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MINIMAL AND MAXIMAL SOLUTIONS FOR A FRACTIONAL BOUNDARY VALUE PROBLEM AT RESONANCE ON THE HALF LINE

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Abstract. This paper is devoted to the study of a Riemann-Liouville fractional boundary value problem on an unbounded inteval. The problem is assumed to be at resonance and the boundary conditions are of nonlocal type. We obtain some existence results for the maximal and minimal solutions by means of a fixed point theorem for an increasing operator and lower and upper solutions.

1. Introduction

In this paper, we are concerned with the existence of solutions for the following boundary value problem (P) at resonance

$$
\begin{gathered}
D_{0+}^{q}x(t) = f(t,x(t)),\ \ t \in (0,\infty),\\
I_{0+}^{2-q}x(0) = 0, \quad D_{0+}^{q-1}x(\infty) = D_{0+}^{q-1}x(0),
\end{gathered}
$$

where D_{0+}^{q} denotes the Riemann-Liouville fractional derivative of order $q,\ 1<q<2$ and $f:[0,\infty)\times\mathbb{R}\to\mathbb{R}_{+}$ is a given function satisfying some conditions that will be specified here after. We prove the existence of maximal and minimal positive solutions for problem (P) by means of a fixed point theorem for an increasing operator and lower and upper solutions. Since the equation $Lx = D_{0+}^{q}x = 0$ together with the boundary conditions has nontrivial solutions, then problem (P) is at resonance.

Similar boundary value problems at resonance for ordinary and fractional differential equations have been studied recently in [3], [4], [7], [8]–[10], [14], [18], [20] by using coincidence degree theory due to Mawhin.

Different from these works, we will focus on the existence of maximal and minimal positive solutions, between the so called upper and lower solutions for the problem (P) on the half-line.

Fixed point theorems are an important tool in the study of differential equations, in particular the fixed point theorem for increasing operators has been used to investigate some boundary value problems, see [1], [2], [6], [15], [18], [20].

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However, few papers in literature are devoted to the existence of solutions of fractional differential equations on the half-line [1], [3], [9], [10], [11], [14], [17], [19].

This paper is organized as follows. In Section 2, we give some necessary notations, definitions and lemmas. In Section 3, we study the existence of maximal and minimal positive solutions between the lower and upper solutions of problem (P), by a fixed point theorem for an increasing operator.

2. Preliminaries

We recall the definitions of Riemann-Liouville fractional derivative and Riemann-Liouville fractional integral, that we can find their properties in [13].

DEFINITION 1. The Riemann-Liouville fractional integral of order $p>0$ of a function $g\in L_1(0,\infty)$ is given by

$$I_{0^+}^{p}g(t)=\frac{1}{\Gamma(p)}\int_0^t\frac{g(s)}{(t-s)^{1-p}}ds,\ \text{ a.e. } t\in(0,\infty).$$

DEFINITION 2. The Riemann-Liouville fractional derivative of order $p>0$ of a function $g\in AC^n[0,b]$ for a certain $b>0$ is given by

$$D_{0^+}^{p}g(t)=\frac{1}{\Gamma(n-p)}\left(\frac{d}{dt}\right)^n\int_0^t\frac{g(s)}{(t-s)^{p-n+1}}ds,$$

where $n=[p]+1$, ($[p]$ is the integer part of p) and

$$AC^n[0,b]=\left\{g\in C^{n-1}[0,b],g^{(n-1)}\in AC[0,b]\right\}$$

with $AC[0,b]$ denotes the space of absolutely continuous functions on the interval $[0,b]$.

LEMMA 1. *The homogenous fractional differential equation $D_{a^+}^{q}g(t)=0$ has a solution*

$$g(t)=c_1t^{q-1}+c_2t^{q-2}+\ldots+c_nt^{q-n},$$

where, $c_i\in\mathbb{R}$, $i=1,\ldots,n$ and $n=[q]+1$.

Now we give some concepts from coincidence degree theory [16].

Let X and Y be two real Banach spaces and let $L: domL\subset X\to Y$ be a linear operator which is a Fredholm map of index zero. Define the continuous projections P and Q respectively by $P:X\to X$, $Q:Y\to Y$ such that $ImP=\ker L$, $\ker Q=ImL$. Then $X=\ker L\oplus\ker P$ and $Y=ImL\oplus ImQ$, thus $L\mid_{domL\cap\ker P}: domL\cap\ker P\to ImL$ is invertible. Denote its inverse by K_P. There exists an isomorphism $J: ImQ\to\ker L$. It is known that the coincidence equation $Lx=Nx$ is equivalent to $x=(P+JQN+K_p(I-Q)N)x$, furthermore $P+JQN+K_p(I-Q)N=\left(L+J^{-1}P\right)^{-1}\left(N+J^{-1}P\right)$.

Let $(X,\|.\|)$ be a real Banach space. A closed convex set K in X is called a cone if $\lambda K \subset K$ for all $\lambda \geqslant 0$ and $K \cap \{-K\} = \{\theta\}$, where θ denotes the zero element of X. The real Banach space $(X,\|.\|)$ is said partially ordered by a cone K of X, if

$$x, y \in X, \ \ x \preceq y \Leftrightarrow y - x \in K.$$

DEFINITION 3. A cone K in a Banach space $(X,\|.\|)$ is called normal if there exists a constant $N > 0$, such that $\theta \preceq x \preceq y$ implies $\|x\| \leqslant N\|y\|$.

DEFINITION 4. Let $D \subseteq X$. An operator $A : D \to X$ is said to be increasing if

$$x, y \in D, \ \ x \preceq y \Rightarrow Ax \preceq Ay.$$

In view of Theorem 2.1.1 in [6], it is easy to obtain the following result.

THEOREM 1. *Let K be a normal cone in a Banach space X and let $u_0, v_0 \in X$, $u_0 \preceq v_0$ and $A : [u_0, v_0] \to X$ be an increasing operator such that : $u_0 \preceq Au_0$, $Av_0 \preceq v_0$. If the operator A is completely continuous, then A has a maximal fixed point x^* and a minimal fixed point x_* in $[u_0, v_0]$, moreover $x^* = \lim_{n\to\infty} v_n$, $x_* = \lim_{n\to\infty} u_n$, where $u_n = Au_{n-1}$, $v_n = Av_{n-1}$, $n = 1,2,3\ldots$, and*

$$u_0 \preceq u_1 \preceq u_2 \preceq \ldots \preceq u_n \preceq \ldots \preceq v_n \preceq \ldots \preceq v_2 \preceq v_1 \preceq v_0.$$

We need the following compactness criteria:

THEOREM 2. [5] *Let $C_\infty = \{y \in C([0,+\infty)),\ \lim_{t\to\infty} y(t) \text{ exists}\}$ equipped with the norm $\|y\|_\infty = \sup_{t\in[0,+\infty)} |y(t)|$. Let $F \subset C_\infty$. Then F is relatively compact if the following conditions hold:*

(1) *F is bounded in C_∞.*

(2) *The functions from F are equicontinuous on any compact sub-interval of $[0,\infty)$, that is, for any $y \in F$ and for any $t_1, t_2 \in [0,T]$, $t_1 < t_2$ with $T > 0$, we have $|y(t_1) - y(t_2)| \to 0$, as $t_1 \to t_2$.*

(3) *The functions from F are equiconvergent at $+\infty$, that is, for any $\varepsilon > 0$, there exists a $T = T(\varepsilon) > 0$ such that, $|y(t) - \lim_{t\to\infty} y(t)| < \varepsilon$ for all $t \geqslant T$ and $y \in F$.*

3. Main results

Let us define the function space that will be used in the sequel. Let X be the space

$$X = \Big\{ x \in C([0,+\infty)),\ D_{0^+}^{q-1} x \in C([0,+\infty)),\ I_{0^+}^{2-q} x(0) = 0$$
$$\lim_{t\to\infty} e^{-t} x(t) \text{ and } \lim_{t\to\infty} D_{0^+}^{q-1} x(t) \text{ exist} \Big\}$$

endowed with the norm $\|x\| = \sup_{t\geqslant 0} e^{-t} |x(t)| + \sup_{t\geqslant 0} \left| D_{0^+}^{q-1} x(t) \right|$.

LEMMA 2. *The normed space $(X,\|.\|)$ is a Banach space.*

Proof. Let (x_n) be a Cauchy sequence in X and denote $\overline{x}_n(t) = e^{-t}x_n(t)$ for $t \in [0,+\infty)$. Since $\|x_n - x_m\| = \|\overline{x}_n - \overline{x}_m\|_\infty + \left\|D_{0^+}^{q-1}x_n - D_{0^+}^{q-1}x_m\right\|_\infty$, then $(\overline{x}_n)$ and $\left(D_{0^+}^{q-1}x_n\right)$ are Cauchy sequences in the Banach space C_∞ defined in Theorem 2. Consequently $\overline{x}_n \to \overline{x}$ and $D_{0^+}^{q-1}x_n \to y$ in C_∞, from which yields $x_n(t) \to x(t) = e^t\overline{x}(t)$ and $D_{0^+}^{q-1}x_n(t) \to y(t)$, $t \in [0,+\infty)$. Since $I_{0^+}^{2-q}x_n(0) = 0$, then

$$I_{0^+}^{q-1}D_{0^+}^{q-1}x_n(t) = \left[x_n(t) - \frac{t^{q-2}}{\Gamma(q-1)}I_{0^+}^{2-q}x_n(0)\right] = x_n(t) \to I_{0^+}^{q-1}y(t).$$

Hence $x(t) = I_{0^+}^{q-1}y(t)$ and consequently $D_{0^+}^{q-1}x(t) = y(t)$. This shows that $x_n \to x$ in X. The proof is complete. □

Let $Y = L^1[0,+\infty)$ with the norm $\|y\|_1 = \int_0^{+\infty}|y(t)|dt$. Define the operator $L : domL \subset X \to Y$ by $Lx = D_{0^+}^{q}x$, where

$$\begin{aligned} domL = \Big\{ & x \in X,\ D_{0^+}^{q-1}x \in AC[0,b] \text{ for all } b>0, \\ & I_{0^+}^{2-q}x(0) = 0,\ D_{0^+}^{q-1}x(\infty) = D_{0^+}^{q-1}x(0)\Big\} \subset X, \end{aligned}$$

then L maps $domL$ into Y. Let $N : X \to Y$ be the operator $Nx(t) = f(t,x(t))$, $t \in [0,+\infty)$. The problem (P) can be written as $Lx = Nx$. Define the linear projections P and Q as

$$Px(t) = \frac{D_{0^+}^{q-1}x(0)}{\Gamma(q)}t^{q-1}, \qquad Qy(t) = \frac{t^{q-1}e^{-t}}{\Gamma(q)}\int_0^\infty y(s)\,ds.$$

By computations, we obtain

$$\ker L = \left\{x \in domL : x(t) = at^{q-1},\ a \in \mathbb{R},\ t \in [0,\infty)\right\}.$$

and its image

$$ImL = \left\{y \in Y : \int_0^\infty y(s)\,ds = 0\right\}.$$

From here, we conclude that the *operator $L : domL \subset X \to Y$ is a Fredholm operator of index zero*. In addition, the generalized inverse K_P of L is given by

$$K_p y(t) = I_{0^+}^{q}y(t),\ \ y \in ImL.$$

Set $J : ImQ \to \ker L$ the linear isomorphism given by $J\left(ct^{q-1}e^{-t}\right) = ct^{q-1}$, for any $c \in \mathbb{R},\ t \geqslant 0$.

LEMMA 3. *The cone*

$$K = \left\{x \in X, x(t) \geqslant 0,\ D_{0^+}^{q-1}x(t) \geqslant 0,\ t \in [0,\infty)\right\}.$$

is normal in X.

Proof. Let $x,y \in K$, be such that $x \preceq y$. Since $y-x \in K$, then $e^{-t}y(t) \geqslant e^{-t}x(t)$ and $D_{0^+}^{q-1}y(t) \geqslant D_{0^+}^{q-1}x(t)$, $t \in [0,\infty)$. Hence

$$\sup_{t\geqslant 0} e^{-t}y(t) + \sup_{t\geqslant 0} D_{0^+}^{q-1}y(t) \geqslant \sup_{t\geqslant 0} e^{-t}x(t) + \sup_{t\geqslant 0} D_{0^+}^{q-1}x(t)$$

that is $\|x\| \leqslant N\|y\|$ where the normal constant of K is $N=1$. □

The operator $L+J^{-1}P : K\cap domL \to K_1 = \left(L+J^{-1}P\right)(K\cap domL)$ is a linear bijection with bounded inverse. From [6], K_1 is a cone in Y and we have the following Lemma:

LEMMA 4. [6] *The following two assertions are equivalent:*
i) $A = P + JQN + K_p(I-Q)N$ *maps* $K\cap domL$ *to* $K\cap domL$.
ii) $\left(N+J^{-1}P\right)$ *maps* $K\cap domL$ *into* $K_1 = \left(L+J^{-1}P\right)(K\cap domL)$.

Define the lower and upper solutions for problem (P) by

DEFINITION 5. [5] Let K be a normal cone in a Banach space X, $u_0 \preceq v_0$, and u_0, $v_0 \in K\cap dom(L)$ are said to be coupled of lower and upper solutions of the equation $Lu = Nu$ if $Lu_0 \leqslant Nu_0$ and $Lv_0 \geqslant Nv_0$.

Now we give the main result.

THEOREM 3. *Let* u_0, $v_0 \in K\cap dom(L)$ *be a coupled of lower and upper solutions of problem (P) such that* $u_0 \preceq v_0$. *Assume that the following conditions are satisfied:*

(H_1) *There exist nonnegative functions* α, $\beta \in L^1[0,\infty)$, *such that for all* $x \in \mathbb{R}$, $t \in [0,\infty)$, *we have*

$$f(t,x) \leqslant e^{-t}\alpha(t)|x| + \beta(t).$$

(H_2) *The function* f *is increasing according to the second variable, i.e., if* $x \leqslant y$ *then* $f(t,x) \leqslant f(t,y)$, *for* $x,y \in \mathbb{R}$, $t \in [0,\infty)$.

Then, the problem (P) has at least a maximal solution x^* *and a minimal solution* x_* *in* $[u_0,v_0]$. *Moreover* $x^* = \lim_{n\to\infty} v_n$, $x_* = \lim_{n\to\infty} u_n$, *where* $u_n = Au_{n-1}$, $v_n = Av_{n-1}$, $n = 1,2,3\ldots$, *with* $A = P + JQN + K_p(I-Q)N$, *and*

$$u_0 \preceq u_1 \preceq u_2 \preceq \ldots \preceq u_n \preceq \ldots \preceq v_n \preceq \ldots \preceq v_2 \preceq v_1 \preceq v_0.$$

REMARK 1. Since the coincidence equation $Lx = Nx$ is equivalent to $x = Ax$, where $A = P + JQN + K_p(I-Q)N = \left(L+J^{-1}P\right)^{-1}\left(N+J^{-1}P\right)$, then the maximal and minimal fixed points of A in $[u_0,v_0]$ are the maximal and a minimal solutions in $[u_0,v_0]$ of problem (P). Consequently to prove Theorem 3, it suffices to prove that all hypotheses of Theorem 1 are satisfied.

Proof of Theorem 3. The proof will be done in some steps.

Step 1. The operator A is completely continuous. Let $\Omega = B(0,r)$, then by condition (H_1), we can see that $N(\Omega)$ and $K_p(I-Q)N(\Omega)$ are bounded. So, $P+JQN$ is completely continuous. In view of the compactness criteria theorem, we need only to prove that $K_p(I-Q)N(\Omega)$ and $D_{0^+}^{q-1}(K_P(I-Q)N)(\Omega)$ are equicontinuous on every compact subinterval of $[0,\infty)$ and equiconvergent at infinity. In fact, for any $x\in\overline{\Omega}$, and any $t_1,t_2\in[0,T]$, $t_1<t_2$ with $T>0$, we have

$$\begin{aligned}
&\left|e^{-t_1}\left(K_P(I-Q)Nx\right)(t_1)-e^{-t_2}\left(K_P(I-Q)Nx\right)(t_2)\right|\\
&=\left|\int_{t_1}^{t_2}\left(e^{-s}K_P(I-Q)Nx(s)\right)'ds\right|\\
&\leqslant\left[\int_{t_1}^{t_2}\left|e^{-s}K_P(I-Q)Nx(s)\right|ds+\int_{t_1}^{t_2}\left|e^{-s}I_{0^+}^{q-1}(I-Q)Nx(s)\right|ds\right]\to 0,\ \text{ as } t_1\to t_2.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&\left|D_0^{q-1}\left(K_P(I-Q)Nx\right)(t_1)-D_0^{q-1}\left(K_P(I-Q)Nx\right)(t_2)\right|\\
&=\left|\int_0^{t_1}(I-Q)Nx(s)\,ds-\int_0^{t_2}(I-Q)Nx(s)\,ds\right|\\
&\leqslant\int_{t_1}^{t_2}|(I-Q)Nx(s)|\,ds\to 0,\quad \text{as } t_1\to t_2.
\end{aligned}$$

So $K_P(I-Q)N\left(\overline{\Omega}\right)$ and $D_0^{q-1}(K_P(I-Q)N)\left(\overline{\Omega}\right)$ are equicontinuous on every compact subinterval of $[0,\infty)$.

In addition, $K_P(I-Q)N\left(\overline{\Omega}\right)$ and $D_0^{q-1}(K_P(I-Q)N)\left(\overline{\Omega}\right)$ are equiconvergent at infinity. In fact, from condition (H_1), it yields

$$\|(I-Q)Nx\|_1<b,$$

where $b=2\left(r\|\alpha\|_1+\|\beta\|_1\right)$. Consequently

$$\begin{aligned}
\left|e^{-t}\left(K_p(I-Q)Nx\right)(t)\right|&\leqslant\frac{1}{\Gamma(q)}\int_0^t e^{-t}(t-s)^{q-1}|(I-Q)Nx(s)|\,ds\\
&\leqslant\frac{be^{-t}t^{q-1}}{\Gamma(q)}\to 0,\ \text{as } t\to\infty
\end{aligned}$$

and

$$\begin{aligned}
&\left|D_{0^+}^{q-1}\left(K_p(I-Q)Nx\right)(t)-\lim_{t\to\infty}D_{0^+}^{q-1}\left(K_p(I-Q)Nx\right)(t)\right|\\
&=\left|\int_0^t(I-Q)Nx(s)\,ds-\int_0^\infty(I-Q)Nx(s)\,ds\right|\\
&\leqslant\int_t^\infty|(I-Q)Nx(s)|\,ds\to 0,\ \text{as } t\to\infty,
\end{aligned}$$

thus $K_P(I-Q)N\left(\overline{\Omega}\right)$ and $D_{0^+}^{q-1}(K_P(I-Q)N)\left(\overline{\Omega}\right)$ are equiconvergent at infinity.

Step 2. A maps $K \cap domL$ to $K \cap domL$. In fact, let $x \in K \cap domL$, then

$$\begin{aligned} Ax(t) &= Px(t) + JQNx(t) + K_p(I-Q)Nx(t) = \frac{D_{0^+}^{q-1}x(0)}{\Gamma(q)}t^{q-1} \\ &+ \frac{t^{q-1}}{\Gamma(q)}\left(\int_0^\infty f(s,x(s))\,ds\right) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x(s))\,ds \\ &- \frac{1}{[\Gamma(q)]^2}\left(\int_0^t (t-s)^{q-1} s^{q-1} e^{-s} ds\right)\left(\int_0^\infty f(s,x(s))\,ds\right). \end{aligned}$$

Taking the following estimates into account

$$-\frac{1}{[\Gamma(q)]^2}\left(\int_0^t (t-s)^{q-1} s^{q-1} e^{-s} ds\right) \geqslant -\frac{t^{q-1}}{[\Gamma(q)]^2}\left(\int_0^\infty s^{q-1} e^{-s} ds\right) = -\frac{t^{q-1}}{\Gamma(q)}$$

we see that

$$Ax(t) \geqslant \frac{D_{0^+}^{q-1}x(0)}{\Gamma(q)}t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x(s))\,ds \geqslant 0.$$

On the other hand we have

$$\begin{aligned} D_0^{q-1}(Ax)(t) &= D_0^{q-1}x(0) + \int_0^\infty f(s,x(s))\,ds + \int_0^t f(s,x(s))\,ds \\ &\quad - \frac{1}{\Gamma(q)}\left(\int_0^t s^{q-1} e^{-s} ds\right)\left(\int_0^\infty f(s,x(s))\,ds\right) \\ &\geqslant D_0^{q-1}x(0) + \int_0^\infty f(s,x(s))\,ds + \int_0^t f(s,x(s))\,ds \\ &\quad - \frac{1}{\Gamma(q)}\left(\int_0^\infty s^{q-1} e^{-s} ds\right)\left(\int_0^\infty f(s,x(s))\,ds\right) \\ &= D_0^{q-1}x(0) + \int_0^t f(s,x(s))\,ds \geqslant 0. \end{aligned}$$

Step 3. A is an increasing operator and $u_0 \preceq Au_0$, $Av_0 \preceq v_0$. Indeed, since $Lu_0 \leqslant Nu_0$, then

$$\left(L + J^{-1}P\right) u_0 \leqslant \left(N + J^{-1}P\right) u_0,$$

that implies

$$u_0 \preceq \left(L + J^{-1}P\right)^{-1}\left(N + J^{-1}P\right) u_0$$

i.e. $u_0 \preceq Au_0$. Similarly, we prove $Av_0 \preceq v_0$.

Now, to prove that $A = \left(L + J^{-1}P\right)^{-1}\left(N + J^{-1}P\right)$ is increasing, it suffices to prove that both operators $\left(L + J^{-1}P\right)^{-1}$ and $N + J^{-1}P$ are increasing. Let $x, y \in [u_0, v_0] \subset$

$K \cap domL$ be such that $x \preceq y$. Using condition (H_2) and the fact that $y - x \in K$, we get

$$\begin{aligned}
&\left[\left(N+J^{-1}P\right)y\right](t)-\left[\left(N+J^{-1}P\right)x\right](t)\\
&=\left(f(t,y(t))+\frac{D_{0^+}^{q-1}y(0)}{\Gamma(q)}t^{q-1}e^{-t}\right)-\left(f(t,x(t))+\frac{D_{0^+}^{q-1}x(0)}{\Gamma(q)}t^{q-1}e^{-t}\right)\\
&=f(t,y(t))-f(t,x(t))+\left(D_{0^+}^{q-1}y(0)-D_{0^+}^{q-1}x(0)\right)\frac{t^{q-1}e^{-t}}{\Gamma(q)}\geqslant 0,
\end{aligned}$$

which implies $N + J^{-1}P$ is increasing. From step 2 and Lemma 4, we conclude that $\left(N+J^{-1}P\right)$ maps $K \cap domL$ into the cone K_1, thus the linear operator $\left(L+J^{-1}P\right)^{-1}$ maps the cone K_1 into the cone K. Hence $\left(L+J^{-1}P\right)^{-1}$ is increasing, this implies that $A = \left(L+J^{-1}P\right)^{-1}\left(N+J^{-1}P\right) : K \cap domL \to K$ is increasing.

From the previous steps, we conclude that the operator A is completely continuous, increasing and $u_0 \preceq Au_0$, $Av_0 \preceq v_0$. By Theorem 1, we deduce that A has a maximal fixed point x^* and a minimal fixed point x_* in $[u_0, v_0]$, such that $x^* = \lim_{n\to\infty} v_n$ and $x_* = \lim_{n\to\infty} u_n$, where the monotone sequences (u_n) and (v_n) are defined by $u_n = Au_{n-1}$, $v_n = Av_{n-1}$, $n = 1,2,3\ldots$, and

$$u_0 \preceq u_1 \preceq u_2 \preceq \ldots \preceq u_n \preceq \ldots \preceq v_n \preceq \ldots \preceq v_2 \preceq v_1 \preceq v_0.$$

The proof is complete. □

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