SOME k–FRACTIONAL INTEGRAL INEQUALITIES
FOR HARMONICALLY CONVEX FUNCTIONS

WAQAS AYUB AND GHULAM FARID

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Abstract. The celebrated Hadamard inequality has been studied extensively since it is established. We have found a weighted version of the Hadamard inequality for harmonically convex functions via Riemann-Liouville k-fractional integrals. Also, we have obtained some bounds of its difference. These results have some connection with fractional integral inequalities for Riemann-Liouville fractional integrals.

1. Introduction

In recent years the study of fractional integral inequalities is an important research subject in mathematical analysis. During the last two decades or so, several interesting and useful extensions of many of the fractional integral inequalities have been considered by several researchers. We are interested to give new Hadamard type inequalities for harmonically convex functions via Riemann-Liouville k-fractional integrals.

Let $f \in L_1[a,b]$. The Riemann-Liouville fractional integrals of order $\alpha > 0$ with $a \geqslant 0$ are defined as follows:

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \ \ x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \ \ x < b,$$

where $\Gamma(.)$ is the Gamma function defined as

$$\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du,$$

one can note that

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

and

$$J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x).$$

For further details one may see [3, 5, 6].

In [4], there is given definition of Riemann-Liouville k-fractional integrals as follows.

Let $f \in L_1[a,b]$. Then Riemann-Liouville k-fractional integrals of order $\alpha, k > 0$ with $a \geqslant 0$ are defined as

$$J_{a+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)\,dt, \ \ x \geqslant a$$

and

$$J_{b-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)\,dt, \ \ x \leqslant b$$

where $\Gamma_k(\alpha)$ is the k-Gamma function defined as

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}}\,dt.$$

One can note that

$$\Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha)$$

and

$$J_{a+}^{0,1} f(x) = J_{b-}^{0,1} f(x) = f(x).$$

For $k = 1$, k-fractional integrals give Riemann-Liouville fractional integrals.

In the following we define the Beta and the Hypergeometric functions, respectively:

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1} (1-t)^{y-1}\,dt, \ \ x,y > 0,$$

$$_2F_1(a,b;c;z) = \frac{1}{\beta(b,c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a}\,dt,$$

$c > b > 0$, $|z| < 1$. For more details of above functions one may see [12].

Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function defined on the interval I and $a,b \in I$, $a < b$. Then the following inequality holds

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

It is well known in the literature as the Hadamard inequality.

In [8], Fejér established the following inequality which is the weighted generalization of the Hadamard inequality.

THEOREM 1.1. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a convex function. Then the inequalities*

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x)\,dx \leqslant \int_a^b f(x)g(x)\,dx \leqslant \frac{f(a)+f(b)}{2} \int_a^b g(x)\,dx, \tag{2}$$

hold, where $g : [a,b] \longrightarrow \mathbb{R}$ is nonnegative, integrable and symmteric with respect to $\frac{a+b}{2}$.

In [7], İşcan gave the definition of harmonically convex functions.

DEFINITION 1.2. Let $I \subset \mathbb{R} - \{0\}$ be a real interval. A function $f : I \longrightarrow \mathbb{R}$ is said to be harmonically convex, if

$$f \left(\frac{xy}{tx + (1-t)y} \right) \leqslant tf(y) + (1-t)f(x) \tag{3}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (3) is reversed, then f is said to be harmonically concave.

DEFINITION 1.3. [9] A function $g : [a,b] \subset \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ is said to be harmonically symmetric with respect to $\frac{2ab}{a+b}$, if

$$g(x) = g \left(\frac{1}{\frac{1}{a} + \frac{1}{b} - x} \right) \tag{4}$$

holds for all $x \in [a,b]$.

We will use the following lemma in sequel to prove some of our results.

LEMMA 1.4. [10, 11] *For $0 < \alpha \leqslant 1$ and $0 \leqslant a < b$ we have*

$$|a^\alpha - b^\alpha| \leqslant (b-a)^\alpha.$$

In this paper we give a weighted version of the Hadamard inequality for harmonically convex functions via Riemann-Liouville k-fractional integrals. We also find some bounds of a difference of this inequality by using harmonically convexity of functions $|f'|^q$, $q \geqslant 1$, via Riemann-Liouville k-fractional integrals. Also we find some results of [2] in particular.

2. Main results

LEMMA 2.1. *If $g : [a,b] \subseteq \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ is integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then*

$$J_{\frac{1}{b}+}^{\alpha,k}(g \circ h) \left(\frac{1}{a} \right) = J_{\frac{1}{a}-}^{\alpha,k}(g \circ h) \left(\frac{1}{b} \right) \tag{5}$$

$$= \frac{1}{2} \left[J_{\frac{1}{b}+}^{\alpha,k}(g \circ h) \left(\frac{1}{a} \right) + J_{\frac{1}{a}-}^{\alpha,k}(g \circ h) \left(\frac{1}{b} \right) \right]$$

with $\alpha, k > 0$ and $h(x) = \frac{1}{x}$, $x \in [\frac{1}{b}, \frac{1}{a}]$.

Proof. Since g is harmonically symmetric with respect to $\frac{2ab}{a+b}$. By substitution $t = \frac{1}{a} + \frac{1}{b} - x$ in the following integral and doing some calculation we get

$$J_{\frac{1}{b}+}^{\alpha,k}(g \circ h)\left(\frac{1}{a}\right) = \frac{1}{k\Gamma_k(\alpha)} \int_{\frac{1}{b}}^{\frac{1}{a}} \left(\frac{1}{a} - t\right)^{\frac{\alpha}{k}-1} g\left(\frac{1}{t}\right) dt$$

$$= -\frac{1}{k\Gamma_k(\alpha)} \int_{\frac{1}{b}}^{\frac{1}{b}} \left(x - \frac{1}{b}\right)^{\frac{\alpha}{k}-1} g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right) dx$$

$$= \frac{1}{k\Gamma_k(\alpha)} \int_{\frac{1}{b}}^{\frac{1}{b}} \left(x - \frac{1}{b}\right)^{\frac{\alpha}{k}-1} g\left(\frac{1}{x}\right) dx$$

$$= J_{\frac{1}{a}-}^{\alpha,k}(g \circ h)\left(\frac{1}{b}\right).$$

From which we get equality in (5). □

REMARK 2.2. If we put $k = 1$ in (5), then we get [2, Lemma 2].

THEOREM 2.3. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a harmonically convex function with $a < b$ and $f \in L[a,b]$. If $g : [a,b] \longrightarrow \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then the following inequalities for k-fractional integrals hold*

$$f\left(\frac{2ab}{a+b}\right)\left[J_{\frac{1}{b}+}^{\alpha,k}(g \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(g \circ h)\left(\frac{1}{b}\right)\right]$$

$$\leqslant \left[J_{\frac{1}{b}+}^{\alpha,k}(fg \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(fg \circ h)\left(\frac{1}{b}\right)\right] \qquad (6)$$

$$\leqslant \frac{f(a)+f(b)}{2}\left[J_{\frac{1}{b}+}^{\alpha,k}(g \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(g \circ h)\left(\frac{1}{b}\right)\right]$$

with $\alpha, k > 0$ and $h(x) = \frac{1}{x}$, $x \in [\frac{1}{b}, \frac{1}{a}]$.

Proof. Since f is harmonically convex. Using (3) for $t = 1/2$ and making substitution $x = \frac{ab}{tb+(1-t)a}$, $y = \frac{ab}{ta+(1-t)b}$, then multiplying resulting inequality by $2t^{\frac{\alpha}{k}-1}g\left(\frac{ab}{tb+(1-t)a}\right)$, and integrating over $[0,1]$ we have

$$2\int_0^1 t^{\frac{\alpha}{k}-1}g\left(\frac{ab}{tb+(1-t)a}\right)f\left(\frac{2ab}{a+b}\right)dt$$

$$\leqslant \int_0^1 t^{\frac{\alpha}{k}-1}g\left(\frac{ab}{tb+(1-t)a}\right)f\left(\frac{ab}{tb+(1-t)a}\right)dt$$

$$+ \int_0^1 t^{\frac{\alpha}{k}-1}g\left(\frac{ab}{tb+(1-t)a}\right)f\left(\frac{ab}{ta+(1-t)b}\right)dt.$$

Setting $\frac{tb+(1-t)a}{ab} = x$ and using that g is harmonically symmetric we have

$$2f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\frac{\alpha}{k}}\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-1}g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)dx$$

$$\leqslant \int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\frac{\alpha}{k}}\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-1}f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)g\left(\frac{1}{x}\right)dx$$

$$+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\frac{\alpha}{k}}\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-1}g\left(\frac{1}{x}\right)f\left(\frac{1}{x}\right)dx$$

$$2\left(\frac{ab}{b-a}\right)^{\frac{\alpha}{k}}f\left(\frac{2ab}{a+b}\right)k\Gamma_k(\alpha)J_{\frac{1}{a}-}^{\alpha,k}(g\circ h)\left(\frac{1}{b}\right)$$

$$\leqslant \left(\frac{ab}{b-a}\right)^{\frac{\alpha}{k}}k\Gamma_k(\alpha)\left[J_{\frac{1}{b}+}^{\alpha,k}(fg\circ h)\left(\frac{1}{a}\right)+J_{\frac{1}{a}-}^{\alpha,k}(fg\circ h)\left(\frac{1}{b}\right)\right].$$

Using Lemma 2.1 on left hand side of above inequality we have

$$\left(\frac{ab}{b-a}\right)^{\frac{\alpha}{k}}f\left(\frac{2ab}{a+b}\right)k\Gamma_k(\alpha)\left[J_{\frac{1}{b}+}^{\alpha,k}(g\circ h)\left(\frac{1}{a}\right)+J_{\frac{1}{a}-}^{\alpha,k}(g\circ h)\left(\frac{1}{b}\right)\right]$$

$$\leqslant \left(\frac{ab}{b-a}\right)^{\frac{\alpha}{k}}k\Gamma_k(\alpha)\left[J_{\frac{1}{b}+}^{\alpha,k}(fg\circ h)\left(\frac{1}{a}\right)+J_{\frac{1}{a}-}^{\alpha,k}(fg\circ h)\left(\frac{1}{b}\right)\right]$$

and the first inequality of (6) is proved.

On the other hand by harmonically convexity of f we have

$$f\left(\frac{ab}{tb+(1-t)a}\right)+f\left(\frac{ab}{ta+(1-t)b}\right)\leqslant f(a)+f(b). \qquad (7)$$

For the proof of the second inequality in (6), multiplying (7) by $t^{\frac{\alpha}{k}-1}g\left(\frac{ab}{tb+(1-t)a}\right)$, then integrating over $[0,1]$ we obtain

$$\int_0^1 t^{\frac{\alpha}{k}-1}g\left(\frac{ab}{tb+(1-t)a}\right)f\left(\frac{ab}{tb+(1-t)a}\right)dt$$

$$+\int_0^1 t^{\frac{\alpha}{k}-1}g\left(\frac{ab}{tb+(1-t)a}\right)f\left(\frac{ab}{ta+(1-t)b}\right)dt$$

$$\leqslant [f(a)+f(b)]\int_0^1 t^{\frac{\alpha}{k}-1}g\left(\frac{ab}{tb+(1-t)a}\right)dt.$$

Again setting $\frac{tb+(1-t)a}{ab} = x$ and doing some calculation we get the second inequality in (6). $\square$

REMARK 2.4. (i) If we put $k = 1$ in (6), then we get [2, Theorem 5].
(ii) If we put $\alpha, k = 1$ in (6), then we get [2, Theorem 4].
(iii) If we put $g(x) = 1$ along with $k = 1$ in (6), then we get [2, Theorem 3].
(iv) If we put $g(x) = 1$ along with $\alpha, k = 1$ in (6), then we get [2, Theorem 2].

LEMMA 2.5. *Let $f : I \subset (0, \infty) \longrightarrow \mathbb{R}$ be a differentiable function on I^o, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $g : [a,b] \longrightarrow \mathbb{R}$ is integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then the following equality for k-fractional integrals holds*

$$\frac{f(a) + f(b)}{2} \left[J_{\frac{1}{b}+}^{\alpha,k} (g \circ h) \left(\frac{1}{a} \right) + J_{\frac{1}{a}-}^{\alpha,k} (g \circ h) \left(\frac{1}{b} \right) \right]$$

$$- \left[J_{\frac{1}{b}+}^{\alpha,k} (fg \circ h) \left(\frac{1}{a} \right) + J_{\frac{1}{a}-}^{\alpha,k} (fg \circ h) \left(\frac{1}{b} \right) \right]$$

$$= \frac{1}{k\Gamma_k(\alpha)} \left[\int_{\frac{1}{b}}^{\frac{1}{a}} \left(\int_{\frac{1}{b}}^{t} \left(\frac{1}{a} - s \right)^{\frac{\alpha}{k}-1} (g \circ h)(s)\, ds \right) (f \circ h)'(t)\, dt \right] \tag{8}$$

$$- \frac{1}{k\Gamma_k(\alpha)} \left[\int_{\frac{1}{b}}^{\frac{1}{a}} \left(\int_{t}^{\frac{1}{a}} \left(s - \frac{1}{b} \right)^{\frac{\alpha}{k}-1} (g \circ h)(s)\, ds \right) (f \circ h)'(t)\, dt \right]$$

with $\alpha, k > 0$ and $h(x) = \frac{1}{x}$, $x \in [\frac{1}{b}, \frac{1}{a}]$.

Proof. Taking and solving the terms on right hand side as follows

$$\int_{\frac{1}{b}}^{\frac{1}{a}} \left(\int_{\frac{1}{b}}^{t} \left(\frac{1}{a} - s \right)^{\frac{\alpha}{k}-1} (g \circ h)(s)\, ds \right) (f \circ h)'(t)\, dt$$

$$= \left(\int_{\frac{1}{b}}^{t} \left(\frac{1}{a} - s \right)^{\frac{\alpha}{k}-1} (g \circ h)(s)\, ds \right) (f \circ h)(t) \Bigg|_{\frac{1}{b}}^{\frac{1}{a}}$$

$$- \int_{\frac{1}{b}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\frac{\alpha}{k}-1} (g \circ h)(t) (f \circ h)(t)\, dt$$

$$= (f \circ h) \left(\frac{1}{a} \right) \int_{\frac{1}{b}}^{\frac{1}{a}} \left(\frac{1}{a} - s \right)^{\frac{\alpha}{k}-1} (g \circ h)(s)\, ds$$

$$- \int_{\frac{1}{b}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\frac{\alpha}{k}-1} (fg \circ h)(t)\, dt$$

$$= f(a) k\Gamma_k(\alpha) J_{\frac{1}{b}+}^{\alpha,k} (g \circ h) \left(\frac{1}{a} \right) - k\Gamma_k(\alpha) J_{\frac{1}{b}+}^{\alpha,k} (fg \circ h) \left(\frac{1}{a} \right)$$

$$= k\Gamma_k(\alpha) \left[f(a) J_{\frac{1}{b}+}^{\alpha,k} (g \circ h) \left(\frac{1}{a} \right) - J_{\frac{1}{b}+}^{\alpha,k} (fg \circ h) \left(\frac{1}{a} \right) \right],$$

using Lemma 2.1 we have

$$\int_{\frac{1}{b}}^{\frac{1}{a}} \left(\int_{\frac{1}{b}}^{t} \left(s - \frac{1}{b} \right)^{\frac{\alpha}{k}-1} (g \circ h)(s)\, ds \right) (f \circ h)'(t)\, dt \tag{9}$$

$$= k\Gamma_k(\alpha) \left[f(a) \frac{J_{\frac{1}{b}+}^{\alpha,k} (g \circ h) (\frac{1}{a}) + J_{\frac{1}{a}-}^{\alpha,k} (g \circ h) (\frac{1}{b})}{2} - J_{\frac{1}{b}+}^{\alpha,k} (fg \circ h) \left(\frac{1}{a} \right) \right],$$

and

$$\int_{\frac{1}{b}}^{\frac{1}{a}} \left(\int_{t}^{\frac{1}{a}} \left(s - \frac{1}{b}\right)^{\frac{\alpha}{k}-1} (g \circ h)(s)\, ds \right) (f \circ h)'(t)\, dt$$

$$= \left(\int_{t}^{\frac{1}{a}} \left(s - \frac{1}{b}\right)^{\frac{\alpha}{k}-1} (g \circ h)(s)\, ds \right) (f \circ h)(t) \,\bigg|_{\frac{1}{b}}^{\frac{1}{a}} \qquad (10)$$

$$+ \int_{\frac{1}{b}}^{\frac{1}{a}} \left(t - \frac{1}{b}\right)^{\frac{\alpha}{k}-1} (g \circ h)(t)(f \circ h)(t)\, dt$$

$$= -f(b) \int_{\frac{1}{b}}^{\frac{1}{a}} \left(s - \frac{1}{b}\right)^{\frac{\alpha}{k}-1} (g \circ h)(s)\, ds + \int_{\frac{1}{b}}^{\frac{1}{a}} \left(t - \frac{1}{b}\right)^{\frac{\alpha}{k}-1} (fg \circ h)(t)\, dt$$

$$= -f(b)\, k\Gamma_k(\alpha)\, J_{\frac{1}{a}-}^{\alpha,k}(g \circ h)\left(\frac{1}{b}\right) + k\Gamma_k(\alpha)\, J_{\frac{1}{a}-}^{\alpha,k}(fg \circ h)\left(\frac{1}{b}\right),$$

using Lemma 2.1 we have

$$\int_{\frac{1}{b}}^{\frac{1}{a}} \left(\int_{t}^{\frac{1}{a}} \left(s - \frac{1}{b}\right)^{\frac{\alpha}{k}-1} (g \circ h)(s)\, ds \right) (f \circ h)'(t)\, dt \qquad (11)$$

$$= -k\Gamma_k(\alpha) \left[f(b) \left(\frac{J_{\frac{1}{b}+}^{\alpha,k}(g \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(g \circ h)\left(\frac{1}{b}\right)}{2} \right) - J_{\frac{1}{a}-}^{\alpha,k}(fg \circ h)\left(\frac{1}{b}\right) \right].$$

Using (9) and (11) in right hand side of (8) we get the left hand side of (8). Hence (8) is established. □

REMARK 2.6. (i) If we put $k = 1$ in (8), then we get [2, Lemma 3].
(ii) If we put $g(x) = 1$ along with $k = 1$ in (8), then we get [1, Lemma 3].

THEOREM 2.7. *Let $f : I \subset (0, \infty) \longrightarrow \mathbb{R}$ be a differentiable function on I^o, the interior of I, such that $f' \in L[a,b]$, where $a,b \in I$ and $a < b$. If $|f'|$ is harmonically convex on $[a,b]$, $g : [a,b] \longrightarrow \mathbb{R}$ is continuous and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then the following inequality for k-fractional integrals holds*

$$\left| \frac{f(a)+f(b)}{2} \left[J_{\frac{1}{b}+}^{\alpha,k}(g \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(g \circ h)\left(\frac{1}{b}\right) \right] \qquad (12)$$

$$- \left[J_{\frac{1}{b}+}^{\alpha,k}(fg \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(fg \circ h)\left(\frac{1}{b}\right) \right] \right|$$

$$\leqslant \frac{\|g\|_\infty ab(b-a)}{\Gamma_k(\alpha+k)} \left(\frac{b-a}{ab} \right)^{\frac{\alpha}{k}} \left[C_1\left(\frac{\alpha}{k}\right) |f'(a)| + C_2\left(\frac{\alpha}{k}\right) |f'(b)| \right]$$

where $\| g \|_{\infty} = \sup_{t \in [a,b]} | g(t) |$ *and*

$$C_1 \left(\frac{\alpha}{k} \right) = \frac{b^{-2}}{\left(\frac{\alpha}{k} + 2 \right)} {}_2F_1 \left(2, 1; \frac{\alpha}{k} + 3; 1 - \frac{a}{b} \right)$$

$$- \frac{b^{-2}}{\left(\frac{\alpha}{k} + 1 \right) \left(\frac{\alpha}{k} + 2 \right)} {}_2F_1 \left(2, \frac{\alpha}{k} + 1; \frac{\alpha}{k} + 3; 1 - \frac{a}{b} \right),$$

$$+ \frac{2 \left(a + b \right)^{-2}}{\left(\frac{\alpha}{k} + 1 \right) \left(\frac{\alpha}{k} + 2 \right)} {}_2F_1 \left(2, \frac{\alpha}{k} + 1; \frac{\alpha}{k} + 3; \frac{b - a}{b + a} \right),$$

$$C_2 \left(\frac{\alpha}{k} \right) = \frac{b^{-2}}{\left(\frac{\alpha}{k} + 1 \right) \left(\frac{\alpha}{k} + 2 \right)} {}_2F_1 \left(2, 2; \frac{\alpha}{k} + 3; 1 - \frac{a}{b} \right)$$

$$- \frac{b^{-2}}{\left(\frac{\alpha}{k} + 2 \right)} {}_2F_1 \left(2, \frac{\alpha}{k} + 2; \frac{\alpha}{k} + 3; 1 - \frac{a}{b} \right)$$

$$+ \frac{\left(\frac{a+b}{2} \right)^{-2}}{\left(\frac{\alpha}{k} + 1 \right)} {}_2F_1 \left(2, \frac{\alpha}{k} + 1; \frac{\alpha}{k} + 2; \frac{b - a}{b + a} \right),$$

$$- \frac{2 \left(a + b \right)^{-2}}{\left(\frac{\alpha}{k} + 1 \right) \left(\frac{\alpha}{k} + 2 \right)} {}_2F_1 \left(2, \frac{\alpha}{k} + 1; \frac{\alpha}{k} + 3; \frac{b - a}{b + a} \right),$$

with $\alpha \leqslant k$ *and* $h(x) = \frac{1}{x}$, $x \in [\frac{1}{b}, \frac{1}{a}]$.

Proof. Using Lemma 2.5 we have

$$\left| \frac{f(a) + f(b)}{2} \left[J_{\frac{1}{b}+}^{\alpha,k} (g \circ h) \left(\frac{1}{a} \right) + J_{\frac{1}{a}-}^{\alpha,k} (g \circ h) \left(\frac{1}{b} \right) \right] \right. \tag{13}$$

$$\left. - \left[J_{\frac{1}{b}+}^{\alpha,k} (fg \circ h) \left(\frac{1}{a} \right) + J_{\frac{1}{a}-}^{\alpha,k} (fg \circ h) \left(\frac{1}{b} \right) \right] \right|$$

$$\leqslant \frac{1}{k \Gamma_k (\alpha)} \int_{\frac{1}{b}}^{\frac{1}{a}} \left| \int_{\frac{1}{b}}^{t} \left(\frac{1}{a} - s \right)^{\frac{\alpha}{k} - 1} (g \circ h)(s) \, ds \right.$$

$$\left. - \int_{t}^{\frac{1}{a}} \left(s - \frac{1}{b} \right)^{\frac{\alpha}{k} - 1} (g \circ h)(s) \, ds \right| |(f \circ h)'(t)| \, dt.$$

Using harmonically symmetricity of g with respect to $\frac{2ab}{a+b}$ and computing modulus in the integrand of right hand side as follows

$$\left| \int_{\frac{1}{b}}^{t} \left(\frac{1}{a} - s \right)^{\frac{\alpha}{k} - 1} (g \circ h)(s) \, ds - \int_{t}^{\frac{1}{a}} \left(s - \frac{1}{b} \right)^{\frac{\alpha}{k} - 1} (g \circ h)(s) \, ds \right|$$

$$= \left| \int_{\frac{1}{a} + \frac{1}{b} - t}^{\frac{1}{a}} \left(s - \frac{1}{b} \right)^{\frac{\alpha}{k} - 1} (g \circ h)(s) \, ds + \int_{\frac{1}{a}}^{t} \left(s - \frac{1}{b} \right)^{\frac{\alpha}{k} - 1} (g \circ h)(s) \, ds \right|$$

$$= \left| \int_{\frac{1}{a}+\frac{1}{b}-t}^{t} \left(s-\frac{1}{b}\right)^{\frac{\alpha}{k}-1} (g \circ h)(s)\, ds \right|$$

$$\leqslant \begin{cases} \left| \int_{t}^{\frac{1}{a}+\frac{1}{b}-t} \left(s-\frac{1}{b}\right)^{\frac{\alpha}{k}-1} (g \circ h)(s)\, ds \right|, & t \in \left[\frac{1}{b}, \frac{a+b}{2ab}\right] \\[4mm] \left| \int_{\frac{1}{a}+\frac{1}{b}-t}^{t} \left(s-\frac{1}{b}\right)^{\frac{\alpha}{k}-1} (g \circ h)(s)\, ds \right|, & t \in \left[\frac{a+b}{2ab}, \frac{1}{a}\right]. \end{cases} \tag{14}$$

Therefore using (14) in (13) we get

$$\left| \frac{f(a)+f(b)}{2} \left[J_{\frac{1}{b}+}^{\alpha,k}(g \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(g \circ h)\left(\frac{1}{b}\right) \right] \right.$$

$$\left. - \left[J_{\frac{1}{b}+}^{\alpha,k}(fg \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(fg \circ h)\left(\frac{1}{b}\right) \right] \right|$$

$$\leqslant \frac{1}{k\Gamma_k(\alpha)} \int_{\frac{1}{b}}^{\frac{a+b}{2ab}} \left(\int_{t}^{\frac{1}{a}+\frac{1}{b}-t} \left| \left(s-\frac{1}{b}\right)^{\frac{\alpha}{k}-1} (g \circ h)(s) \right| ds \right) |(f \circ h)'(t)|\, dt$$

$$+ \frac{1}{k\Gamma_k(\alpha)} \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{t} \left| \left(s-\frac{1}{b}\right)^{\frac{\alpha}{k}-1} (g \circ h)(s) \right| ds \right) |(f \circ h)'(t)|\, dt$$

$$\leqslant \frac{\|g\|_\infty}{k\Gamma_k(\alpha)} \int_{\frac{1}{b}}^{\frac{a+b}{2ab}} \left(\int_{t}^{\frac{1}{a}+\frac{1}{b}-t} \left(s-\frac{1}{b}\right)^{\frac{\alpha}{k}-1} ds \right) \cdot \frac{1}{t^2} \left| f'\left(\frac{1}{t}\right) \right| dt$$

$$+ \frac{\|g\|_\infty}{k\Gamma_k(\alpha)} \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{t} \left(s-\frac{1}{b}\right)^{\frac{\alpha}{k}-1} ds \right) \cdot \frac{1}{t^2} \left| f'\left(\frac{1}{t}\right) \right| dt$$

$$= \frac{\|g\|_\infty}{\alpha\Gamma_k(\alpha)} \int_{\frac{1}{b}}^{\frac{a+b}{2ab}} \left[\left(\frac{1}{a}-t\right)^{\frac{\alpha}{k}} - \left(t-\frac{1}{b}\right)^{\frac{\alpha}{k}} \right] \cdot \frac{1}{t^2} \left| f'\left(\frac{1}{t}\right) \right| dt$$

$$+ \frac{\|g\|_\infty}{\alpha\Gamma_k(\alpha)} \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \left[\left(t-\frac{1}{b}\right)^{\frac{\alpha}{k}} - \left(\frac{1}{a}-t\right)^{\frac{\alpha}{k}} \right] \cdot \frac{1}{t^2} \left| f'\left(\frac{1}{t}\right) \right| dt.$$

Setting $\frac{ub+(1-u)a}{ab} = t$ in the resulting inequality obtained from above we have

$$\left| \frac{f(a)+f(b)}{2} \left[J_{\frac{1}{b}+}^{\alpha,k}(g \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(g \circ h)\left(\frac{1}{b}\right) \right] \right. \tag{15}$$

$$\left. - \left[J_{\frac{1}{b}+}^{\alpha,k}(fg \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(fg \circ h)\left(\frac{1}{b}\right) \right] \right|$$

$$\leqslant \frac{\|g\|_\infty (ab)^2}{\alpha\Gamma_k(\alpha)} \left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1} \int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} \left| f'\left(\frac{ab}{ub+(1-u)a}\right) \right| du$$

$$+ \frac{\|g\|_\infty (ab)^2}{\alpha\Gamma_k(\alpha)} \left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1} \int_{\frac{1}{2}}^1 \frac{u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} \left| f'\left(\frac{ab}{ub+(1-u)a}\right) \right| du.$$

Using harmonically convexity of $|f'|$ on $[a,b]$, we have

$$\left| \frac{f(a)+f(b)}{2} \left[J_{\frac{1}{b}+}^{\alpha,k}(g\circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(g\circ h)\left(\frac{1}{b}\right) \right] \right. \tag{16}$$

$$\left. - \left[J_{\frac{1}{b}+}^{\alpha,k}(fg\circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(fg\circ h)\left(\frac{1}{b}\right) \right] \right|$$

$$\leqslant \frac{\| g \|_\infty (ab)^2}{\alpha\Gamma_k(\alpha)} \left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1}$$

$$\times \left[\int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2}udu + \int_{\frac{1}{2}}^1 \frac{u^{\frac{\alpha}{k}}-(1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2}udu \right] |f'(a)|$$

$$+ \frac{\| g \|_\infty (ab)^2}{\alpha\Gamma_k(\alpha)} \left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1}$$

$$\times \left[\int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2}(1-u)du + \int_{\frac{1}{2}}^1 \frac{u^{\frac{\alpha}{k}}-(1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2}(1-u)du \right] |f'(b)|.$$

We calculate the integrals appeared on the right hand side of (16) as follows

$$\int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2}udu + \int_{\frac{1}{2}}^1 \frac{u^{\frac{\alpha}{k}}-(1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2}udu$$

$$= \int_0^1 \frac{u^{\frac{\alpha}{k}}-(1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2}udu + 2\int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2}udu$$

$$\leqslant \int_0^1 \frac{u^{\frac{\alpha}{k}+1}}{(ub+(1-u)a)^2}du - \int_0^1 \frac{(1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2}udu$$

$$+ 2\int_0^{\frac{1}{2}} \frac{(1-2u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2}udu, \tag{17}$$

from here we take the right hand side as

$$\int_0^1 \frac{(1-u)^{\frac{\alpha}{k}+1}}{(ua+(1-u)b)^2}du - \int_0^1 \frac{(1-u)u^{\frac{\alpha}{k}}}{(ua+(1-u)b)^2}du$$

$$+ \frac{1}{2}\int_0^1 \frac{(1-u)^{\frac{\alpha}{k}}}{(\frac{u}{2}b+(1-\frac{u}{2})a)^2}udu$$

$$= b^{-2}\int_0^1 (1-u)^{\frac{\alpha}{k}+1}\left(1-u\left(1-\frac{a}{b}\right)\right)^{-2}du$$

$$- b^{-2}\int_0^1 (1-u)u^{\frac{\alpha}{k}}\left(1-u\left(1-\frac{a}{b}\right)\right)^{-2}du$$

$$+ \frac{\left(\frac{a+b}{2}\right)^{-2}}{2}\int_0^1 (1-v)v^{\frac{\alpha}{k}}\left(1-v\left(\frac{b-a}{b+a}\right)\right)^{-2}dv$$

$$= b^{-2} \beta \left(1, \frac{\alpha}{k} + 2\right) {}_2F_1 \left(2, 1; \frac{\alpha}{k} + 3; 1 - \frac{a}{b}\right)$$

$$- b^{-2} \beta \left(\frac{\alpha}{k} + 1, 2\right) {}_2F_1 \left(2, \frac{\alpha}{k} + 1; \frac{\alpha}{k} + 3; 1 - \frac{a}{b}\right)$$

$$+ \frac{\left(\frac{a+b}{2}\right)^{-2}}{2} \beta \left(\frac{\alpha}{k} + 1, 2\right) {}_2F_1 \left(2, \frac{\alpha}{k} + 1; \frac{\alpha}{k} + 3; \frac{b-a}{b+a}\right).$$

Using the definition of β-function we have

$$\int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub + (1-u)a)^2} u \, du + \int_{\frac{1}{2}}^1 \frac{u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}}{(ub + (1-u)a)^2} u \, du \tag{18}$$

$$\leqslant \frac{b^{-2}}{\left(\frac{\alpha}{k} + 2\right)} {}_2F_1 \left(2, 1; \frac{\alpha}{k} + 3; 1 - \frac{a}{b}\right)$$

$$- \frac{b^{-2}}{\left(\frac{\alpha}{k} + 1\right)\left(\frac{\alpha}{k} + 2\right)} {}_2F_1 \left(2, \frac{\alpha}{k} + 1; \frac{\alpha}{k} + 3; 1 - \frac{a}{b}\right)$$

$$+ \frac{2\,(a+b)^{-2}}{\left(\frac{\alpha}{k} + 1\right)\left(\frac{\alpha}{k} + 2\right)} {}_2F_1 \left(2, \frac{\alpha}{k} + 1; \frac{\alpha}{k} + 3; \frac{b-a}{b+a}\right)$$

$$= C_1 \left(\frac{\alpha}{k}\right),$$

and similarly we get

$$\int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub + (1-u)a)^2} (1-u) \, du + \int_{\frac{1}{2}}^1 \frac{u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}}{(ub + (1-u)a)^2} (1-u) \, du$$

$$\leqslant \int_0^1 \frac{(1-u)^{\frac{\alpha}{k}}}{(ua + (1-u)b)^2} u \, du - \int_0^1 \frac{u^{\frac{\alpha}{k} + 1}}{(ua + (1-u)b)^2} du$$

$$+ \int_0^1 \frac{(1-u)^{\frac{\alpha}{k}}}{\left(\frac{u}{2}b + \left(1 - \frac{u}{2}\right)a\right)^2} du - \frac{1}{2} \int_0^1 \frac{u(1-u)^{\frac{\alpha}{k}}}{\left(\frac{u}{2}b + \left(1 - \frac{u}{2}\right)a\right)^2} du$$

$$= b^{-2} \int_0^1 u(1-u)^{\frac{\alpha}{k}} \left(1 - u\left(1 - \frac{a}{b}\right)\right)^{-2} du$$

$$- b^{-2} \int_0^1 u^{\frac{\alpha}{k} + 1} \left(1 - u\left(1 - \frac{a}{b}\right)\right)^{-2} du$$

$$+ \left(\frac{a+b}{2}\right)^{-2} \int_0^1 v^{\frac{\alpha}{k}} \left(1 - v\left(\frac{b-a}{b+a}\right)\right)^{-2} dv$$

$$- \frac{\left(\frac{a+b}{2}\right)^{-2}}{2} \int_0^1 v^{\frac{\alpha}{k}} (1-v) \left(1 - v\left(\frac{b-a}{b+a}\right)\right)^{-2} dv$$

$$= b^{-2} \beta \left(2, \frac{\alpha}{k} + 1\right) {}_2F_1 \left(2, 2; \frac{\alpha}{k} + 3; 1 - \frac{a}{b}\right)$$

$$- b^{-2} \beta \left(\frac{\alpha}{k} + 2, 1\right) {}_2F_1 \left(2, \frac{\alpha}{k} + 2; \frac{\alpha}{k} + 3; 1 - \frac{a}{b}\right)$$

$$+ \left(\frac{a+b}{2}\right)^{-2} \beta\left(\frac{\alpha}{k}+1,1\right) {}_2F_1\left(2,\frac{\alpha}{k}+1;\frac{\alpha}{k}+2;\frac{b-a}{b+a}\right)$$

$$- \frac{\left(\frac{a+b}{2}\right)^{-2}}{2} \beta\left(\frac{\alpha}{k}+1,2\right) {}_2F_1\left(2,\frac{\alpha}{k}+1;\frac{\alpha}{k}+3;\frac{b-a}{b+a}\right).$$

Using the definition of β-function we have

$$\int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2}(1-u)\,du + \int_{\frac{1}{2}}^1 \frac{u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2}(1-u)\,du \qquad (19)$$
$$\leqslant C_2\left(\frac{\alpha}{k}\right).$$

Using (18) and (19) in (16), we get (12). $\quad\square$

REMARK 2.8. (i) If we put $k = 1$ in (12), then we get [2, Theorem 6].
(ii) If we put $\alpha, k = 1$ in (12), then we get [2, Corollary 1 (1)].
(iii) If we put $g(x), k = 1$ in (12), then we get [2, Corollary 1 (2)].
(iv) If we put $\alpha, k = 1$ and $g(x) = 1$ in (12), then we get [2, Corollary 1 (3)].

THEOREM 2.9. *Let $f : I \subset (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I^o, such that $f' \in L[a,b]$, where $a,b \in I$ and $a < b$. If $|f'|^q$, $q \geqslant 1$, is harmonically convex on $[a,b]$, $g : [a,b] \longrightarrow \mathbb{R}$ is continuous and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then the following inequality for k-fractional integrals holds*

$$\left| \frac{f(a)+f(b)}{2} \left[J^{\alpha,k}_{\frac{1}{b}+}(g \circ h)\left(\frac{1}{a}\right) + J^{\alpha,k}_{\frac{1}{a}-}(g \circ h)\left(\frac{1}{b}\right) \right] \right. \qquad (20)$$

$$\left. - \left[J^{\alpha,k}_{\frac{1}{b}+}(fg \circ h)\left(\frac{1}{a}\right) + J^{\alpha,k}_{\frac{1}{a}-}(fg \circ h)\left(\frac{1}{b}\right) \right] \right|$$

$$\leqslant \frac{\|g\|_\infty ab(b-a)}{\Gamma_k(\alpha+k)} \left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}} \left[C_3^{1-\frac{1}{q}}\left(\frac{\alpha}{k}\right) \left[C_4\left(\frac{\alpha}{k}\right)|f'(a)|^q + C_5\left(\frac{\alpha}{k}\right)|f'(b)|^q \right]^{\frac{1}{q}} \right.$$

$$\left. + C_6^{1-\frac{1}{q}}\left(\frac{\alpha}{k}\right) \left[C_7\left(\frac{\alpha}{k}\right)|f'(a)|^q + C_8\left(\frac{\alpha}{k}\right)|f'(b)|^q \right]^{\frac{1}{q}} \right]$$

where $\| g \|_\infty = sup_{t\in[a,b]} | g(t) |$ and

$$C_3\left(\frac{\alpha}{k}\right) = \frac{2(a+b)^{-2}}{\left(\frac{\alpha}{k}+1\right)} {}_2F_1\left(2,\frac{\alpha}{k}+1;\frac{\alpha}{k}+3;\frac{b-a}{b+a}\right),$$

$$C_4\left(\frac{\alpha}{k}\right) = \frac{(a+b)^{-2}}{\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)} {}_2F_1\left(2,\frac{\alpha}{k}+1;\frac{\alpha}{k}+3;\frac{b-a}{b+a}\right),$$

$$C_5\left(\frac{\alpha}{k}\right) = C_3\left(\frac{\alpha}{k}\right) - C_4\left(\frac{\alpha}{k}\right),$$

$$C_6\left(\frac{\alpha}{k}\right) = \frac{b^{-2}}{\left(\frac{\alpha}{k}+1\right)} {}_2F_1\left(2,1;\frac{\alpha}{k}+2;1-\frac{a}{b}\right)$$

$$-\frac{b^{-2}}{\left(\frac{\alpha}{k}+1\right)} {}_2F_1\left(2,\frac{\alpha}{k}+1;\frac{\alpha}{k}+2;1-\frac{a}{b}\right) + C_3\left(\frac{\alpha}{k}\right),$$

$$C_7\left(\frac{\alpha}{k}\right) = \frac{b^{-2}}{\left(\frac{\alpha}{k}+2\right)} {}_2F_1\left(2,1;\frac{\alpha}{k}+3;1-\frac{a}{b}\right)$$

$$-\frac{b^{-2}}{\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)} {}_2F_1\left(2,\frac{\alpha}{k}+1;\frac{\alpha}{k}+3;1-\frac{a}{b}\right) + C_4\left(\frac{\alpha}{k}\right),$$

$$C_8\left(\frac{\alpha}{k}\right) = C_6\left(\frac{\alpha}{k}\right) - C_7\left(\frac{\alpha}{k}\right),$$

with $\alpha \leqslant k$ *and* $h(x) = \frac{1}{x}$, $x \in \left[\frac{1}{b},\frac{1}{a}\right]$.

Proof. Using (15), power mean inequality and the harmonically convexity of $|f'|^q$

$$\left| \frac{f(a)+f(b)}{2} \left[J^{\alpha,k}_{\frac{1}{b}+}(g\circ h)\left(\frac{1}{a}\right) + J^{\alpha,k}_{\frac{1}{a}-}(g\circ h)\left(\frac{1}{b}\right) \right] \right.$$

$$\left. - \left[J^{\alpha,k}_{\frac{1}{b}+}(fg\circ h)\left(\frac{1}{a}\right) + J^{\alpha,k}_{\frac{1}{a}-}(fg\circ h)\left(\frac{1}{b}\right) \right] \right|$$

$$\leqslant \frac{\|g\|_\infty (ab)^2}{\alpha\Gamma_k(\alpha)}\left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1} \int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right| du$$

$$+ \frac{\|g\|_\infty (ab)^2}{\alpha\Gamma_k(\alpha)}\left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1} \int_{\frac{1}{2}}^1 \frac{u^{\frac{\alpha}{k}}-(1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right| du$$

from here we take the right hand side as

$$\leqslant \frac{\|g\|_\infty (ab)^2}{\Gamma_k(\alpha+k)}\left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1} \left(\int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right|^q du\right)^{\frac{1}{q}}$$

$$+ \frac{\|g\|_\infty (ab)^2}{\Gamma_k(\alpha+k)}\left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1} \left(\int_{\frac{1}{2}}^1 \frac{u^{\frac{\alpha}{k}}-(1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}$$

$$\times \left(\int_{\frac{1}{2}}^1 \frac{u^{\frac{\alpha}{k}}-(1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right| du\right)^{\frac{1}{q}}.$$

Using harmonically convexity of $|f'|^q$ we have

$$\left| \frac{f(a)+f(b)}{2} \left[J_{\frac{1}{b}+}^{\alpha,k}(g \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(g \circ h)\left(\frac{1}{b}\right) \right] \right.$$

$$\left. - \left[J_{\frac{1}{b}+}^{\alpha,k}(fg \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(fg \circ h)\left(\frac{1}{b}\right) \right] \right|$$

$$\leqslant \frac{\|g\|_\infty (ab)^2}{\Gamma_k(\alpha+k)} \left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1} \left(\int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} du \right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} \left[u|f'(a)|^q + (1-u)|f'(b)|^q \right] du \right)^{\frac{1}{q}}$$

$$+ \frac{\|g\|_\infty (ab)^2}{\Gamma_k(\alpha+k)} \left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1} \left(\int_{\frac{1}{2}}^1 \frac{u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} du \right)^{1-\frac{1}{q}}$$

$$\times \left(\int_{\frac{1}{2}}^1 \frac{u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} \left[u|f'(a)|^q + (1-u)|f'(b)|^q \right] du \right)^{\frac{1}{q}}.$$

From which we have

$$\left| \frac{f(a)+f(b)}{2} \left[J_{\frac{1}{b}+}^{\alpha,k}(g \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(g \circ h)\left(\frac{1}{b}\right) \right] \right. \tag{21}$$

$$\left. - \left[J_{\frac{1}{b}+}^{\alpha,k}(fg \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(fg \circ h)\left(\frac{1}{b}\right) \right] \right|$$

$$\leqslant \frac{\|g\|_\infty (ab)^2}{\Gamma_k(\alpha+k)} \left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1} \left(\int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} du \right)^{1-\frac{1}{q}}$$

$$\times \left(|f'(a)|^q \int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} u\,du + |f'(b)|^q \int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} (1-u)\,du \right)^{\frac{1}{q}}$$

$$+ \frac{\|g\|_\infty (ab)^2}{\Gamma_k(\alpha+k)} \left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1} \left(\int_{\frac{1}{2}}^1 \frac{u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} du \right)^{1-\frac{1}{q}}$$

$$\times \left(|f'(a)|^q \int_{\frac{1}{2}}^1 \frac{u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} u\,du + |f'(b)|^q \int_{\frac{1}{2}}^1 \frac{u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} (1-u)\,du \right)^{\frac{1}{q}}.$$

By calculating the appearing integrals on right hand side of (21) we have

$$\int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} du \leqslant \int_0^{\frac{1}{2}} \frac{(1-2u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} du$$

$$= \frac{1}{2} \int_0^1 \frac{(1-u)^{\frac{\alpha}{k}}}{\left(\frac{u}{2}b + \left(1-\frac{u}{2}\right)a\right)^2} du$$

$$= 2(a+b)^{-2} \int_0^1 v^{\frac{\alpha}{k}} \left(1 - v\left(\frac{b-a}{b+a}\right)\right)^{-2} dv$$

$$= 2(a+b)^{-2} \beta\left(\frac{\alpha}{k}+1,1\right) {}_2F_1\left(2,\frac{\alpha}{k}+1;\frac{\alpha}{k}+2;\frac{b-a}{b+a}\right).$$

Using the definition of β-function we have

$$\int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} du \leqslant \frac{2(a+b)^{-2}}{\frac{\alpha}{k}+1} {}_2F_1\left(2,\frac{\alpha}{k}+1;\frac{\alpha}{k}+2;\frac{b-a}{b+a}\right)$$

$$= C_3\left(\frac{\alpha}{k}\right), \tag{22}$$

using Lemma 1.4 we have

$$\int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} u\,du \leqslant \int_0^{\frac{1}{2}} \frac{(1-2u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} u\,du$$

$$= \frac{1}{4} \int_0^1 \frac{u(1-u)^{\frac{\alpha}{k}}}{\left(\frac{u}{2}b + \left(1-\frac{u}{2}\right)a\right)^2} u\,du$$

$$= \frac{\left(\frac{a+b}{2}\right)^{-2}}{4} \int_0^1 v^{\frac{\alpha}{k}}(1-v)\left(1 - v\left(\frac{b-a}{b+a}\right)\right)^{-2} dv$$

$$= (a+b)^{-2} \beta\left(\frac{\alpha}{k}+1,2\right) {}_2F_1\left(2,\frac{\alpha}{k}+1;\frac{\alpha}{k}+3;\frac{b-a}{b+a}\right)$$

$$= \frac{(a+b)^{-2}}{\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)} {}_2F_1\left(2,\frac{\alpha}{k}+1;\frac{\alpha}{k}+3;\frac{b-a}{b+a}\right)$$

$$= C_4\left(\frac{\alpha}{k}\right), \tag{23}$$

$$\int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2}(1-u)\,du \leqslant C_3\left(\frac{\alpha}{k}\right) - C_4\left(\frac{\alpha}{k}\right) = C_5\left(\frac{\alpha}{k}\right) \tag{24}$$

$$\int_{\frac{1}{2}}^1 \frac{u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} du = \int_0^1 \frac{u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} du + \int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} u\,du$$

$$= \int_0^1 \frac{u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} du - \int_0^1 \frac{(1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} du + \int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} u\,du$$

$$= \int_0^1 \frac{(1-v)^{\frac{\alpha}{k}}}{(va+(1-v)b)^2} dv - \int_0^1 \frac{v^{\frac{\alpha}{k}}}{(va+(1-v)b)^2} dv + \int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} u du$$

$$= b^{-2} \int_0^1 (1-v)^{\frac{\alpha}{k}} \left(1 - v\left(1 - \frac{a}{b}\right)\right)^{-2} dv - b^{-2} \int_0^1 v^{\frac{\alpha}{k}} \left(1 - v\left(1 - \frac{a}{b}\right)\right)^{-2} dv$$

$$+ \int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} u du$$

$$= b^{-2} \beta \left(1, \frac{\alpha}{k} + 1\right) {}_2F_1 \left(2, 1; \frac{\alpha}{k} + 2; 1 - \frac{a}{b}\right)$$

$$- b^{-2} \beta \left(\frac{\alpha}{k} + 1, 1\right) {}_2F_1 \left(2, \frac{\alpha}{k} + 1; \frac{\alpha}{k} + 2; 1 - \frac{a}{b}\right) + \int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} u du.$$

Using the definition of β-function and (22) we have

$$\int_{\frac{1}{2}}^1 \frac{u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} du \tag{25}$$

$$\leqslant \frac{b^{-2}}{(\frac{\alpha}{k}+1)} {}_2F_1 \left(2, 1; \frac{\alpha}{k} + 2; 1 - \frac{a}{b}\right)$$

$$- \frac{b^{-2}}{(\frac{\alpha}{k}+1)} {}_2F_1 \left(2, \frac{\alpha}{k} + 1; \frac{\alpha}{k} + 2; 1 - \frac{a}{b}\right) + C_3 \left(\frac{\alpha}{k}\right)$$

$$= C_6 \left(\frac{\alpha}{k}\right)$$

$$\int_{\frac{1}{2}}^1 \frac{u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} u du \tag{26}$$

$$= \int_0^1 \frac{u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} u du + \int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} u du$$

$$= \int_0^1 \frac{u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} u du - \int_0^1 \frac{(1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} u du + \int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} u du$$

$$= \int_0^1 \frac{(1-v)^{\frac{\alpha}{k}+1}}{(va+(1-v)b)^2} dv - \int_0^1 \frac{(1-v)v^{\frac{\alpha}{k}}}{(va+(1-v)b)^2} dv + \int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} u du$$

$$= b^{-2} \int_0^1 (1-v)^{\frac{\alpha}{k}+1} \left(1 - v\left(1 - \frac{a}{b}\right)\right)^{-2} dv - b^{-2} \int_0^1 v^{\frac{\alpha}{k}} (1-v) \left(1 - v\left(1 - \frac{a}{b}\right)\right)^{-2} dv$$

$$+ \int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} u du$$

$$= b^{-2} \beta \left(1, \frac{\alpha}{k} + 2\right) {}_2F_1 \left(2, 1; \frac{\alpha}{k} + 3; 1 - \frac{a}{b}\right)$$

$$- b^{-2} \beta \left(\frac{\alpha}{k} + 1, 2\right) {}_2F_1 \left(2, \frac{\alpha}{k} + 1; \frac{\alpha}{k} + 3; 1 - \frac{a}{b}\right) + \int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} u du$$

Using the definition of β-function and (23) we have

$$\int_{\frac{1}{2}}^{1} \frac{u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} u\, du \leqslant \frac{b^{-2}}{\left(\frac{\alpha}{k}+2\right)} {}_2F_1\left(2,1;\frac{\alpha}{k}+3;1-\frac{a}{b}\right) \tag{27}$$

$$- \frac{b^{-2}}{\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)} {}_2F_1\left(2,\frac{\alpha}{k}+1;\frac{\alpha}{k}+3;1-\frac{a}{b}\right) + C_4\left(\frac{\alpha}{k}\right)$$

$$= C_7\left(\frac{\alpha}{k}\right),$$

and

$$\int_{\frac{1}{2}}^{1} \frac{u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} (1-u)\, du \leqslant C_6\left(\frac{\alpha}{k}\right) - C_7\left(\frac{\alpha}{k}\right) = C_8\left(\frac{\alpha}{k}\right). \tag{28}$$

Using (22), (23), (24), (25), (27) and (28) in (21), we get (20).   □

REMARK 2.10. (i) If we put $k=1$ in (20), then we get [2, Theorem 7].
(ii) If we put $\alpha, k=1$ in (20), then we get [2, Corollary 2 (1)].
(iii) If we put $g(x)=1$ along with $k=1$ in (20), then we get [2, Corollary 2 (2)].
(iv) If we put $g(x)=1$ and $\alpha, k=1$ in (20), then we get [2, Corollary 2 (3)].

THEOREM 2.11. *Let $f : I \subset (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I^o such that $f' \in L[a,b]$, where $a,b \in I$ and $a < b$. If $|f'|^q, q > 1$, is harmonically convex on $[a,b]$, $g : [a,b] \longrightarrow \mathbb{R}$ is continuous and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then the following inequality for k-fractional integrals holds*

$$\left| \frac{f(a)+f(b)}{2} \left[J_{\frac{1}{b}+}^{\alpha,k}(g \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(g \circ h)\left(\frac{1}{b}\right) \right] \right. \tag{29}$$

$$\left. \left[J_{\frac{1}{b}+}^{\alpha,k}(fg \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(fg \circ h)\left(\frac{1}{b}\right) \right] \right|$$

$$\leqslant \frac{\|g\|_\infty ab(b-a)}{\Gamma_k(\alpha+k)} \left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}}$$

$$\left[C_9^{\frac{1}{p}}\left(\frac{\alpha}{k}\right) \left[\frac{|f'(a)|^q + 3|f'(b)|^q}{8} \right]^{\frac{1}{q}} + C_{10}^{\frac{1}{p}}\left(\frac{\alpha}{k}\right) \left[\frac{3|f'(a)|^q + |f'(b)|^q}{8} \right]^{\frac{1}{q}} \right]$$

where $\| g \|_\infty = \sup_{t \in [a,b]} | g(t) |$ and

$$C_9\left(\frac{\alpha}{k}\right) = \frac{\left(\frac{a+b}{2}\right)^{-2p}}{2\left(\frac{\alpha}{k}p+1\right)} {}_2F_1\left(2p,\frac{\alpha}{k}p+1;\frac{\alpha}{k}p+2;\frac{b-a}{b+a}\right),$$

$$C_{10}\left(\frac{\alpha}{k}\right) = \frac{b^{-2p}}{2\left(\frac{\alpha}{k}p+1\right)} {}_2F_1\left(2p,1;\frac{\alpha}{k}p+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right),$$

with $\alpha \leqslant k$ and $h(x) = \frac{1}{x}$, $x \in \left[\frac{1}{b},\frac{1}{a}\right]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using (15) and Hölder's inequality and the harmonically convexity of $|f'|^q$

$$\left| \frac{f(a)+f(b)}{2} \left[J_{\frac{1}{b}+}^{\alpha,k}(g \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(g \circ h)\left(\frac{1}{b}\right) \right] \right.$$

$$\left. - \left[J_{\frac{1}{b}+}^{\alpha,k}(fg \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(fg \circ h)\left(\frac{1}{b}\right) \right] \right|$$

$$\leqslant \frac{\|g\|_\infty (ab)^2}{\alpha \Gamma_k(\alpha)} \left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1} \int_0^{\frac{1}{2}} \frac{(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} \left| f'\left(\frac{ab}{ub+(1-u)a}\right) \right| du$$

$$+ \frac{\|g\|_\infty (ab)^2}{\alpha \Gamma_k(\alpha)} \left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1} \int_{\frac{1}{2}}^1 \frac{u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}}{(ub+(1-u)a)^2} \left| f'\left(\frac{ab}{ub+(1-u)a}\right) \right| du,$$

from here we take the right hand side as

$$\leqslant \frac{\|g\|_\infty (ab)^2}{\Gamma_k(\alpha+k)} \left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1} \left(\int_0^{\frac{1}{2}} \frac{\left[(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}\right]^p}{(ub+(1-u)a)^{2p}} du \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^{\frac{1}{2}} \left| f'\left(\frac{ab}{ub+(1-u)a}\right) \right|^q du \right)^{\frac{1}{q}}$$

$$+ \frac{\|g\|_\infty (ab)^2}{\Gamma_k(\alpha+k)} \left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1} \left(\int_{\frac{1}{2}}^1 \frac{\left[u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}\right]^p}{(ub+(1-u)a)^{2p}} du \right)^{\frac{1}{p}}$$

$$\times \left(\int_{\frac{1}{2}}^1 \left| f'\left(\frac{ab}{ub+(1-u)a}\right) \right|^q du \right)^{\frac{1}{q}}.$$

Using harmonically convexity of $|f'|^q$ we have

$$\left| \frac{f(a)+f(b)}{2} \left[J_{\frac{1}{b}+}^{\alpha,k}(g \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(g \circ h)\left(\frac{1}{b}\right) \right] \right.$$

$$\left. - \left[J_{\frac{1}{b}+}^{\alpha,k}(fg \circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(fg \circ h)\left(\frac{1}{b}\right) \right] \right|$$

$$\leqslant \frac{\|g\|_\infty (ab)^2}{\Gamma_k(\alpha+k)} \left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1} \left(\int_0^{\frac{1}{2}} \frac{\left[(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}\right]^p}{(ub+(1-u)a)^{2p}} du \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^{\frac{1}{2}} \left[u|f'(a)|^q + (1-u)|f'(b)|^q \right] du \right)^{\frac{1}{q}}$$

$$+ \frac{\|g\|_\infty (ab)^2}{\Gamma_k(\alpha+k)} \left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1} \left(\int_{\frac{1}{2}}^1 \frac{\left[u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}\right]^p}{(ub+(1-u)a)^{2p}} du\right)^{\frac{1}{p}}$$

$$\times \left(\int_{\frac{1}{2}}^1 \left[u|f'(a)|^q + (1-u)|f'(b)|^q\right] du\right)^{\frac{1}{q}}$$

$$\left| \frac{f(a)+f(b)}{2} \left[J_{\frac{1}{b}+}^{\alpha,k}(g\circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(g\circ h)\left(\frac{1}{b}\right)\right] \right. \tag{30}$$

$$\left. - \left[J_{\frac{1}{b}+}^{\alpha,k}(fg\circ h)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha,k}(fg\circ h)\left(\frac{1}{b}\right)\right] \right|$$

$$\leqslant \frac{\|g\|_\infty (ab)^2}{\Gamma_k(\alpha+k)} \left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1} \left(\int_0^{\frac{1}{2}} \frac{\left[(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}\right]^p}{(ub+(1-u)a)^{2p}} du\right)^{\frac{1}{p}}$$

$$\times \left[\frac{|f'(a)|^q + 3|f'(b)|^q}{8}\right]^{\frac{1}{q}}$$

$$+ \frac{\|g\|_\infty (ab)^2}{\Gamma_k(\alpha+k)} \left(\frac{b-a}{ab}\right)^{\frac{\alpha}{k}+1} \left(\int_{\frac{1}{2}}^1 \frac{\left[u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}\right]^p}{(ub+(1-u)a)^{2p}} du\right)^{\frac{1}{p}}$$

$$\times \left[\frac{3|f'(a)|^q + |f'(b)|^q}{8}\right]^{\frac{1}{q}}.$$

Calculating the appearing integrals on right hand side of (30) and using Lemma 1.4 we have

$$\int_0^{\frac{1}{2}} \frac{\left[(1-u)^{\frac{\alpha}{k}} - u^{\frac{\alpha}{k}}\right]^p}{(ub+(1-u)a)^{2p}} du \leqslant \int_0^{\frac{1}{2}} \frac{(1-2u)^{\frac{\alpha}{k}p}}{(ub+(1-u)a)^{2p}} du$$

$$= \frac{1}{2} \int_0^1 \frac{(1-u)^{\frac{\alpha}{k}p}}{\left(\frac{u}{2}b + \left(1-\frac{u}{2}\right)a\right)^{2p}} du$$

$$= \frac{\left(\frac{a+b}{2}\right)^{-2p}}{2} \int_0^1 v^{\frac{\alpha}{k}p}\left(1 - v\left(\frac{b-a}{b+a}\right)\right)^{-2p} dv$$

$$= \frac{\left(\frac{a+b}{2}\right)^{-2p}}{2} \beta\left(\frac{\alpha}{k}p+1,1\right) {}_2F_1\left(2p, \frac{\alpha}{k}p+1; \frac{\alpha}{k}p+2; \frac{b-a}{b+a}\right)$$

$$= \frac{\left(\frac{a+b}{2}\right)^{-2p}}{2\left(\frac{\alpha}{k}p+1\right)} {}_2F_1\left(2p, \frac{\alpha}{k}p+1; \frac{\alpha}{k}p+2; \frac{b-a}{b+a}\right) = C_9\left(\frac{\alpha}{k}\right), \tag{31}$$

and

$$\int_{\frac{1}{2}}^{1} \frac{\left[u^{\frac{\alpha}{k}} - (1-u)^{\frac{\alpha}{k}}\right]^p}{(ub + (1-u)a)^{2p}} du \leqslant \int_{\frac{1}{2}}^{1} \frac{(2u-1)^{\frac{\alpha}{k}p}}{(ub + (1-u)a)^{2p}} du$$

$$= \int_{0}^{\frac{1}{2}} \frac{(1-2u)^{\frac{\alpha}{k}p}}{(ua + (1-u)b)^{2p}} du$$

$$= \frac{1}{2} \int_{0}^{1} \frac{(1-v)^{\frac{\alpha}{k}p}}{\left(\frac{v}{2}a + \left(1 - \frac{v}{2}\right)b\right)^{2p}} dv$$

$$= \frac{b^{-2p}}{2} \int_{0}^{1} (1-v)^{\frac{\alpha}{k}p} \left(1 - \frac{v}{2}\left(1 - \frac{a}{b}\right)\right)^{-2p} dv$$

$$= \frac{b^{-2p}}{2} \beta\left(1, \frac{\alpha}{k}p + 1\right) {}_2F_1\left(2p, 1; \frac{\alpha}{k}p + 2; \frac{1}{2}\left(1 - \frac{a}{b}\right)\right)$$

$$= \frac{b^{-2p}}{2\left(\frac{\alpha}{k}p + 1\right)} {}_2F_1\left(2p, 1; \frac{\alpha}{k}p + 2; \frac{1}{2}\left(1 - \frac{a}{b}\right)\right)$$

$$= C_{10}\left(\frac{\alpha}{k}\right). \tag{32}$$

Using (31) and (32) in (30), we get (29). □

REMARK 2.12. (i) If we put $k = 1$ in (20), then we get [2, Theorem 8].
(ii) If we put $\alpha, k = 1$ in (20), then we get [2, Corollary 3 (1)].
(iii) If we put $g(x)$ and $k = 1$ in (20), then we get [2, Corollary 3 (2)].
(iv) If we put $g(x) = 1$ and $\alpha, k = 1$ in (20), then we get [2, Corollary 3 (3)].

REFERENCES

[1] İ. İŞCAN, S. WU, *Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals*, Appl. Math. Comput. **238** (2014) 237–244.

[2] İ. İŞCAN, M. KUNT, *Hermite-Hadamard-Fejér type inequalities for harmonically convex functions via fractional integrals*, RGMIA Research Report Collection **18** (2015), 1–19.

[3] R. GORENFLO, F. MAINARDI, *Fractional calculus: Integral and differential equations of fractional order*, Springer Verlag, Wien (1997).

[4] S. MUBEEN, G. M. HABIBULLAH, *k-fractional integrals and applications*, Int. J. Contemp. Math. Sci. **7** (2012), 89–94.

[5] I. PODLUBNY, *Fractional differential equations*, Mathematics in Science and Engineering, Academic Press, San Diego, Calif, USA (1999).

[6] K. MILLER, B. ROSS, *An introduction to the fractional calculus and fractional differential equations*, John Wiley and Sons, Inc, New York (1993).

[7] İ. İŞCAN, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacet. J. Math. Stat. **43** (6) (2014), 935–942.

[8] L. FEJÉR, *Uberdie Fourierreihen, II*, Math. Naturwise. Anz. Ungar. Akad., Wiss, **24** (1906), pp. 369–390, (in Hungarian).

[9] M. A. LATIF, S. S. DRAGOMIR AND E. MOMONIAT, *Some Fejer type inequalities for harmonically-convex functions with applications to special means*, http://rgmia.org/papers/v18/v18a24.pdf.

[10] A. P. PRUDNIKOV, Y. A. BRYCHKOV, O. J. MARICHEV, *Integral and series*, Elementary Functions, Nauka, Moscow, 1 (1981).

[11] J. WANG, C. ZHU AND Y. ZHOU, *New generalized Hermite-Hadamard type inequalities and applications to special means*, J. Inequal. Appl. **2013** (325) (2013), 15 pages.

[12] A. A. KILBAS, H. M. SRIVASTA, J. J. TRUJILLO, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam (2006).

Waqas Ayub
COMSATS
Institute of Information Technology
Attock, Pakistan
e-mail: waqas.pakistan19@gmail.com

Ghulam Farid
COMSATS
Institute of Information Technology
Attock Campus, Pakistan
e-mail: faridphdsms@hotmail.com;
ghlmfarid@ciit-attock.edu.pk

Fractional Differential Calculus
www.ele-math.com
fdc@ele-math.com