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QUALITATIVE RESULTS FOR SOLUTIONS TO NONLINEAR CAPUTO DIFFERENTIAL EQUATIONS SATISFYING THE OSGOOD CONDITION

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Abstract. We consider an initial value problem involving a single-term Caputo fractional differential equation. For those with right-hand sides that satisfy the Osgood condition, we establish novel uniqueness and comparison theorems.

In addition, we discuss a reduction of the fractional order problem to an integer ordered one. We identify inconsistencies in recent work by Demirci and Ozalp regarding this via the use of several counterexamples. Nevertheless, we take a constructive approach by proving that *a priori* estimates for the solution of a fractional order problem can be obtained from that for the corresponding integer order problem. All results are illustrated with examples.

1. Introduction

In this paper we consider nonlinear initial value problems of fractional order. The problem under consideration consists of the following single-term Caputo fractional differential equation:

$$ {}^{C}D^{q}[x](t) = f(t, x(t)) \tag{1} $$

(with ${}^{C}D^{q}$ precisely defined in the next section); coupled with the following initial conditions:

$$ x^{(i)}(0) = a_i, \quad i = 0, 1, \ldots, \lceil q \rceil - 1. \tag{2} $$

Above, f is a real-valued function defined on $[0,a] \times \mathscr{I}$ ($\mathscr{I}$ is an interval in $\mathbb{R}$) and the a_i are constants. For a general function f, classical questions are concerned with the existence and uniqueness of solutions to (1), (2) as well as obtaining qualitative information about solutions.

It is well known (see e.g. [6], [14]) that for every $q > 0$ the continuity of f implies that the initial value problem (1)–(2) has at least one continuous solution on a sufficiently small interval $[0,h]$. The uniqueness of a continuous solution to (1)–(2) was classically established under the assumption that f is Lipschitz [13] in the second variable (see e.g. the paper by Diethelm and Ford [6]).

Since then, there have been a large number of results in the literature extending and complementing the Diethelm-Ford Uniqueness Theorem. In particular, in [7] the result

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was extended to the case of a unique continuously differentiable solution. In [4] the result was generalised to the case when f depends on a lower order derivative. For the case $0 < q < 1$ it was proved for the function f satisfying the Nagumo condition [5]. In [9] the uniqueness theorem was proved for $0 < q < 1$ and the function f satisfying the Osgood condition, which is a far reaching generalisation of Lipschitz condition (see Example 8 and discussion after it).

One of the main results of this paper is the Osgood uniqueness theorem for $q > 1$. On one hand, this result extends that of [9]. On the other hand, this result generalises the classical Lipschitz uniqueness theorem. Of course, it is an extension of the Osgood uniqueness theorem for ODEs to the case of fractional order.

The second main result of this paper is a comparison theorem. Results of this type are powerful tools that provide qualitative information about solutions to two fractional order IVPs without needing to solve them. The result given in Theorem 5 is a generalisation of [14, Theorem 3.1], where it was established for $0 < q < 1$ and Lipschitz right-hand side. We illustrate our results with examples. We also discuss the significance of conditions in theorems.

In Section 5 we discuss a relationship between fractional order problems and integer ordered ones. We identify inconsistencies in recent work by Demirci and Ozalp [3] regarding this via the use of several counterexamples. Although this relation cannot be used to solve fractional order problems, it is useful in obtaining qualitative information about solutions. In particular, we show that *a priori* estimates for the solution of an integer order problem provide that for the original fractional order problem.

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2. Preliminaries

Let $\mathbb{N} := \{1,2,3,\ldots\}$ be the set of all natural numbers. Let $L_p[0,a]$, $p \geqslant 1$, be the space of all Lebesgue integrable functions on $[0,a]$ and for $n \in \mathbb{N}$, let $A^n[0,a]$ be the space of functions on $[0,a]$ with an absolutely continuous $(n-1)$-st derivative.

The Riemann–Liouville fractional integral of order $q > 0$ of $f \in L_1[0,a]$ is defined by the formula

$$I^q[f](t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\, ds, \quad t \in [0,a], \tag{3}$$

where Γ is the Gamma function. For $q = 0$, we set I^0 to be an identity operator. Throughout the paper we denote $m := \lceil q \rceil$.

The Caputo fractional derivative of order $q > 0$ of $f \in A^m[0,a]$ is defined by

$${}^C D^q[f](t) = I^{m-q} \left[\frac{d^m f}{dt^m} \right](t). \tag{4}$$

One well-known advantage of the Caputo derivative over the Riemann–Liouville one is that the former applied to a constant function gives zero.

We define the two parameter Mittag–Leffler Function $E_{\alpha,\beta}:\mathbb{C}\to\mathbb{C}$ by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \tag{5}$$

where $\Re(\alpha),\Re(\beta)>0$. When $\beta=1$ we obtain the classical Mittag–Leffler Function $E_\alpha := E_{\alpha,1}$ [12]. By ψ we denote the digamma function given by the following formula (see e.g. [1]):

$$\psi(x)=\frac{d}{dx}\ln(\Gamma(x)).$$

3. Main results

We begin with a simple variant of Osgood's result [11] (see also [2, Lemma 1.4.1]).

LEMMA 1. *Let $g:[0,\infty)\to[0,\infty)$ be a continuous, non-decreasing function such that: $g(0)=0$; $g(z)>0$ if $z>0$; and*

$$\int_0^1 \frac{dz}{g(z)}=\infty. \tag{6}$$

Let $a>0$ and let $(t,s)\mapsto k(t,s)$ be a real-valued function bounded on a triangular region $0\leqslant t\leqslant a$, $0\leqslant s\leqslant t$. Let $\phi:[0,a]\to[0,\infty)$ be a continuous function. If

$$\phi(t)\leqslant\int_0^t k(t,s)g(\phi(s))\,ds,\quad 0\leqslant t\leqslant a, \tag{7}$$

then $\phi\equiv 0$ on $[0,a]$.

Proof. Since the function k is bounded, it follows that $|k(t,s)|\leqslant K$ for all $0\leqslant t\leqslant a$, $0\leqslant s\leqslant t$ and some $K>0$. Hence

$$\phi(t)\leqslant K\int_0^t g(\phi(s))\,ds,\quad 0\leqslant t\leqslant a,$$

and the result follows from the classical Osgood theorem (see e.g. [2, Lemma 1.4.1]). □

In the following definition we introduce the class of functions that we will be dealing with.

DEFINITION 2. Let $\mathscr{I}$ be an interval in $\mathbb{R}$ and $a>0$. We say that $f:[0,a]\times\mathscr{I}\to\mathbb{R}$ satisfies the Osgood condition if it is continuous and such that for every $t\in[0,a]$, $u,v\in\mathscr{I}$ one has

$$|f(t,u)-f(t,v)|\leqslant\psi(t)\cdot g(|u-v|), \tag{8}$$

where g is any function satisfying conditions of Lemma 1 and ψ is a continuous, non-negative function on $[0,a]$.

REMARK 3. The Osgood condition introduced above extends the classical Lipschitz condition [13]. It is less general than the Montel-Tonelli condition (where the function ψ is merely integrable).

The following theorem is an extension of Osgood's Uniqueness Theorem [2, Theorem 1.4.2] to fractional differential equations. It is also a partial extension of the more general Montel-Tonelli Uniqueness Theorem [2, Theorem 1.5.1].

THEOREM 4. *Let $\mathscr{I}$ be an interval in $\mathbb{R}$ with $a_0 \in \mathscr{I}$ and $a > 0$. Let $f : [0,a] \times \mathscr{I} \to \mathbb{R}$ be a function satisfying the Osgood condition (8) for some g and ψ. If $q \geqslant 1$, then the initial value problem (1), (2) has, at most, one solution.*

Proof. Denote $m := \lceil q \rceil$. Suppose that x and y are solutions to the IVP (1), (2). Then both x and y solve the integral equation

$$z(t) = \sum_{i=0}^{m-1} \frac{a_i}{i!} t^i + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, z(s))\, ds, \quad t \in [0,a]. \tag{9}$$

Hence

$$\begin{aligned} |x(t) - y(t)| &\leqslant \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s,x(s)) - f(s,y(s))|\, ds \\ &\leqslant \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \psi(s) g(|x(s) - y(s)|)\, ds \end{aligned}$$

by (8). Since $q \geqslant 1$ and ψ is continuous on $[0,a]$, the function $k(t,s) := (t-s)^{q-1}\psi(s)$ is bounded on the region $0 \leqslant t \leqslant a$, $0 \leqslant s \leqslant t$. Consequently, the non-negative function $\phi(t) := |x(t) - y(t)|$ satisfies all conditions of Lemma 1, with application of the Lemma yielding $\phi(t) = 0$ on $[0,a]$. Thus $x(t) = y(t)$ on $[0,a]$; that is the IVP has at most one solution. □

THEOREM 5. *Let $\mathscr{I}$ be an interval in $\mathbb{R}$ with $a_0, b_0 \in \mathscr{I}$ and $a > 0$. Let $f, h : [0,a] \times \mathscr{I} \to \mathbb{R}$ be such that at least one of them satisfies condition (8) and at least one of them is non-decreasing in the second variable. Let $q > 1$ and $m := \lceil q \rceil$.*

Suppose that $x = x(t)$ satisfies

$${}^{C}D^q[x](t) = h(t, x(t)), \quad x^{(i)}(0) = a_i, \quad i = 0, 1, \ldots, m-1$$

and $y = y(t)$ satisfies

$${}^{C}D^q[y](t) = f(t, y(t)), \quad y^{(i)}(0) = b_i, \quad i = 0, 1, \ldots, m-1.$$

Further, if $h(t,u) \leqslant f(t,u)$ for all $(t,u) \in [0,a] \times \mathscr{I}$ and $a_i \leqslant b_i$ for $i = 0, 1, \ldots, m-1$, then $x \leqslant y$ on $[0,a]$.

Proof. Suppose, to the contrary, that there exists $t_0 \in [0,a]$ such that $x(t_0) > y(t_0)$. Define $r(t) := x(t) - y(t)$. Since $r(0) \leqslant 0$, without loss of generality we can assume that $r(t) \leqslant 0$ on some interval $[0,c]$ and $r(t) > 0$ on $(c,t_0]$.

Since x and y are solutions to the IVPs, they satisfy integral equations of the form (9). Thus, for every $t \in (c,t_0]$ we obtain

$$\begin{aligned} r(t) &= \sum_{i=0}^{m-1} \frac{(a_i - b_i)}{i!} t^i + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [h(s,x(s)) - f(s,y(s))]\, ds \\ &\leqslant \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [h(s,x(s)) - f(s,y(s))]\ ds, \end{aligned}$$

since $a_i \leqslant b_i$ for $i = 0,1,\ldots,m-1$.

Since $r(t) \leqslant 0$ on $[0,c]$, $x(s) \leqslant y(s)$ for $t \in [0,c]$. If h is non-decreasing in the second variable, then we use the fact that $h(t,u) \leqslant f(t,u)$ for all $t \in [0,a]$ to obtain

$$h(s,x(s)) - f(s,y(s)) \leqslant h(s,x(s)) - h(s,y(s)) \leqslant 0.$$

If f is non-decreasing in the second variable, then similarly we obtain

$$h(s,x(s)) - f(s,y(s)) \leqslant f(s,x(s)) - f(s,y(s)) \leqslant 0.$$

Therefore, in any case

$$\int_0^c (t-s)^{q-1} [h(s,x(s)) - f(s,y(s))] \quad ds \leqslant 0$$

and so

$$r(t) \leqslant \frac{1}{\Gamma(q)} \int_c^t (t-s)^{q-1} [h(s,x(s)) - f(s,y(s))]\ ds,\ t \in (c,t_0].$$

Next, if h satisfies (8), then for $s \in (c,t_0]$ we obtain

$$\begin{aligned} h(s,x(s)) - f(s,y(s)) &\leqslant h(s,x(s)) - h(s,y(s)) \leqslant |h(s,x(s)) - h(s,y(s))| \\ &\leqslant \psi(s) g(|x(s) - y(s)|). \end{aligned}$$

Similarly if f satisfies (8), then for $s \in (c,t_0]$ we obtain

$$\begin{aligned} h(s,x(s)) - f(s,y(s)) &\leqslant f(s,x(s)) - f(s,y(s)) \leqslant |f(s,x(s)) - f(s,y(s))| \\ &\leqslant \psi(s) g(|x(s) - y(s)|). \end{aligned}$$

Therefore, in any case

$$h(s,x(s)) - f(s,y(s)) \leqslant \psi(s) g(|r(s)|) \leqslant \psi(s) g(r(s)),$$

since $r(t) > 0$ on $(c,t_0]$. Thus

$$r(t) \leqslant \frac{1}{\Gamma(q)} \int_c^t (t-s)^{q-1} \psi(s) g(r(s)) ds, \quad t \in (c,t_0].$$

Equivalently,

$$r(z+c) \leqslant \frac{1}{\Gamma(q)} \int_{c}^{z+c} (z+c-s)^{q-1} \psi(s) g(r(s))\, ds$$
$$\leqslant \frac{1}{\Gamma(q)} \int_{0}^{z} (z-v)^{q-1} \psi(v+c) g(r(v+c))\, dv, \quad z \in (0, t_0 - c].$$

Note that the function $k(z,v) := (z-v)^{q-1}\psi(v+c)$ satisfies the conditions of Lemma 1. Since $r(t) \geqslant 0$ on $[c,t_0]$, it follows that the function $\phi(z) = r(z+c)$ is non-negative on $[0,t_0-c]$. By Lemma 1, we conclude that $\phi(z) = 0$ on $[0,t_0-c]$, that is $r(t) = 0$ on $[c,t_0]$. This contradicts the assumption. Hence, $x(t) \leqslant y(t)$ for all $t \in [0,a]$. □

REMARK 6. Theorem 5 above can be proved with a less restrictive condition on h and f. This condition is that

$$u \leqslant v \Longrightarrow h(t,u) \leqslant f(t,v), \quad \text{for all } (t,u),(t,v) \in [0,a] \times \mathscr{I}. \tag{10}$$

The proof of Theorem 5 with this condition is identical to the one given above and therefore omitted.

It is clear that if either h or f is non-decreasing and $h(t,u) \leqslant f(t,u)$ for all $(t,u) \in [0,a] \times \mathscr{I}$ then (10) is satisfied. The following example shows that using condition (10) we can extend the class of IVPs compared to that in Theorem 5.

EXAMPLE 7. *Consider* $h, f : [0,a] \times \mathbb{R} \to \mathbb{R}$ *where*

$$h(t,u) = 1 - (u-1)^2 \text{ and } f(t,u) = 1 + (u-4)^2.$$

Then for $u \leqslant v$, $h(t,u) - f(t,v) = -(u-1)^2 - (v-4)^2 \leqslant 0$ *and thus we have that the condition*

$h(t,u) \leqslant f(t,v)$ *for* $u \leqslant v$ *is satisfied. However, neither* h *nor* f *is non-decreasing in the second variable.*

Thus taking Theorem 5 at face value one could not make a conclusion on the relative behaviour of the solutions $x(t)$ *and* $y(t)$ *to (say)*

$${}^{C}D^{q}[x](t) = 1 - (u-1)^2$$

$$x(0) = 0, \quad x'(0) = 1$$

and

$${}^{C}D^{q}[y](t) = (u-2)^2 + 1$$

$$y(0) = 0, \quad y'(0) = 1.$$

However with Remark 6 in mind we can conclude that the solutions satisfy $x(t) \leqslant y(t)$ *on* $t \in [0,a]$.

4. Examples and counterexamples

The theorems proved in the previous section provide qualitative information on IVPs with the right-hand side satisfying a condition much more general than Lipschitz's condition.

EXAMPLE 8. *Let $g : [0,\infty) \to [0,\infty)$ be defined by the following formula:*

$$g(u) = \begin{cases} 0, & u = 0 \\ -u\ln u, & 0 < u \leqslant 1/e \\ 1/e, & u > 1/e. \end{cases}$$

Let $\mathscr{I}$ be an interval in $\mathbb{R}$ with $a_0 \in \mathscr{I}$ and $a > 0$. Let $f : [0,a] \times \mathscr{I} \to \mathbb{R}$ be defined by the formula $f(t,u) = |\sin t| \cdot g(u)$. The function $\psi(t) = |\sin t|$ is non-negative and continuous. The function g is positive, continuous and non-decreasing with: $g(0) = 0$; $g(u) > 0$ if $u > 0$; and

$$\int_0^1 \frac{dz}{g(z)} = \infty;$$

that is, g satisfies all of the conditions of Lemma 1. Hence f satisfies the Osgood condition. We note that f is not Lipschitz in the second variable.

Therefore, it follows from Theorem 4 that the IVP

$${}^{C}D^{q}[x](t) = f(t,x(t)), \quad x^{(i)}(0) = a_i, \quad i = 0,1,\ldots,\lceil q\rceil - 1$$

has at most one solution.

Note that the functions $f(t,u) = u$, $u|\ln u|$, $u|\ln u|\cdot|\ln|\ln u||\ldots$ satisfy the Osgood condition. In particular, the Osgood condition extends that of Lipschitz.

Next we shall discuss the conditions of Theorem 5. Recall that in the classical case of the comparison theorem for first order IVPs, the right-hand sides are not necessarily non-decreasing. The next example illustrates that, in general, Theorem 5 fails for the second order problem if f and h are not non-decreasing even when they are Lipschitz.

EXAMPLE 9. *Let $R = [0,2\pi] \times \mathbb{R}$. Define $h, f : R \to \mathbb{R}$ by*

$$h(t,u) = -u = f(t,u).$$

Then h, f satisfy

$$h(t,u) \leqslant f(t,u)$$

and are Lipschitz, but both h and f are decreasing in the second variable.

Consider the second order initial value problems

$$\begin{aligned} x''(t) &= -x(t) \\ x(0) = 1, &\quad x'(0) = 0 \end{aligned}$$

and

$$\begin{aligned} y''(t) &= -y(t) \\ y(0) = 1, \quad & y'(0) = 1, \end{aligned}$$

for which the solutions are $x(t) = \cos t$ *and* $y(t) = \cos t + \sin t$.

Thus we see that all of the conditions of Theorem 5 apart from the nondecreasingness of either h *or* f *are satisfied. But* $x(\frac{3\pi}{2}) - y(\frac{3\pi}{2}) = 1 > 0$, *so there exists* $t \in [0, 2\pi]$ *such that* $x(t) > y(t)$.

Next, we extend Example 9 to the case of IVPs of order greater than or equal to three. We shall prove an auxiliary result first.

LEMMA 10. *For* $q \geqslant 3$ *there exists* $t_0 \in [0, a]$ *such that* $E_{q,2}(-t_0^q) < 0$, *where* a *depends on* q *and is given by* (13).

Proof. It follows from the definition of $E_{q,2}$ that

$$E_{q,2}(-t^q) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{qk}}{\Gamma(qk+2)}.$$

Using the estimation for the remainder of an alternating series we obtain

$$\left| E_{q,2}(-t^q) - \left(1 - \frac{t^q}{\Gamma(q+2)}\right) \right| \leqslant \frac{t^{2q}}{\Gamma(2q+2)}.$$

In particular,

$$E_{q,2}(-t^q) \leqslant 1 - \frac{t^q}{\Gamma(q+2)} + \frac{t^{2q}}{\Gamma(2q+2)}. \tag{11}$$

Hence to prove the assertion it is sufficient to show that the right-hand side of (11) is negative for some value of $t \in [0, a]$. Consider the following quadratic equation corresponding to the right-hand side of (11):

$$1 - \frac{x}{\Gamma(q+2)} + \frac{x^2}{\Gamma(2q+2)} = 0,$$

or equivalently,

$$\Gamma(q+2) \cdot x^2 - \Gamma(2q+2) \cdot x + \Gamma(2q+2) \cdot \Gamma(q+2) = 0. \tag{12}$$

The determinant of this equation is

$$\Delta = \Gamma(2q+2) \cdot \left(\Gamma(2q+2) - 4 \cdot \Gamma^2(q+2)\right).$$

Consider the function $f(q) = \frac{\Gamma(2q+2)}{\Gamma^2(q+2)}$ on $[3,\infty)$. Using the fact that $\frac{d}{dz}\Gamma(z) = \Gamma(z)\psi(z)$, where ψ is a digamma function, we obtain

$$\begin{aligned}\frac{df}{dq} &= \frac{\frac{d}{dq}(\Gamma(2q+2))\Gamma^2(q+2) - \Gamma(2q+2)\cdot 2\cdot\Gamma(q+2)\cdot\frac{d}{dq}(\Gamma(q+2))}{\Gamma^4(q+2)} \\ &= \frac{2\cdot\Gamma(2q+2)\psi(2q+2)\Gamma^2(q+2) - \Gamma(2q+2)\cdot 2\cdot\Gamma(q+2)\cdot\Gamma(q+2)\psi(q+2)}{\Gamma^4(q+2)} \\ &= \frac{2\cdot\Gamma(2q+2)\Gamma^2(q+2)}{\Gamma^4(q+2)}(\psi(2q+2)-\psi(q+2)) > 0,\end{aligned}$$

since the digamma function ψ is increasing on the positive semi-axis.

Hence the function f is strictly increasing. Thus

$$\frac{\Gamma(2q+2)}{\Gamma^2(q+2)} \geqslant \frac{\Gamma(2\cdot 3+2)}{\Gamma^2(3+2)} = \frac{7!}{(4!)^2} = \frac{35}{4} > 4.$$

Hence, the determinant of the equation (12) is positive. The largest root is

$$x^* = \frac{\Gamma(2q+2)+\sqrt{\Gamma(2q+2)\cdot(\Gamma(2q+2)-4\cdot\Gamma^2(q+2))}}{2\Gamma(q+2)} > 0.$$

Since the function $y:\mathbb{R}\to\mathbb{R}$ given by

$$y(x) = 1 - \frac{x}{\Gamma(q+2)} + \frac{x^2}{\Gamma(2q+2)}$$

is convex down, it follows that there exists $x_0 \in [0,x^*]$ such that $y(x_0) < 0$.

Setting $t_0 = x_0^{\frac{1}{q}}$ and

$$a = (x^*)^{\frac{1}{q}} = \left(\frac{\Gamma(2q+2)+\sqrt{\Gamma(2q+2)\cdot(\Gamma(2q+2)-4\cdot\Gamma^2(q+2))}}{2\Gamma(q+2)}\right)^{1/q}, \tag{13}$$

we conclude that there exists $t_0 \in [0,a]$ such that the right-hand side of (11) is negative. Therefore $E_{q,2}(-t_0^q) < 0$. □

EXAMPLE 11. *Let $q \geqslant 3$ and let a be defined by the formula* (13). *Define $f,h : [0,a]\times\mathbb{R}\to\mathbb{R}$ by $f(t,u) = h(t,u) = -u$. Clearly, f and h are Lipschitz and they are decreasing. Consider the initial value problems*

$${}^C D^q[x](t) = h(t,x(t)), \quad x(0)=1, \; x^{(i)}(0)=0, \; i=1,\dots,m-1$$

and

$${}^C D^q[y](t) = f(t,y(t)), \quad y(0)=1, \; y'(0)=1, \; y^{(i)}(0)=0, \; i=2,\dots,m-1.$$

We have that $h(t,u) \leqslant f(t,u)$ *for all* $(t,u) \in [0,a] \times \mathbb{R}$ *and that* $x^{(i)}(0) \leqslant y^{(i)}(0)$ *for* $i = 0,1,\ldots,m-1$.

It follows from [8, Theorem 4.3], *that the solutions to these IVPs are* $x(t) = E_{q,1}(-t^q)$ *and* $y(t) = E_{q,1}(-t^q) + tE_{q,2}(-t^q)$ *respectively.*

It follows from Lemma 10 that there exists $t_0 \in [0,a]$ *such that* $E_{q,2}(-t_0^q) < 0$. *This means that* $x(t_0) > y(t_0)$, *that is, the conclusion of Theorem 5 fails.*

5. Relation between fractional and ordinary DEs

While working on examples in the previous section, we came across the paper [3] which attempts to reduce a fractional problem of order $q \in (0,1)$ to a first order problem.

The main result of that paper, namely [3, Theorem 5], is incorrect. Their result is also illustrated therein with several examples. However, [3, Example 10], [3, Example 11], [3, Example 12] are incorrect as the "solutions" obtained do not satisfy their respective IVPs. We provide a simple counterexample to their technique.

EXAMPLE 12. *Consider the following IVP on* $[0,1]$*:*

$$^{C}D^{\frac{1}{2}}[x](t) = x(t), \quad x(0) = 1.$$

By the Demirci-Ozalp "method" the solution to this IVP is

$$x(t) = e^{\frac{t^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}}.$$

However, straightforward substitution of this function into the IVP shows that it does not satisfy the differential equation. We know in fact from [8, Theorem 4.3] *that the correct solution to this IVP is* $x(t) = E_{\frac{1}{2}}(t^{\frac{1}{2}})$.

The statement of the main result [3, Theorem 5] in the Demirci-Ozalp paper is given in a somewhat vague manner and may be misinterpreted. We provide one more counterexample confirming that their result is incorrect regardless of the interpretation.

EXAMPLE 13. *By the Demirci-Ozalp "method" the IVPs*

$$^{C}D^{\frac{1}{2}}[x](t) = x(t), \quad x(0) = 1, \quad t \in [0,1]$$

and

$$^{C}D^{\frac{1}{4}}[x](t) = x(t), \quad x(0) = 1 \quad t \in [0,1]$$

both correspond to the same integer order problem $x_*'(v) = g(v, x_*(v))$, $x_*(0) = 1$.

If x_* *is a solution to this problem, then solutions to the fractional IVPs are*

$$x(t) = x_*\left(\frac{t^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}\right) \text{ and } x(t) = x_*\left(\frac{t^{\frac{1}{4}}}{\Gamma\left(\frac{5}{4}\right)}\right),$$

respectively. However, we know that the correct solutions are $E_{\frac{1}{2}}(t^{\frac{1}{2}})$ *and* $x(t) = E_{\frac{1}{4}}(t^{\frac{1}{4}})$ *respectively. Hence if the Demirci-Ozalp method is correct, it must be that*

$$E_{\frac{1}{2}}(t^{\frac{1}{2}}) = x_*\left(\frac{t^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}\right) \text{ and } E_{\frac{1}{4}}(t^{\frac{1}{4}}) = x_*\left(\frac{t^{\frac{1}{4}}}{\Gamma\left(\frac{5}{4}\right)}\right),$$

or equivalently, that

$$E_{\frac{1}{2}}\left(s\Gamma\left(\frac{3}{2}\right)\right) = x_*(s) \text{ and } E_{\frac{1}{4}}\left(s\Gamma\left(\frac{5}{4}\right)\right) = x_*(s).$$

Thus

$$E_{\frac{1}{2}}\left(s\Gamma\left(\frac{3}{2}\right)\right) = E_{\frac{1}{4}}\left(s\Gamma\left(\frac{5}{4}\right)\right),$$

which is clearly not the case for $s \neq 0$.

It should be pointed out that some problems involving fractional differential equations can be reduced to that of an ordinary differential equation. The results presented below first appeared in the book [10] for the Riemann-Liouville derivative. We extend them to the case of Caputo derivative.

Let $0 < q < 1$, $f : [0,a] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and let x be a solution to the following IVP:

$$^{C}D^{q}[x](t) = f(t,x(t)), \quad x(t)t^{1-q}|_{t=0} = a. \tag{14}$$

From the definition of the Riemann-Liouville derivative and its relation to the Caputo derivative we have

$$^{C}D^{q}[x](t) = D^{q}[x-a](t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t (t-s)^{-q}(x(s)-a)\, ds.$$

Let us introduce a function ϕ defined on a triangular region as follows:

$$\phi(t,s) = x(t) - x(s), \quad 0 \leqslant s \leqslant t \leqslant a.$$

Hence,

$$\begin{aligned} ^{C}D^{q}[x](t) &= \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t (t-s)^{-q}(x(t)-\phi(t,s)-a)\, ds \\ &= \frac{d}{dt}\left[t \mapsto \frac{x(t)}{\Gamma(1-q)}\frac{t^{1-q}}{1-q}\right] - \eta(t), \end{aligned}$$

where

$$\eta(t) = \eta(t,q,\phi) := \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t (t-s)^{-q}(\phi(t,s)+a)\, ds.$$

Define $y:[0,1]\to\mathbb{R}$ by the formula

$$y(t)=\frac{x(t)}{\Gamma(2-q)}t^{1-q}.$$

We have that y is a solution of the integer order problem

$$\frac{dy}{dt}(t)=f(t,\Gamma(2-q)t^{-1+q}y(t))+\eta(t),\ y(0)=b:=\frac{a}{\Gamma(2-q)}. \tag{15}$$

REMARK 14. It should be pointed out that the IVP (15) cannot be used to solve the IVP (14), since the right-hand side of (15) depends on η, which, in turn, depends on the solution x to the IVP (14). Thus, to solve the IVP (14) by reduction to the IVP (15) we would need to know the solution of (14) in advance.

However, the technique outlined above can be used to obtain an *a priori* estimate for x using perturbation theory for the IVP (15).

EXAMPLE 15. *Let f be such that*

$$|f(t,u)|\leqslant k(t)\frac{t^{1-q}}{\Gamma(2-q)}|u|,\quad \textit{for all } (t,u)\in[0,a]\times\mathbb{R}.$$

If y is a solution of (15) then it also solves

$$y(t)=b+\int_0^t[f(s,\Gamma(2-q)t^{-1+q}y(s))+\eta(s)]\,ds. \tag{16}$$

Hence,

$$|y(t)|\leqslant|b|+\int_0^t[k(s)|y(s)|+|\eta(s)|]\,ds. \tag{17}$$

If we further assume that

$$|\eta(s)|\leqslant Mt^{\alpha},\quad t\in[0,a],$$

then

$$|y(t)|\leqslant|b|+\int_0^t[k(s)|y(s)|+Ms^{\alpha}]\,ds. \tag{18}$$

Denoting the right-hand side of the above inequality by $v(t)$ and differentiating we obtain

$$v'(t)\leqslant k(t)|y(t)|+Mt^{\alpha}\leqslant k(t)v(t)+Mt^{\alpha}.$$

The solution to the IVP $u'(t)=k(t)u(t),\ u(0)=b$ is

$$u(t)=b\exp\Big\{\int_0^t k(s)\,ds\Big\}.$$

Replacing b with $b(t)$ and using the variation of parameters method we obtain that the solution to the perturbed IVP $u'(t) = k(t)u(t) + Mt^{\alpha}$, $u(0) = b$ is

$$u(t) = e^{\int_0^t k(s)ds} \left[b + M \int_0^t s^{\alpha} e^{-\int_0^s k(z)dz} ds \right].$$

Using the classical comparison theorem we obtain

$$|y(t)| \leqslant v(t) \leqslant e^{\int_0^t k(s)ds} \left[b + M \int_0^t s^{\alpha} e^{-\int_0^s k(z)dz} ds \right].$$

Therefore, we obtain the following estimate for the solution of the fractional IVP:

$$|x(t)| \leqslant t^{q-1} e^{\int_0^t k(s)ds} \left[a + M\Gamma(2-q) \int_0^t s^{\alpha} e^{-\int_0^s k(z)dz} ds \right].$$

6. Conclusions

In this paper we established uniqueness and comparison theorems for the initial value problem involving the single-term Caputo fractional differential equation with right-hand sides satisfying the Osgood condition. These results generalise and extend known results in the literature. They allow the treatment of FDEs with "very" nonlinear right-hand sides. The further question in this direction is whether the analogous theorems hold for the right-hand sides satisfying more general Montel-Tonelli condition.

In the second part of the paper we established a relationship between fractional differential equations and ODEs. This relation provides a powerful tool in analysis of fractional differential equations. In general, one can use much more advanced techniques to extract qualitative information about solutions to the ODE first, and then apply the result of Section 5 (that is, the relation between (14) and (15)) to obtain such information for solutions to the corresponding fractional differential equation. Doing this without a mediation of ODE can be problematic, since the theory of fractional differential equations is less developed.

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