical solutions of the optimal control problem for multi-strain TB model are presented. Modified parameters are introduced to account for the fractional order model. Four controls functions $u_1$, $u_2$, $u_3$, and $u_4$, are introduced, these controls are given to reduce the number of active infected and latent TB individuals of first strain. The controls $u_2$, $u_3, and$ $u_4$, represents the effort that prevents the failure of treatment in active TB infectious individuals $I_s$, $I_m$, and $I_x$, e.g., supervising the patients, helping them to take the TB medications regularly and to complete the TB treatment, while the control $u_1$ governs the latent individuals $L_s$ under treatment with anti-TB drugs. Necessary and sufficient conditions that guarantee the existence and the uniqueness of the solution of the resulting systems are given. The optimality system is approximated by shifted Chebyshev polynomials which transformed the model problem to a system of algebraic equations with unknown coefficients. It is solved numerically using Newton's iteration method. Some figures are given to demonstrate how the fractional model is a generalization of the integer order model. Comparative studies are implemented between SCSM and GEM, It can be concluded from the numerical results presented in this paper that, the proposed method is better than GEM. Moreover, It can be concluded that fractional models have the potential to describe more complex dynamics than the integer models and can include easily the memory effect present in many real world phenomena.

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(Received May 4, 2017)

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