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EXISTENCE OF SOLUTIONS FOR A CLASS OF FRACTIONAL HAMILTONIAN SYSTEMS WITH IMPULSIVE EFFECTS

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Abstract. In this paper, we are concerned with a class of fractional Hamiltonian systems containing right Riemann-Liouville fractional derivatives and left Caputo fractional derivatives with impulsive effects. Under certain conditions, the existence of solutions are obtained for this class of systems by means of the least action principle, the saddle point theorem as well as some skills of inequalities. One of the innovations of this paper is that the variational functional of these problems are established in a proper fractional derivative space. Moreover, in order to show the feasibility and effectiveness of our results, we present two examples.

1. Introduction

In this paper, we study the following fractional Hamiltonian system with impulsive effects

$$\begin{cases} {}_tD_T^{\alpha}\big({}_0^cD_t^{\alpha}u(t)\big) = \nabla F(t,u(t)), \qquad \text{a.e. } t\in[0,T],\\ u(0)=u(T)=0,\\ \Delta\big({}_tD_T^{\alpha-1}\big({}_0^cD_t^{\alpha}u^i\big)\big)(t_j) = I_{ij}(u^i(t_j)),\ \ i=1,2,\dots,N,\ \ j=1,2,\dots,p, \end{cases} \tag{1}$$

where ${}_tD_T^{\alpha}$ is the right Riemann-Liouville fractional derivatives of order $\frac{1}{2}<\alpha\leqslant 1$, ${}_0^cD_t^{\alpha}$ is the left Caputo fractional derivatives of order $\frac{1}{2}<\alpha\leqslant 1$, $t_0=0<t_1<t_2<\dots<t_p<t_{p+1}=T$, $u(t)=\big(u^1(t),u^2(t),\dots,u^N(t)\big)$ and

$$\Delta\big({}_tD_T^{\alpha-1}\big({}_0^cD_t^{\alpha}u^i\big)\big)(t_j) = {}_tD_T^{\alpha-1}\big({}_0^cD_t^{\alpha}u\big)(t_j^+) - {}_tD_T^{\alpha-1}\big({}_0^cD_t^{\alpha}u\big)(t_j^-),$$

$${}_tD_T^{\alpha-1}\big({}_0^cD_t^{\alpha}u\big)(t_j^+) = \lim_{t\to t_j^+} {}_tD_T^{\alpha-1}\big({}_0^cD_t^{\alpha}u\big)(t),$$

$${}_tD_T^{\alpha-1}\big({}_0^cD_t^{\alpha}u\big)(t_j^-) = \lim_{t\to t_j^-} {}_tD_T^{\alpha-1}\big({}_0^cD_t^{\alpha}u\big)(t),$$

$I_{ij}:\mathbb{R}\to\mathbb{R}$ $(i=1,2,\dots,N,\ j=1,2,\dots,p)$ are continuous and $F:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ satisfies the following assumption:

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(A) $F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$ and there exist $a \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $b \in L^1(0,T;\mathbb{R}^+)$ such that
$$\left|F(t,x)\right| \leqslant a(|x|)b(t), \quad \left|\nabla F(t,x)\right| \leqslant a(|x|)b(t)$$
for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, where $\nabla F(t,x)$ denotes the gradient of $F(t,x)$ in x.

For the sake of convenience, in the sequel, we denote $\Lambda_1 = \{1,2,\dots,N\}$ and $\Lambda_2 = \{1,2,\dots,p\}$.

When $\alpha = 1$ and $I_{ij} \equiv 0$, $i \in \Lambda_1$, $j \in \Lambda_2$, (1) is the second order Hamiltonian system
$$\begin{cases} -\ddot{u}(t) = \nabla F(t,u(t)), \text{ a.e. } t \in [0,T]; \\ u(0) = u(T) = 0. \end{cases} \tag{2}$$
Many solvability conditions for problem (2) are obtained in [1] by some critical point theorems.

When $\alpha = 1$ and $I_{ij} \not\equiv 0$, $i \in \Lambda_1$, $j \in \Lambda_2$, (1) is the second order Hamiltonian system with impulsive effects
$$\begin{cases} -\ddot{u}(t) = \nabla F(t,u(t)), & \text{a.e. } t \in [0,T]; \\ u(0) = u(T) = 0, & \\ \Delta \dot{u}^i(t_j) = \dot{u}^i(t_j^+) - \dot{u}^i(t_j^-) = I_{ij}(u^i(t_j)), \, i \in \Lambda_1, j \in \Lambda_2. & \end{cases} \tag{3}$$
Zhou and Li in [2] studied the existence of solutions for (3) using variational methods.

Moreover, when $0 < \beta < 1$, $I_{ij} \not\equiv 0$, $i \in \Lambda_1$, $j \in \Lambda_2$, up to now, problem (1) has received considerably less attention.

On the one hand, fractional differential equations involving the Riemann-Liouville fractional derivative or the Caputo fractional derivative have recently been proved to be valuable tools in the modeling of many phenomena in various fields of science, engineering, physics and economics. Indeed, fractional differential equations have applications in many areas including fluid flow, electrical networks, probability and statistics, viscoelasticity, chemical physics and signal processing, and so on, see [3–7] and references therein. There has been a significant development in fractional differential equations in recent years, since the behavior of physical systems can be properly described by using fractional order system theory. So fractional differential equations got the attention of many researchers and considerable work has been done in this regard, see [8–11], and the references therein. Especially, fractional Hamiltonian systems have received increasing attention in various fields of science and engineering, with a growing number of applications in electrochemistry, physics, rheology and biology, probability, fluid flow, control theory, etc., for instance see the monographs of Kilbas et al. [12], Podlubny [13], Zhou [14], the papers [15, 16] and the references therein. As stated in [13], the qualitative theory of differential equations can be very highly useful in applications. When $N = 1$, the authors in [17] investigated the existence of solutions for a class of impulsive fractional Hamiltonian systems
$$\begin{cases} {}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right) + \kappa(t)u(t) = f(t,u), \quad t \in [0,T], \ t \neq t_j, \\ u(0) = u(T) = 0, \\ \Delta\left({}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u^i\right)\right)(t_j) = I_{ij}(u^i(t_j)), \ i = 1,2,\dots,N, \ j = 1,2,\dots,m, \end{cases} \tag{4}$$

by using Morse theory coupled with local linking arguments.

On the other hand, the study of boundary value problems for fractional differential equations is an intensively developed area. Recently, there have appeared a very large number of papers which are devoted to the existence of solutions of boundary value problems for fractional differential equations (see [18–21]). Especially, impulsive boundary value problems for fractional differential equations are intensively studied recently. Such problems appear in mathematical models with sudden changes of their states in population dynamics, pharmacology, optimal control (see [22–24]). There have been many approaches to study solutions of boundary value problems for fractional differential equations with or without impulses, such as lower and upper solution method, monotone iterative method ([25, 26]), fixed-point theorems ([11, 27]), Leray-Schauder theory ([22–29]), and so on. However, there have been very few papers published on the existence of solutions for fractional Hamiltonian systems (when $N > 1$) with or without impulses which are done by using the variational method ([30, 31]), since it is often very difficult to establish a suitable space and an appropriate variational functional for fractional differential equations boundary value problems. Since variational method is, to the best of our knowledge, a novel, powerful and promising approach to deal with nonlinear boundary value problems for fractional differential equations with some type of discontinuities such as impulses, it is interesting and necessary for us to continue to explore using the variational method to study such problems.

Motivated by the above, we investigate the existence of a variational construction for problem (1) in an appropriate space of functions in this paper. Then, we study the existence of solutions for (1) using some critical point theorems. All of our these results are new.

The rest of this paper is organized as follows. In Section 2, we give some necessary notation, definitions and properties of the fractional calculus involving the Riemann-Liouville fractional derivative and the Caputo fractional derivative. In Section 3, we introduce critical point theorems as variational tools. In Section 4, we make the variational structure of problem (1). In Section 5, based on some critical point theorems, we establish three theorems on the existence of solutions of (1). In Section 6, we give two examples to show the feasibility and effectiveness of the existence results.

2. Preliminaries

Now, we introduce some basic definitions and properties of the fractional calculus involving the Riemann-Liouville fractional derivative and the Caputo fractional derivative which will be used in later sections. We first briefly recall some basic definitions and results concerning fractional calculus.

DEFINITION 1. [32, 33] Let $f(t)$ be a function defined on $[a,b]$ and $\gamma > 0$. The left and right Riemann-Liouville fractional integrals of order γ for function $f(t)$ denoted by ${}_aD_t^{-\gamma}f(t)$ and ${}_tD_b^{-\gamma}f(t)$, respectively, are defined by

$$ {}_aD_t^{-\gamma}f(t) = \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}f(s)ds, \quad t \in [a,b], \quad \gamma > 0 $$

and

$$ {}_tD_b^{-\gamma}f(t)=\frac{1}{\Gamma(\gamma)}\int_t^b (s-t)^{\gamma-1}f(s)ds, \quad t\in[a,b], \quad \gamma>0, $$

provided the right hand sides are pointwise defined on $[a,b]$, where $\Gamma>0$ is the gamma function.

DEFINITION 2. [32, 33] Let $f(t)$ be a function defined on $[a,b]$ and $\gamma>0$. The left and right Riemann-Liouville fractional derivatives of order γ for function $f(t)$ denoted by ${}_aD_t^{\gamma}f(t)$ and ${}_tD_b^{\gamma}f(t)$, respectively, are defined by

$$ {}_aD_t^{\gamma}f(t)=\frac{d^n}{dt^n}{}_aD_t^{\gamma-n}f(t)=\frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\Big(\int_a^t (t-s)^{n-\gamma-1}f(s)ds\Big) $$

and

$$ {}_tD_b^{\gamma}f(t)=(-1)^n\frac{d^n}{dt^n}{}_tD_b^{\gamma-n}f(t)=\frac{1}{\Gamma(n-\gamma)}(-1)^n\frac{d^n}{dt^n}\Big(\int_t^b (s-t)^{n-\gamma-1}f(s)ds\Big), $$

where $t\in[a,b]$, $n-1\leqslant\gamma<n$ and $n\in\mathbb{N}$. In particular, if $0\leqslant\gamma<1$, then

$$ {}_aD_t^{\gamma}f(t)=\frac{d}{dt}{}_aD_t^{\gamma-1}f(t)=\frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\Big(\int_a^t (t-s)^{-\gamma}f(s)ds\Big), \quad t\in[a,b] $$

and

$$ {}_tD_b^{\gamma}f(t)=-\frac{d}{dt}{}_tD_b^{\gamma-1}f(t)=-\frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\Big(\int_t^b (s-t)^{-\gamma}f(s)ds\Big), \quad t\in[a,b]. \tag{1} $$

The left and right Caputo fractional derivatives are defined according to the above Riemann-Liouville fractional derivatives (see [32], p. 91). Especially, they are defined for the function belonging to the space of absolutely continuous functions.

DEFINITION 3. ([32]) Let $\gamma\geqslant 0$ and $n\in\mathbb{N}$.

(i) If $\gamma\in[n-1,n)$ and $f\in AC^n([a,b],\mathbb{R}^N)$, then the left and right Caputo fractional derivatives of order γ for function $f(t)$ denoted by ${}_a^cD_t^{\gamma}f(t)$ and ${}_t^cD_b^{\gamma}f(t)$, respectively, exist almost everywhere on $[a,b]$. ${}_a^cD_t^{\gamma}f(t)$ and ${}_t^cD_b^{\gamma}f(t)$ are represented by

$$ {}_a^cD_t^{\gamma}f(t)={}_aD_t^{\gamma-n}f^{(n)}(t)=\frac{1}{\Gamma(n-\gamma)}\left(\int_a^t (t-s)^{n-\gamma-1}f^{(n)}(s)ds\right), \quad t\in[a,b] $$

and

$$ {}_t^cD_b^{\gamma}f(t)=(-1)^n{}_tD_b^{\gamma-n}f^{(n)}(t)=\frac{(-1)^n}{\Gamma(n-\gamma)}\left(\int_t^b (s-t)^{n-\gamma-1}f^{(n)}(s)ds\right), \quad t\in[a,b], $$

respectively. In particular, if $0<\gamma<1$, then

$$ {}_a^cD_t^{\gamma}f(t)={}_aD_t^{\gamma-1}f'(t)=\frac{1}{\Gamma(1-\gamma)}\left(\int_a^t (t-s)^{n-\gamma-1}f^{(n)}(s)ds\right), \quad t\in[a,b] \tag{2} $$

and

$$ {}_t^cD_b^{\gamma}f(t)=(-1)^n{}_tD_b^{\gamma-1}f'(t)=\frac{(-1)^n}{\Gamma(1-\gamma)}\left(\int_t^b (s-t)^{n-\gamma-1}f^{(n)}(s)ds\right), \quad t\in[a,b]. $$

(ii) If $\gamma = n-1$ and $f \in AC^{n-1}([a,b],\mathbb{R}^N)$, then ${}_a^cD_t^{n-1}f(t)$ and ${}_t^cD_b^{n-1}f(t)$ are represented by

$$ {}_a^cD_t^{n-1}f(t) = f^{(n-1)}(t) \text{ and } {}_t^cD_b^{n-1}f(t) = (-1)^{n-1}f^{(n-1)}(t), t \in [a,b]. $$

In particular, ${}_a^cD_t^0 f(t) = {}_t^cD_b^0 f(t) = f(t), t \in [a,b]$.

In view of these definitions, it should be noted some of the properties of the Riemann-Liouville fractional integral and derivative operators.

PROPERTY 1. ([32]) We have the following property of fractional integration

$$ \int_a^b \left[{}_aD_t^{-\gamma}f(t)\right]g(t)dt = \int_a^b \left[{}_tD_b^{-\gamma}g(t)\right]f(t)dt, \; \gamma > 0, $$

provided that $f \in L^p([a,b],\mathbb{R}^N)$, $g \in L^q([a,b],\mathbb{R}^N)$ and $p \geqslant 1$, $q \geqslant 1$, $\frac{1}{p} + \frac{1}{q} \leqslant 1+\gamma$ or $p \neq 1$, $q \neq 1$, $\frac{1}{p} + \frac{1}{q} = 1+\gamma$.

Throughout this paper, we denote by the norm of the space $L^p([0,T],\mathbb{R}^N)$ for $1 \leqslant p \leqslant +\infty$ as $\|u\|_{L^p} = \left(\int_0^T |u(t)|^p dt\right)^{\frac{1}{p}}$ and $\|u\|_\infty = \max_{u\in[0,T]} |u(t)|$.

DEFINITION 4. ([30]) Let $0 < \alpha \leqslant 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^\infty([0,T],\mathbb{R}^N)$ with respect to the norm

$$ \|u\|_{\alpha,p} = \left(\int_0^T |u(t)|^p dt + \int_0^T |{}_0^cD_t^\alpha u(t)|^p dt\right)^{\frac{1}{p}}, \quad \forall u \in E_0^{\alpha,p}, $$

where

$$ C_0^\infty([0,T],\mathbb{R}^N) = \{u : [0,T] \longrightarrow \mathbb{R}^N \big| u(0) = u(T) = 0, \; u \in C^\kappa([0,T],\mathbb{R}^N), \kappa \in \mathbb{R}\}. $$

REMARK 1. ([30])

(i) It is obviously that this fractional derivative space $E_0^{\alpha,p}$ is the space of functions $u \in L^p([0,T],\mathbb{R}^N)$ having an α-order Caputo fractional derivative ${}_0^cD_t^\alpha u \in L^p([0,T],\mathbb{R}^N)$ and $u(0) = u(T) = 0$.

(ii) For any $u \in E_0^{\alpha,p}$, noting the fact that $u(0) = 0$, we have ${}_0^cD_t^\alpha u(t) = {}_0D_t^\alpha u(t)$, $t \in [0,T]$.

PROPOSITION 1. ([30], Proposition 3.1) *Let* $0 < \alpha \leqslant 1$ *and* $1 < p < \infty$. *The fractional derivative spaces* $E_0^{\alpha,p}$ *are reflexive and separable Banach spaces.*

PROPOSITION 2. ([30], Proposition 3.2) *Let* $0 < \alpha \leqslant 1$ *and* $1 < p < \infty$. $E_0^{\alpha,p} \hookrightarrow L^p([0,T],\mathbb{R}^N)$ *is compact and for all* $u \in E_0^{\alpha,p}$, *we have*

$$ \|u\|_{L^p} \leqslant \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0^cD_t^\alpha u\|_{L^p}. $$

Moreover, if $\alpha > \frac{1}{p}$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\|u\|_\infty \leqslant \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\big((\alpha-1)q+1\big)^{\frac{1}{q}}}\|{}_0^cD_t^\alpha u\|_{L^p}.$$

PROPOSITION 3. ([30], Proposition 3.3) *Let* $0<\alpha\leqslant 1$ *and* $1<p<\infty$. *Assume that* $\alpha > \frac{1}{p}$ *and the sequence* $\{u_n\}$ *converges weakly to* u *in* $E_0^{\alpha,p}$, *i.e.,* $u_k \rightharpoonup u$. *Then* $\{u_k\}$ *converges strongly to* u *in* $C([0,T],\mathbb{R}^N)$, *i.e.,* $\|u_k-u\|_\infty \to 0$, *as* $k\to+\infty$.

In this paper, we study problem (1) in the Hilbert space $E^\alpha \doteq E_0^{\alpha,2}$ with the inner product and the corresponding norm defined by

$$\langle u,v\rangle = \int_0^T \big(u(t),v(t)\big)\mathrm{d}t + \int_0^T \big({}_0^cD_t^\alpha u(t),{}_0^cD_t^\alpha v(t)\big)\mathrm{d}t \quad \forall u,v\in E^\alpha$$

and

$$\|u\|_\alpha = |u\|_{\alpha,2} = \left(\int_0^T |u(t)|^2\mathrm{d}t + \int_0^T |{}_0^cD_t^\alpha u(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}, \quad \forall u\in E^\alpha. \tag{3}$$

Now, we will prove a continuous differentiability theorem for a class of functionals in space E^α.

THEOREM 1. *Let* $0<\alpha\leqslant 1$, $\frac{1}{p}<\alpha\leqslant 1$, $L:[0,T]_{\mathbb{T}}\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R},(t,x,y)\to L(t,x,y)$ *be measurable in* t *for each* $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$ *and continuously differentiable in* (x,y) *for almost every* $t\in[0,T]$. *If there exist* $a\in C(\mathbb{R}^+,\mathbb{R}^+)$, $b\in L^1([0,T],\mathbb{R}^+)$ *and* $c\in L^q([0,T],\mathbb{R}^+)$ $(1<q<+\infty)$ *such that for almost* $t\in[0,T]$ *and every* $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$, *one has*

$$\begin{aligned} |L(t,x,y)| &\leqslant a(|x|)\big(b(t)+|y|^p\big),\\ |L_x(t,x,y)| &\leqslant a(|x|)\big(b(t)+|y|^p\big),\\ |L_y(t,x,y)| &\leqslant a(|x|)\big(c(t)+|y|^{p-1}\big), \end{aligned} \tag{4}$$

where $\frac{1}{p}+\frac{1}{q}=1$, *then the functional* $E_0^{\alpha,p}\to\mathbb{R}$ *defined as*

$$\Phi(u) = \int_0^T L\big(t,u(t),{}_0^cD_t^\alpha u(t)\big)\,\mathrm{d}t$$

is continuously differentiable on $E_0^{\alpha,p}$ *and*

$$\big\langle\Phi'(u),v\big\rangle = \int_0^T \Big[\big(L_x\big(t,u(t),{}_0^cD_t^\alpha u(t)\big),v(t)\big) + \big(L_y\big(t,u(t),{}_0^cD_t^\alpha u(t)\big),{}_0^cD_t^\alpha v(t)\big)\Big]\,\mathrm{d}t. \tag{5}$$

Proof. It suffices to prove that Φ has at every point u a directional derivative $\Phi'(u)\in\big(E_0^{\alpha,p}\big)^*$ given by (5) and that the mapping

$$\Phi' : E_0^{\alpha,p} \to \big(E_0^{\alpha,p}\big)^*$$

is continuous.

Firstly, it follows from (4) that Φ is everywhere finite on $E_0^{\alpha,p}$. We define, for u and v fixed in $E_0^{\alpha,p}, t \in [0,T], \lambda \in [-1,1]$,

$$G(\lambda,t) = L\big(t, u(t)+\lambda v(t), {}_0^cD_t^\alpha u(t) + \lambda\, {}_0^cD_t^\alpha v(t)\big)$$

and

$$\Psi(\lambda) = \int_{[0,T)_{\mathbb{T}}} G(\lambda,t)\,\Delta t = \Phi(u+\lambda v).$$

From (4), we have

$$\begin{aligned}
\big|D_\lambda G(\lambda,t)\big| &\leqslant \big|\big(D_x L\big(t,u(t)+\lambda v(t), {}_0^cD_t^\alpha u(t)+\lambda\, {}_0^cD_t^\alpha v(t)\big), v(t)\big)\big| \\
&\quad + \big|\big(D_y L\big(t,u(t)+\lambda v(t), {}_0^cD_t^\alpha u(t)+\lambda\, {}_0^cD_t^\alpha v(t)\big), {}_0^cD_t^\alpha v(t)\big)\big| \\
&\leqslant a\big(|u(t)+\lambda v(t)|\big)\big(b(t)+|{}_0^cD_t^\alpha u(t)+\lambda\, {}_0^cD_t^\alpha v(t)|^p\big)|v(t)| \\
&\quad + a\big(|u(t)+\lambda v(t)|\big)\big(c(t)+|{}_0^cD_t^\alpha u(t)+\lambda\, {}_0^cD_t^\alpha v(t)|^{p-1}\big)|{}_0^cD_t^\alpha v(t)| \\
&\leqslant \overline{a}\big(b(t)+\big(|{}_0^cD_t^\alpha u(t)|+|{}_0^cD_t^\alpha v(t)|\big)^p\big)|v(t)| \\
&\quad + \overline{a}\big(c(t)+\big(|{}_0^cD_t^\alpha u(t)|+|{}_0^cD_t^\alpha v(t)|\big)^{p-1}\big)|{}_0^cD_t^\alpha v(t)| \\
&\triangleq d(t),
\end{aligned} \tag{6}$$

where

$$\overline{a} = \max_{(\lambda,t)\in[-1,1]\times[0,T]} a\big(|u(t)+\lambda v(t)|\big),$$

thus, $d \in L^1([0,T],\mathbb{R}^+)$. Since $b \in L^1([0,T],\mathbb{R}^+)$, $(|{}_0^cD_t^\alpha u| + |{}_0^cD_t^\alpha v|)^p \in L^1([0,T],\mathbb{R})$, $c \in L^q([0,T],\mathbb{R}^+)$, we have

$$|D_\lambda G(\lambda,t)| \leqslant d(t),$$

and

$$\begin{aligned}
\Psi'(0) &= \int_0^T D_\lambda G(0,t)\,\mathrm{d}t \\
&= \int_0^T \big[\big(L_x\big(t,u(t),{}_0^cD_t^\alpha u(t)\big), v(t)\big) + \big(L_y\big(t,u(t),{}_0^cD_t^\alpha u(t)\big), {}_0^cD_t^\alpha v(t)\big)\big]\,\mathrm{d}t.
\end{aligned} \tag{7}$$

On the other hand, it follows from (4) that

$$|D_x L\big(t,u(t),{}_0^cD_t^\alpha u(t)\big)| \leqslant a(|u(t)|)\big(b(t)+|{}_0^cD_t^\alpha u(t)|^p\big) \triangleq \psi_1(t), \tag{8}$$

$$|D_y L\big(t,u(t),{}_0^cD_t^\alpha u(t)\big)| \leqslant a(|u(t)|)\big(c(t)+|{}_0^cD_t^\alpha u(t)|^{p-1}\big) \triangleq \psi_2(t), \tag{9}$$

thus $\psi_1 \in L^1([0,T],\mathbb{R}^+)$, $\psi_2 \in L^q([0,T],\mathbb{R}^+)$. Thereby, by Theorem 2.7, (7), (8) and (9), there exists positive constants C_2, C_3, C_4 such that

$$\begin{aligned}
&\int_0^T \big[\big(L_x\big(t,u(t),{}_0^cD_t^\alpha u(t)\big), v(t)\big) + \big(L_y\big(t,u(t),{}_0^cD_t^\alpha u(t)\big), {}_0^cD_t^\alpha v(t)\big)\big]\,\mathrm{d}t \\
&\leqslant C_1\|v\|_\infty + C_2\|{}_0^cD_t^\alpha v\|_{L^p} \\
&\leqslant C_3\|v\|_{\alpha,p}
\end{aligned}$$

and Φ has a directional derivative at u and $\Phi'(u) \in (E^{\alpha,p})^*$ given by (5).

Moreover, (4) implies that the mapping from $E^{\alpha,p}$ into $L^1([0,T],\mathbb{R}^N) \times L^q([0,T],\mathbb{R}^N)$ defined by

$$u \to \left(D_x L\left(\cdot,u,{}_0^cD_t^\alpha u\right), D_y L\left(\cdot,u,{}_0^cD_t^\alpha u\right)\right)$$

is continuous, so that Φ' is continuous from $E^{\alpha,p}$ into $(E^{\alpha,p})^*$. The proof is complete. $\square$

3. Variational tools

For the sake of the proof for the existence of solutions to problem (1), the following definitions and critical point theorems are needed as tools.

DEFINITION 5. ([1], p. 81) Let X be a real Banach space and $I \in C^1(X,\mathbb{R})$. I is said to satisfy (PS) condition on X if any sequence $\{x_n\} \subseteq X$ for which $I(x_n)$ is bounded and $I'(x_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence in X.

DEFINITION 6. ([1]) Let X be a real Banach space and $I \in C^1(X,\mathbb{R})$. I is said to satisfy (PS) condition on X if any sequence $\{x_n\} \subseteq X$ for which $I(x_n) \to c$ and $I'(x_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence in X.

REMARK 2. It is clear that the (PS) condition implies the $(\mathrm{PS})_c$ condition for any constant.

THEOREM 2. (Theorem 1.2, [1]) *If X is a normed space and $\varphi : X \to (-\infty,+\infty)$ is lower semi-continuous and convex, then φ is weakly lower semi-continuous.*

THEOREM 3. (Theorem 4.7, [1]) *Let X be a Banach space and let $\Phi \in C^1(X,\mathbb{R})$. Assume that X splits into a direct sum of closed subspace $X = X^- \oplus X^+$ with*

$$\dim X^- < \infty$$

and

$$\sup_{S_R^-} \Phi < \inf_{X^+} \Phi,$$

where $S_R^- = \{u \in X^- : \|u\| = R\}$. Let

$$B_R^- = \{u \in X^- : \|u\| \leqslant R\},$$
$$M = \{h \in C(B_R^-, X) : h(s) = s \text{ if } s \in S_R^-\}$$

and

$$c = \inf_{h \in M} \max_{s \in B_R^-} \Phi(h(s)).$$

Then, if Φ satisfies the $(\mathrm{PS})_c$ condition, c is a critical value of Φ.

4. Variational framework

In this section, we will establish a variational framework that enables us to reduce the existence of solutions of problem (1) to the one of finding critical points of corresponding functional.

Take $v \in E^{\alpha}$ and multiply the two sides of the equality

$$ {}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right) = \nabla F(t,u(t)) $$

by v and integrate from 0 to T, we have

$$ \int_0^T \left[{}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right) - \nabla F(t,u(t))\right]v(t)\,\mathrm{d}t = 0. \tag{1} $$

Moreover, by Remark 1, 1, (1) and (2), one has

$$ \begin{aligned} &\int_0^T \left({}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right), v(t)\right)\mathrm{d}t \\ &= -\sum_{j=0}^{p}\int_{t_j}^{t_{j+1}} \left(\frac{d}{dt}\left({}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u(t)\right)\right), v(t)\right)\mathrm{d}t \\ &= -\sum_{j=0}^{p}\left[\left({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t_{j+1}^-), v(t_{j+1}^-)\right) - \left({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t_j^+), v(t_j^+)\right)\right] \\ &\quad + \sum_{j=0}^{p}\int_{t_j}^{t_{j+1}} \left({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t), v'(t)\right)\mathrm{d}t \\ &= -\sum_{j=0}^{p}\sum_{i=1}^{N}\left[\left({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u^i)(t_{j+1}^-), v^i(t_{j+1}^-)\right) - \left({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u^i)(t_j^+), v^i(t_j^+)\right)\right] \\ &\quad + \sum_{j=0}^{p}\int_{t_j}^{t_{j+1}} \left({}_0^cD_t^{\alpha}u(t), {}_0D_t^{\alpha-1}v'(t)\right)\mathrm{d}t\Big] \\ &= {}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(0)v(0) - {}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(T)v(T) \\ &\quad + \sum_{j=1}^{p}\sum_{i=1}^{N}\left[{}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u^i)(t_j^+) - {}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u^i)(t_j^-)\right]v^i(t_j) \\ &\quad + \int_{t_j}^{t_{j+1}} \left({}_0^cD_t^{\alpha}u(t), {}_0^cD_t^{\alpha}v(t)\right)\mathrm{d}t \\ &= \sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}u^i(t_j))v^i(t_j) + \int_0^T \left({}_0^cD_t^{\alpha}u(t), {}_0^cD_t^{\alpha}v(t)\right)\mathrm{d}t. \end{aligned} $$

By means of (1), we have

$$ \int_0^T \left({}_0^cD_t^{\alpha}u(t), {}_0^cD_t^{\alpha}v(t)\right)\mathrm{d}t + \sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}u^i(t_j))v^i(t_j) - \int_0^T \left(\nabla F(t,u(t)), v(t)\right)\mathrm{d}t = 0. $$

On account of above, we give the following concept solution for problem (1).

DEFINITION 7. A function $u \in E^{\alpha}$ is a weak solution of problem (1) if the identity

$$\int_0^T \left({}_0^cD_t^{\alpha}u(t), {}_0^cD_t^{\alpha}v(t)\right) \mathrm{d}t + \sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}u^i(t_j))v^i(t_j) - \int_0^T \left(\nabla F(t,u(t)), v(t)\right) \mathrm{d}t = 0$$

holds for any $v \in E^{\alpha}$.

We define the functional $\Phi : E^{\alpha} \to \mathbb{R}$ as

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_0^T \left|{}_0^cD_t^{\alpha}u(t)\right|^2 \mathrm{d}t + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)\,\mathrm{d}t - \int_0^T F(t,u(t))\mathrm{d}t \\ &= \psi(u) + \phi(u), \end{aligned} \tag{2}$$

where

$$\psi(u) = \frac{1}{2}\int_0^T \left|{}_0^cD_t^{\alpha}u(t)\right|^2 \mathrm{d}t - \int_0^T F(t,u(t))\,\mathrm{d}t$$

and

$$\phi(u) = \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)\,\mathrm{d}t.$$

Then, we can prove the following facts.

THEOREM 4. *The functional Φ is continuously differentiable on E^{α} and*

$$\begin{aligned}\langle \Phi'(u), v\rangle = &\int_0^T \left({}_0^cD_t^{\alpha}u(t), {}_0^cD_t^{\alpha}v(t)\right) \mathrm{d}t + \sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}(u^i(t_j))v^i(t_j) \\ &- \int_0^T \left(\nabla F(t,u(t)), v(t)\right) \mathrm{d}t.\end{aligned}$$

The proof of this theorem is very simple, we omit it.

THEOREM 5. *If $u \in E^{\alpha}$ is a critical point of Φ in E^{α}, i.e., $\Phi'(u) = 0$, then u is a weak solution of problem* (1).

Proof. Since $\Phi'(u) = 0$, in the light of Theorem4, one has

$$\int_0^T \left({}_0^cD_t^{\alpha}u(t), {}_0^cD_t^{\alpha}v(t)\right) \mathrm{d}t + \sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}(u^i(t_j))v^i(t_j) - \int_0^T \left(\nabla F(t,u(t)), v(t)\right) \mathrm{d}t = 0.$$

By means of Definition 7, u is a weak solution of problem (1). $\square$

5. Existence results

In order to use the saddle point theorem (Theorem 3), we decompose the space E^α. For $u \in E^\alpha$, let $\overline{u} = \frac{1}{T}\int_0^T u(t)\,\mathrm{d}t$ and $\widetilde{u}(t) = u(t) - \overline{u}$, then

$$E^\alpha = \mathbb{R}^N \oplus \widetilde{E^\alpha},$$

where $\widetilde{E^\alpha} = \left\{u \in E^\alpha \middle| \int_0^T u(t)\,\mathrm{d}t = 0\right\}$

LEMMA 1. *The functional $H : E^\alpha \to \mathbb{R}$ denoted by*

$$H(u) = \frac{1}{2}\int_0^T \left|{}_0^cD_t^\alpha u(t)\right|^2 \mathrm{d}t$$

is convex and continuous on E^α.

This lemma is trivial, hence, we omit the proof of this lemma.

LEMMA 2. *Φ is weakly lower semi-continuous on E^α.*

Proof. On the basis of Theorem 2 and Lemma 1, we can deduce that H is weakly lower semi-continuous on E^α. More than this, by Proposition 3, the functional $G : E^\alpha \to \mathbb{R}$ denoted by

$$G(u) = \int_0^T F(t, u(t))\mathrm{d}t$$

is weakly continuous on E^α. Consequently, ψ is weakly lower semi-continuous on E^α. So far, it suffices to show that ϕ is weakly continuous on E^α. In fact, if $\{u_k\}_{k\in\mathbb{N}} \subseteq E^\alpha$, $u_k \rightharpoonup u$, then from Proposition 3, $\{u_k\}_{k\in\mathbb{N}}$ converges uniformly to u on $[0,T]$. Thereby, there exists $C_4 > 0$ such that

$$\|u_k\|_\infty \leqslant C_4, \quad \forall k \in \mathbb{N}.$$

Accordingly, we can assert that

$$\begin{aligned}
\left|\phi(u_k) - \phi(u)\right| &= \left|\sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u_k^i(t_j)} I_{ij}(t)\,\mathrm{d}t - \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)\,\mathrm{d}t\right| \\
&\leqslant \sum_{j=1}^{p}\sum_{i=1}^{N}\left|\int_{u^i(t_j)}^{u_k^i(t_j)} I_{ij}(t)\,\mathrm{d}t\right| \\
&\leqslant \frac{pNC_5}{2}\|u_k - u\|_\infty \to 0,
\end{aligned}$$

where $C_5 = \max\limits_{i\in A, j\in B, |t|\leqslant C_4} \left|I_{ij}(t)\right|$. In view of the above, Φ is weakly lower semi-continuous on E^α. $\square$

THEOREM 6. *Suppose that the following conditions hold.*

(i) *There exist* $f,g\in L^1(0,T;\mathbb{R}^+)$ *and* $\zeta\in[0,1)$ *such that*

$$|\nabla F(t,x)|\leqslant f(t)|x|^{\zeta}+g(t)$$

for all $x\in\mathbb{R}^N$ *and a.e.* $t\in[0,T]$.

(ii) $|x|^{-2\zeta}\int_0^T F(t,x)\,\mathrm{d}t\to-\infty$ *as* $|x|\to\infty$.

(iii) *For any* $i\in\Lambda_1, j\in\Lambda_2$,

$$I_{ij}(t)t\geqslant 0,\ \ \forall t\in\mathbb{R}.$$

Then problem (1) *has at least one weak solution which minimizes the function* Φ.

Proof. Combining (i) with Proposition 2, we can get

$$\begin{aligned}
&\left|\int_0^T \big(F(t,u(t))-F(t,\overline{u})\big)\,\mathrm{d}t\right|\\
\leqslant&\left|\int_0^T\int_0^1\big(\nabla F(t,\overline{u}+s\widetilde{u}(t)),\widetilde{u}(t)\big)\,\mathrm{d}s\,\mathrm{d}t\right|\\
\leqslant&\int_0^T\int_0^1 f(t)|\overline{u}+s\widetilde{u}(t)|^{\zeta}|\widetilde{u}(t)|\,\mathrm{d}s\,\mathrm{d}t+\int_0^T\int_0^1 g(t)|\widetilde{u}(t)|\,\mathrm{d}s\,\mathrm{d}t\\
\leqslant&\,2\big(|\overline{u}|^{\zeta}+\|\widetilde{u}\|_\infty^{\zeta}\big)\|\widetilde{u}\|_\infty\int_0^T f(t)\,\mathrm{d}t+\|\widetilde{u}\|_\infty\int_0^T g(t)\,\mathrm{d}t\\
\leqslant&\,\frac{\Gamma^2(\alpha)(2\alpha-1)}{4T^{2\alpha-1}}\|\widetilde{u}\|_\infty^2+\frac{4T^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}|\overline{u}|^{2\zeta}\left(\int_0^T f(t)\,\mathrm{d}t\right)^2\\
&+2\|\widetilde{u}\|_\infty^{\zeta+1}\int_0^T f(t)\,\mathrm{d}t+\|\widetilde{u}\|_\infty\int_0^T g(t)\,\mathrm{d}t\\
\leqslant&\,\frac14\int_0^T|{}_0^cD_t^{\alpha}u(t)|^2\mathrm{d}t+C_6|\overline{u}|^{2\zeta}+C_7\left(\int_0^T|{}_0^cD_t^{\alpha}u(t)|^2\mathrm{d}t\right)^{\frac{\zeta+1}{2}}+C_8\left(\int_0^T|{}_0^cD_t^{\alpha}u(t)|^2\mathrm{d}t\right)^{\frac12}
\end{aligned}$$

for all $u\in E^{\alpha}$, where $C_6=\frac{4T^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}\big(\int_0^T f(t)\,\mathrm{d}t\big)^2$, $C_7=2\big(\frac{T^{\alpha-\frac12}}{\Gamma(\alpha)(2\alpha-1)}\big)^{\zeta+1}\int_0^T|f(t)|\,\mathrm{d}t$ and $C_8=\frac{T^{\alpha-\frac12}}{\Gamma(\alpha)(2\alpha-1)}\int_0^T|g(t)|\,\mathrm{d}t$.

According to (iii), one has

$$\phi(u)\geqslant 0,\ \ u\in E^{\alpha}.$$

Hence, for any $u\in E^{\alpha}$, we can get

$$\begin{aligned}
\Phi(u)&=\frac12\int_0^T\left|{}_0^cD_t^{\alpha}u(t)\right|^2\mathrm{d}t-\int_0^T F(t,u(t))\,\mathrm{d}t+\phi(u)\\
&=\frac12\int_0^T|{}_0^cD_t^{\alpha}u(t)|^2\,\mathrm{d}t-\int_0^T\big(F(t,u(t))-F(t,\overline{u})\big)\,\mathrm{d}t-\int_0^T F(t,\overline{u})\,\mathrm{d}t+\phi(u)\\
&\geqslant\frac14\int_0^T|{}_0^cD_t^{\alpha}u(t)|^2\,\mathrm{d}t-|\overline{u}|^{2\zeta}\left(|\overline{u}|^{-2\zeta}\int_0^T F(t,\overline{u})\,\mathrm{d}t+C_6\right)\\
&\quad-C_7\left(\int_0^T|{}_0^cD_t^{\alpha}u(t)|^2\,\mathrm{d}t\right)^{\frac{\zeta+1}{2}}-C_8\left(\int_0^T|{}_0^cD_t^{\alpha}u(t)|^2\,\mathrm{d}t\right)^{\frac12}.
\end{aligned}\tag{1}$$

As $\|u\|_\alpha \to \infty$ if and only if $(|\overline{u}|^2 + \int_0^T |{}_0^cD_t^\alpha u(t)|^2\,\mathrm{d}t)^{\frac{1}{2}} \to \infty$, (1) and (ii) imply that

$$\Phi(u) \to +\infty \text{ as } \|u\|_\alpha \to \infty.$$

Combining Proposition 1, Theorem 1.1 in [1] with Lemma 2, Φ has a minimum point on E^α, which is a critical point of Φ. Therefore, problem (1) has at least one weak solution. $\square$

THEOREM 7. *Assume that condition* (i) *of Theorem 6 and the following conditions are satisfied.*

(iv) *There exist a_{ij}, $b_{ij} > 0$ and $\beta_{ij} \in (0,1)$ such that*

$$|I_{ij}(t)| \leqslant a_{ij} + b_{ij}|t|^{\zeta\beta_{ij}} \text{ for every } t \in \mathbb{R}, \quad i \in \Lambda_1, \quad j \in \Lambda_2.$$

(v) *For any $i \in \Lambda_1$, $j \in \Lambda_2$,*

$$I_{ij}(t)\ \ t \leqslant 0, \ \ \forall\, t \in \mathbb{R}.$$

(vi) $|x|^{-2\zeta} \int_0^T F(t,x)\,\mathrm{d}t \to +\infty$ *as* $|x| \to \infty$.

Then problem (1) *has at least one weak solution.*

Before proving Theorem 7, we prove the following lemma firstly,

LEMMA 3. *If the conditions of Theorem 7 hold, then Φ satisfies P.S. condition.*

Proof. Let $\{u_n\} \subseteq E^\alpha$ be a P.S. sequence for Φ, that is, $\{\Phi(u_n)\}$ is bounded and $\Phi'(u_n) \to 0$ as $n \to \infty$. We will prove that $\{u_n\}$ consists of a convergent subsequence in E^α. In deed, for all $n \in \mathbb{N}$, according to (i) and Proposition 2, one has

$$\begin{aligned}
&\left| \int_0^T \big(F(t,u_n(t)) - F(t,\overline{u}_n)\big)\,\mathrm{d}t \right| \\
&\leqslant \left| \int_0^T \int_0^1 \big(\nabla F(t,\overline{u}_n + s\widetilde{u}_n(t)), \widetilde{u}_n(t)\big)\,\mathrm{d}s\,\mathrm{d}t \right| \\
&\leqslant \int_0^T \int_0^1 f(t)|\overline{u}_n + s\widetilde{u}_n(t)|^\zeta |\widetilde{u}_n(t)|\,\mathrm{d}s\,\mathrm{d}t + \int_0^T \int_0^1 g(t)|\widetilde{u}_n(t)|\,\mathrm{d}s\,\mathrm{d}t \\
&\leqslant 2\big(|\overline{u}_n|^\zeta + \|\widetilde{u}_n\|_\infty^\zeta\big)\|\widetilde{u}_n\|_\infty \int_0^T f(t)\,\mathrm{d}t + \|\widetilde{u}_n\|_\infty \int_0^T g(t)\,\mathrm{d}t \\
&\leqslant \frac{\Gamma^2(\alpha)(2\alpha-1)}{4T^{2\alpha-1}}\|\widetilde{u}_n\|_\infty^2 + \frac{4T^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}|\overline{u}_n|^{2\zeta}\left(\int_0^T f(t)\,\mathrm{d}t\right)^2 \\
&\quad +2\|\widetilde{u}_n\|_\infty^{\zeta+1}\int_0^T f(t)\,\mathrm{d}t + \|\widetilde{u}_n\|_\infty \int_0^T g(t)\,\mathrm{d}t \\
&\leqslant \frac{1}{4}\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t + C_6|\overline{u}_n|^{2\zeta} \\
&\quad +C_7\left(\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{\zeta+1}{2}} + C_8\left(\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}.
\end{aligned} \tag{2}$$

For convenience, we let $a=\max_{i\in\Lambda_1,j\in\Lambda_2} a_{ij}$, $b=\max_{i\in\Lambda_1,j\in\Lambda_2} b_{ij}$. From (2), (iv) and Young inequality, for sufficiently large n, we have

$$
\begin{aligned}
&\|\widetilde{u}_n\|_\alpha \\
\geqslant\ & \left\langle \Phi'(u_n),\widetilde{u}_n\right\rangle \\
=\ & \int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t-\int_0^T \left(\nabla F(t,u_n(t),\widetilde{u}_n(t)\right)\mathrm{d}t+\sum_{j=1}^p\sum_{i=1}^N I_{ij}\left(u_n^i(t)\right)\widetilde{u}_n^i(t) \\
\geqslant\ & \frac{3}{4}\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t-C_6|\overline{u}_n|^{2\zeta}-C_7\left(\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{\zeta+1}{2}} \\
&-C_8\left(\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}-\sum_{j=1}^p\sum_{i=1}^N\left(a_{ij}+b_{ij}|u_n^i(t)|^{\zeta\beta_{ij}}\right)|\widetilde{u}_n^i(t)| \\
=\ & \frac{3}{4}\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t-C_3|\overline{u}_n|^{2\zeta}-C_4\left(\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{\zeta+1}{2}} \\
&-C_5\left(\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}-\sum_{j=1}^p\sum_{i=1}^N\left(a_{ij}+b_{ij}\left|\overline{u}_n^i(t)+\widetilde{u}_n^i(t)\right|^{\zeta\beta_{ij}}\right)|\widetilde{u}_n^i(t)| \\
\geqslant\ & \frac{3}{4}\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t-C_6|\overline{u}_n|^{2\zeta}-C_7\left(\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{\zeta+1}{2}} \\
&-C_8\left(\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}-apN\|\widetilde{u}_n\|_\infty-b\sum_{j=1}^p\sum_{i=1}^N 2\left(|\overline{u}_n|^{\zeta\beta_{ij}}+\|\widetilde{u}_n\|_\infty^{\zeta\beta_{ij}}\right)\|\widetilde{u}_n\|_\infty \\
\geqslant\ & \frac{3}{4}\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t-C_6|\overline{u}_n|^{2\zeta}-C_7\left(\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{\zeta+1}{2}} \\
&-C_8\left(\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}-apN\frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2(\alpha-1)+1}}\left(\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&-b\sum_{j=1}^p\sum_{i=1}^N\beta_{ij}|\overline{u}_n|^{2\zeta}-2b\sum_{j=1}^p\sum_{i=1}^N\frac{2-\beta_{ij}}{2}\|\widetilde{u}_n\|_\infty^{\frac{2}{2-\beta_{ij}}}-2b\sum_{j=1}^p\sum_{i=1}^N\|\widetilde{u}_n\|_\infty^{\zeta\beta_{ij}+1} \\
\geqslant\ & \frac{3}{4}\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t-C_6|\overline{u}_n|^{2\zeta}-C_7\left(\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{\alpha+1}{2}} \\
&-C_8\left(\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}-apN\frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2(\alpha-1)+1}}\left(\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&-b\sum_{j=1}^p\sum_{i=1}^N\beta_{ij}|\overline{u}_n|^{2\zeta}-2b\sum_{j=1}^p\sum_{i=1}^N\left(\frac{T^{2\alpha-1}}{(\Gamma(\alpha))^2\left(2(\alpha-1)+1\right)}\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{\zeta\beta_{ij}+1}{2}} \\
&-2b\sum_{j=1}^p\sum_{i=1}^N(2-\beta_{ij})\left(\frac{T^{2\alpha-1}}{(\Gamma(\alpha))^2\left(2(\alpha-1)+1\right)}\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2-\beta_{ij}}}.
\end{aligned}
\tag{3}
$$

According to Proposition 2, one has

$$\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2 \leqslant \|\widetilde{u}_n\|_\alpha^2 \leqslant \left(1+\frac{T^{2\alpha}}{(\Gamma(\alpha+1))^2}\right)\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t. \tag{4}$$

The inequalities (3) and (4) imply that for all large n, there exist some positive constants C_9 and C_{10} such that

$$C_9|\overline{u}_n|^\alpha \geqslant \left(\int_0^T |\dot{u}_n(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}} - C_{10}. \tag{5}$$

The same as the proof of Theorem 6, for all n, we have

$$\begin{aligned}&\left|\int_0^T \big(F(t,u_n(t))-F(t,\overline{u}_n)\big)\,\mathrm{d}t\right|\\ &\leqslant \frac{1}{4}\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t + C_6|\overline{u}_n|^{2\zeta} + C_7\left(\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{\zeta+1}{2}}\\ &\quad + C_8\left(\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}.\end{aligned} \tag{6}$$

It follows from the boundedness of $\{\Phi(u_n)\}$, (v), (5) and (6) that for all large n, there exists constant C_{11} and some constant C_{12} such that

$$\begin{aligned}C_{11} &\leqslant \Phi(u_n)\\ &= \frac{1}{2}\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t - \int_0^T \big(F(t,u_n(t))-F(t,\overline{u}_n)\big)\,\mathrm{d}t - \int_0^T F(t,\overline{u}_n)\,\mathrm{d}t + \phi(u)\\ &\leqslant \frac{1}{2}\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t - \int_0^T \big(F(t,u_n(t))-F(t,\overline{u}_n)\big)\,\mathrm{d}t - \int_0^T F(t,\overline{u}_n)\,\mathrm{d}t\\ &\leqslant \frac{3}{4}\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t + C_6|\overline{u}_n|^{2\zeta} - \int_0^T F(t,\overline{u}_n)\,\mathrm{d}t\\ &\quad + C_7\left(\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{\zeta+1}{2}} + C_8\left(\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\\ &\leqslant -|\overline{u}_n|^{2\zeta}\left(|\overline{u}_n|^{-2\zeta}\int_0^T F(t,\overline{u}_n)\,\mathrm{d}t + C_{12}\right).\end{aligned} \tag{7}$$

According to (7) and (vi) that $\{|\overline{u}_n|\}$ is bounded. Thus, $\{u_n\}$ is bounded in E^α from (4) and (5). Thereby, there exists a subsequence of $\{u_n\}$ (for simplicity denoted again by $\{u_n\}$) such that

$$u_n \rightharpoonup u \ \text{ in } E^\alpha. \tag{8}$$

In view of Proposition 3, we have

$$u_n \to u \ \text{ in } C([0,T],\mathbb{R}^N). \tag{9}$$

Furthemore, one has

$$\begin{aligned}
&\left\langle \Phi'(u_n)-\Phi'(u),u_n-u\right\rangle\\
=&\int_0^T \left|{}_0^cD_t^\alpha u_n(t)-{}_0^cD_t^\alpha u(t)\right|^2\mathrm{d}t+\int_0^T \big(\nabla F(t,u_n(t))-\nabla F(t,u(t)),u_n(t)-u(t)\big)\,\mathrm{d}t\\
&+\sum_{j=1}^{p}\sum_{i=1}^{N}\big(I_{ij}\big(u_n^i(t_j)\big)-I_{ij}\big(u^i(t_j)\big)\big)\big(u_n^i(t_j)-u^i(t_j)\big). \qquad (10)
\end{aligned}$$

From (8), (9), (10), (A) and the continuity of I_{ij}, we admit that $u_n\to u$ in E^α. Hence, Φ satisfies P.S. condition. $\square$

Now, we prove Theorem 7.

Proof. As mentioned earlier,

$$E^\alpha=\mathbb{R}^N\oplus\widetilde{E^\alpha},$$

where $\widetilde{E^\alpha}=\left\{u\in E^\alpha\,\middle|\,\int_0^T u(t)\,\mathrm{d}t=0\right\}$. We show that

$$\Phi(u)\to+\infty \ \text{ as } \ u\in\widetilde{E^\alpha},\ \ \|u\|\to\infty. \qquad (11)$$

In reality, for $u\in\widetilde{E^\alpha}$, then $\overline{u}=0$, similar to the proof of Theorem 6, one has

$$\begin{aligned}
&\left|\int_0^T \big(F(t,u(t))-F(t,0)\big)\,\mathrm{d}t\right|\\
\leqslant&\frac{1}{4}\int_0^T |{}_0^cD_t^\alpha u(t)|^2\,\mathrm{d}t+C_7\left(\int_0^T |{}_0^cD_t^\alpha u(t)|^2\,\mathrm{d}t\right)^{\frac{\zeta+1}{2}}+C_8\left(\int_0^T |{}_0^cD_t^\alpha u(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}. \qquad (12)
\end{aligned}$$

Combining (iv) with Proposition 2, we can find

$$\begin{aligned}
|\phi(u)|&=\left|\sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)\,\mathrm{d}t\right|\\
&\leqslant\sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u^i(t_j)}\big(a_{ij}+b_{ij}|t|^{\zeta\beta_{ij}}\big)\,\mathrm{d}t\\
&\leqslant apN\|u\|_\infty+b\sum_{j=1}^{p}\sum_{i=1}^{N}\|u\|_\infty^{\zeta\beta_{ij}+1}\\
&\leqslant\frac{apNT^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}}\left(\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\\
&\quad+b\sum_{j=1}^{p}\sum_{i=1}^{N}\left(\frac{T^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2\,\mathrm{d}t\right)^{\frac{\zeta\beta_{ij}+1}{2}} \qquad (13)
\end{aligned}$$

for all $u \in \widetilde{E^\alpha}$. From (12) and (13), we can get

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_0^T |{}_0^cD_t^\alpha u(t)|^2\,dt - \int_0^T \big(F(t,u(t)) - F(t,0)\big)\,dt - \int_0^T F(t,0)\,dt + \phi(u)\\ &\geqslant \frac{1}{4}\int_0^T |{}_0^cD_t^\alpha u(t)|^2\,dt - C_7\left(\int_0^T |{}_0^cD_t^\alpha u(t)|^2\,dt\right)^{\frac{\zeta+1}{2}} - C_8\left(\int_0^T |{}_0^cD_t^\alpha u(t)|^2\,dt\right)^{\frac{1}{2}}\\ &\quad -b\sum_{j=1}^{p}\sum_{i=1}^{N}\left(\frac{T^{2\alpha-1}}{(\Gamma(\alpha))^2(2(\alpha-1)+1)}\int_0^T |{}_0^cD_t^\alpha u(t)|^2\,dt\right)^{\frac{\zeta\beta_{ij}+1}{2}}\\ &\quad -\frac{apNT^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}}\left(\int_0^T |{}_0^cD_t^\alpha u(t)|^2\,dt\right)^{\frac{1}{2}} + \int_0^T F(t,0)\,dt \end{aligned} \tag{14}$$

for all $u \in \widetilde{E^\alpha}$. Due to Proposition 2, one has

$$\|u\| \to \infty \Leftrightarrow \|\dot{u}\|_{L^2} \to \infty$$

on $\widetilde{E^\alpha}$. Therefore, (11) follows from (14).

Moreover, from (v), we can see

$$\phi(u) \leqslant 0 \tag{15}$$

for all $u \in E^\alpha$. Thereby, combining (15) with (vi), we obtain

$$\Phi(u) = -\int_0^T F(t,u)\,dt + \phi(u) \leqslant -|u|^{2\zeta}\left(|u|^{-2\zeta}\int_0^T F(t,u)\,dt\right) \to -\infty$$

as $|u| \to \infty$ in $\mathbb{R}^N$. It follows from Theorem 3 and Lemma 3 that problem (1) has at least one weak solution. $\square$

THEOREM 8. *Assume that assumption* (iii) *and the following condition hold.*

(vii) $F(t,\cdot)$ *is concave for a.e.* $t \in [0,T]$ *and that*

$$\int_0^T F(t,x)\,dt \to +\infty \text{ as } |x| \to \infty.$$

Then problem (1) *has at least one weak solution which minimizes the function* Φ.

Proof. Seeing that (iii), we find

$$\phi(u) \geqslant 0 \tag{16}$$

for all $u \in E^\alpha$. In addition, from assumption (A), the function $G : \mathbb{R}^N \to \mathbb{R}$ defined by

$$G(x) = -\int_0^T F(t,x)\,dt$$

has a minimum at some point $\overline{x}$ for which

$$\int_0^T \nabla F(t,\overline{x})\,dt = 0. \tag{17}$$

For any minimizing sequence $\{u_k\}$ of Φ, by Proposition 1.4 in [1], (16) and (17), we claim that

$$\begin{aligned}\Phi(u_k) &= \frac{1}{2}\int_0^T |{}_0^cD_t^\alpha u_k(t)|^2\,\mathrm{d}t - \int_0^T \big(F(t,u_k(t)) - F(t,\overline{x})\big)\,\mathrm{d}t - \int_0^T F(t,\overline{x})\,\mathrm{d}t + \phi(u)\\ &\geqslant \frac{1}{2}\int_0^T |{}_0^cD_t^\alpha u_k(t)|^2\,\mathrm{d}t - \int_0^T F(t,\overline{x})\,\mathrm{d}t - \int_0^T \big(\nabla F(t,\overline{x}), u_k(t)-\overline{x}\big)\,\mathrm{d}t\\ &= \frac{1}{2}\int_0^T |{}_0^cD_t^\alpha u_k(t)|^2\,\mathrm{d}t - \int_0^T F(t,\overline{x})\,\mathrm{d}t - \int_0^T \big(\nabla F(t,\overline{x}), \widetilde{u}_k(t)\big)\,\mathrm{d}t,\end{aligned} \tag{18}$$

where $\widetilde{u}_k(t) = u_k(t) - \overline{u}_k$, $\overline{u}_k = \frac{1}{T}\int_0^T u_k(t)\,\mathrm{d}t$. As a result of (18), (A) and Proposition 2.2, there exist some positive constants C_{13} and C_{14} such that

$$\begin{aligned}\Phi(u_k) &\geqslant \frac{1}{2}\int_0^T |{}_0^cD_t^\alpha u_k(t)|^2\,\mathrm{d}t - \int_0^T F(t,\overline{x})\,\mathrm{d}t - \left(\int_0^T \big|\nabla F(t,\overline{x})\big|\,\mathrm{d}t\right)\|\widetilde{u}_k\|_\infty\\ &\geqslant \frac{1}{2}\int_0^T |{}_0^cD_t^\alpha u_k(t)|^2\,\mathrm{d}t + C_{13} - C_{14}\left(\int_0^T |{}_0^cD_t^\alpha u_k(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}.\end{aligned} \tag{19}$$

Hence, from (19), there exists $C_{15} > 0$ such that

$$\int_0^T |{}_0^cD_t^\alpha u_k(t)|^2\,\mathrm{d}t \leqslant C_{15}. \tag{20}$$

On the other hand, Proposition 2 and (20) imply that there exists $C_{16} > 0$ such that

$$\|\widetilde{u}_k\|_\infty \leqslant C_{16}. \tag{21}$$

According to (vii), we have

$$\begin{aligned}F\left(t,\frac{\overline{u}_k}{2}\right) &= F\left(t,\frac{u_k(t)-\widetilde{u}_k(t)}{2}\right)\\ &\geqslant \frac{1}{2}F(t,u_k(t)) + \frac{1}{2}F(t,-\widetilde{u}_k(t))\end{aligned} \tag{22}$$

for a.e. $t \in [0,T]$ and all $k \in \mathbb{N}$. By means of (16) and (22), we obtain

$$\Phi(u_k) \geqslant \frac{1}{2}\int_0^T |{}_0^cD_t^\alpha u_k(t)|^2\,\mathrm{d}t - 2\int_0^T F\left(t,\frac{\overline{u}_k}{2}\right)\mathrm{d}t + \int_0^T F(t,-\widetilde{u}_k(t))\,\mathrm{d}t. \tag{23}$$

Connecting (21) and (23), there exists $C_{17} > 0$ such that

$$\Phi(u_k) \geqslant -2\int_0^T F\left(t,\frac{\overline{u}_k}{2}\right)\mathrm{d}t - C_{17}. \tag{24}$$

According to inequality (24) and (vii), $\{\overline{u}_k\}$ is bounded. This shows that $\{u_k\}$ is bounded in E^α via Proposition 2.2 and inequality (20). In consideration of Proposition 1, Theorem 1.1 in [1] and Lemma 2, Φ has a minimum point on E^α, which is a critical point of Φ. This means that problem (1) has at least one weak solution which minimizes the function Φ. □

6. Examples

In this section, let's give two examples to illustrate the feasibility and effectiveness of our main results.

EXAMPLE 1. Let $\alpha = \frac{2}{3}$, $T = \pi$, $N = 4$, $t_1 = 2$. Consider the fractional Hamiltonian system with impulsive effects

$$\begin{cases} {}_tD_T^{\alpha}\big({}_0^cD_t^{\alpha}u(t)\big) = \nabla F(t,u(t)), \qquad \text{a.e. } t\in[0,\pi], \\ u(0)=u(\pi)=0, \\ \Delta\big({}_tD_T^{\alpha-1}\big({}_0^cD_t^{\alpha}u^i\big)\big)(2) = I_{ij}(u^i(2)), \;\; i=1,2,3,4, \;\; j=1, \end{cases} \tag{1}$$

where $F(t,x) = (t-6)|x|^{\frac{3}{2}} + \big((1,2,3,4),x\big)$.

In view of $F(t,x) = (t-6)|x|^{\frac{3}{2}} + \big((1,2,3,4),x\big)$, $I_{ij}(t) = t^{\frac{3}{5}}$, $\zeta = \frac{1}{2}$, all of the conditions of Theorem 6 are satisfied. It follows from Theorem 6 that problem (1) has at least one weak solution. Not only this, from Definition 7, 0 is not the weak solution of problem (1). This means that problem (1) has at least one nontrivial weak solution.

EXAMPLE 2. Let $\alpha = \frac{3}{3}$, $T = 1$, $N = 3$, $t_1 = \frac{1}{2}$. Consider the fractional Hamiltonian system with impulsive effects

$$\begin{cases} {}_tD_T^{\alpha}\big({}_0^cD_t^{\alpha}u(t)\big) = \nabla F(t,u(t)), \qquad \text{a.e. } t\in[0,1], \\ u(0)=u(1)=0, \\ \Delta\big({}_tD_T^{\alpha-1}\big({}_0^cD_t^{\alpha}u^i\big)\big)(\tfrac{1}{2}) = I_{ij}(u^i(\tfrac{1}{2})), \;\; i=1,2,3, \;\; j=1, \end{cases} \tag{2}$$

where $F(t,x) = (t-\frac{\pi}{6})|x|^{\frac{4}{3}} + \big((1,1,1),x\big)$, $I_{i1}(t) = -t^{\frac{1}{5}}$.

Owing to $F(t,x) = (t-\frac{\pi}{6})|x|^{\frac{4}{3}} + \big((1,1,1),x\big)$, $I_{i1}(t) = -t^{\frac{1}{5}}$, $\zeta = \frac{1}{3}$, $\beta_{i1} = \frac{1}{3}$, by calculation, every condition of Theorem 7 holds. According to Theorem 7, problem (2) has at least one weak solution. Furthemore, by Definition 7, 0 is not the weak solution of problem (2). Hence, problem (2) has at least one nontrivial weak solution.

REMARK 3. If (2) without impulses, that is, $I_{i1}(t)\equiv 0$, each condition of Theorem 7 holds. As a result of Theorem 7, problem (2) has at least one weak solution. By calculus, 0 is not the weak solution of problem (2). Therefore, problem (2) has at least one nontrivial weak solution.

Conflict of Interest. The authors declare that they have no conflict of interest.

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