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INEQUALITIES OF THE HERMITE–HADAMARD TYPE FOR QUASI–CONVEX FUNCTIONS VIA THE (k,s)–RIEMANN–LIOUVILLE FRACTIONAL INTEGRALS

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Abstract. Recently, Hussain et al. in the paper [Some k-fractional associates of Hermite–Hadamard's inequality for quasi-convex functions and applications to special means, Fractional Differential Calculus, **7**(2) 2017, 301–309] established some new Hermite–Hadamard type inequalities for functions whose absolute values are quasi-convex via the k-Riemann–Liouville fractional integral operators. The purpose of this article is to extend and generalize the results, obtained in the aforementioned paper, via the (k,s)-fractional integrals.

1. Introduction

Let $I\subset\mathbb{R}$ be an interval. We say that a function $f:I\to\mathbb{R}$ is convex if for every $a,b\in I$ and $t\in[0,1]$, we have

$$f(ta+(1-t)b)\leqslant tf(a)+(1-t)f(b).$$

For this class of functions, we have the following relation:

$$f\Big(\frac{a+b}{2}\Big)\leqslant\frac{1}{b-a}\int_a^b f(x)\,dx\leqslant\frac{f(a)+f(b)}{2}.$$

The above inequality is known in the literature as the Hermite–Hadamard integral inequality for convex functions [4]. Since the advent of the above inequality, loads of work have been done around it – ranging from extensions, improvements to generalizations. For the purpose of this work, we will discuss this inequality for the class of quasi-convex functions. We start by presenting the definition in what follows.

DEFINITION 1. A function $f:I\to\mathbb{R}$ is quasi-convex if for every $a,b\in I$ and $t\in[0,1]$, we have

$$f(ta+(1-t)b)\leqslant\max\{f(a),f(b)\}.$$

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It is a general knowledge that every convex function is quasi-convex, but the converse is not necessarily true (see [6]). An analogue of the Hermite–Hadamard inequality for quasi-convex functions was given by Dragomir and Pearce [2, 3]. Specifically, they proved

THEOREM 1. *Let $f : I \to \mathbb{R}$ be a quasi-convex function and $f \in L_1[a,b]$. Then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \max\{f(a), f(b)\}.$$

It has now become a trending aspect of mathematical research to generalize classical known results (in particular, those partaining to inequalities) via fractional integral operators. Worthy of mention is the k-Riemann–Liouville fractional integral operator [9] which generalizes the known Riemann–Liouville fractional operator. For more on this subject, we invite the interested- reader to see the papers [7, 12, 13, 14] and the references given therein.

Recently, Hussain et al. [5] generalized Theorem 1 and other results in the literature by proving the following four theorems via the k-Riemann–Liouville fractional integral operators.

THEOREM 2. *Let $f : [a,b] \to \mathbb{R}$ be a positive function and $f \in L_1[a,b]$. If f is quasi-convex on $[a,b]$, then the subsequent inequality for the k-fractional integrals is valid*

$$\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\Big[{}_k\mathscr{I}^{\alpha}_{a^+}f(b) + {}_k\mathscr{I}^{\alpha}_{b^-}f(a)\Big] \leqslant \max\{f(a), f(b)\}. \tag{1}$$

THEOREM 3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b). If $|f'|$ is quasi-convex on $[a,b]$, $\alpha > 0$, the subsequent inequality for the k-fractional integral is valid*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\Big[{}_k\mathscr{I}^{\alpha}_{a^+}f(b) + {}_k\mathscr{I}^{\alpha}_{b^-}f(a)\Big]\right| \leqslant \frac{b-a}{(1+\frac{\alpha}{k})}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\max\{|f'(a)|, |f'(b)|\}. \tag{2}$$

THEOREM 4. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $f' \in L_1[a,b]$. If $|f'|^q$ is quasi-convex on $[a,b]$ and $q > 1$, then the subsequent inequality for the k-fractional integral is valid*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\Big[{}_k\mathscr{I}^{\alpha}_{a^+}f(b) + {}_k\mathscr{I}^{\alpha}_{b^-}f(a)\Big]\right| \leqslant \frac{b-a}{2\big(1+\frac{\alpha}{k}p\big)^{\frac{1}{p}}}\Big(\max\{|f'(a)|^q, |f'(b)|^q\}\Big)^{\frac{1}{q}}, \tag{3}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $\frac{\alpha}{k}\in[0,1]$.

THEOREM 5. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable function on* (a,b) *such that* $f'\in L_1[a,b]$. *If* $|f'|^q$ *is quasi-convex on* $[a,b]$ *and* $q>1$. *Then the subsequent inequality for the* k*-fractional integral is valid*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[{}_k\mathscr{J}^{\alpha}_{a^+}f(b)+{}_k\mathscr{J}^{\alpha}_{b^-}f(a)\right]\right| \leqslant \frac{b-a}{\left(1+\frac{\alpha}{k}\right)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left(\max\{|f'(a)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}}. \quad (4)$$

In 2016, Sarikaya et al. gave a generalization of the k-Riemann–Liouville fractional integral operators as follows:

DEFINITION 2. ([11]) The (k,s)-Riemann–Liouville fractional integral operators ${}^s_k\mathscr{J}^{\alpha}_{a^+}$ and ${}^s_k\mathscr{J}^{\alpha}_{b^-}$ of order $\alpha>0$ for a real valued continuous function $f(x)$ are defined as

$${}^s_k\mathscr{J}^{\alpha}_{a^+}f(x)=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x (x^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1}t^s f(t)\,dt \quad x>a \quad (5)$$

and

$${}^s_k\mathscr{J}^{\alpha}_{b^-}f(x)=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_x^b (t^{s+1}-x^{s+1})^{\frac{\alpha}{k}-1}t^s f(t)\,dt, \quad x<b, \quad (6)$$

where $k>0$, $s\in\mathbb{R}\setminus\{-1\}$, Γ_k is the k-gamma function given by the following integral

$$\Gamma_k(x):=\int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}\,dt \quad (Re(x)>0)$$

with the properties: $\Gamma_k(x+k)=x\Gamma_k(x)$ and $\Gamma_k(k)=1$.

Using the above defined operators, Agarwal et al. [1] established the following Hermite–Hadamard type result for convex functions.

THEOREM 6. *Let* $\alpha,k>0$, $s\in\mathbb{R}\setminus\{-1\}$. *If* f *is a convex function on* $[a,b]$, *then we have*

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}\left[{}^s_k\mathscr{J}^{\alpha}_{a^+}F(b)+{}^s_k\mathscr{J}^{\alpha}_{b^-}F(a)\right]\leqslant\frac{f(a)+f(b)}{2},$$

where the function F *is defined by* (7) *below.*

Inspired by the above works, it is our purpose in this present paper to obtain some generalizations of Theorems 2, 3, 4 and 5 via the (k,s)-Riemann–Liouville fractional integral operators ${}^s_k\mathscr{J}^{\alpha}_{a^+}$ and ${}^s_k\mathscr{J}^{\alpha}_{b^-}$. Our results reduce to those theorems for the case when $s=0$ (see Remarks 1–4). To the best of our knowledge, the results presented here are new for the case $s\neq 0$.

2. Main results

We start by making the following observations. Given f defined on I with $[a,b] \subset I^\circ$, we define the functions $F, \tilde{f} : [a,b] \to \mathbb{R}$ by

$$\tilde{f}(x) := f(a+b-x) \text{ and } F(x) := f(x) + \tilde{f}(x). \tag{7}$$

For the operators to be well defined, we shall assume throughout that $f \in L_\infty[a,b]$. Now, using the substitutions $u = \frac{t-a}{x-a}$ and $u = \frac{b-t}{b-x}$ in (5) and (6), respectively, one gets that

$$ {}^{s}_{k}\mathscr{J}^{\alpha}_{a^+} f(x) = (x-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_0^1 \frac{(ux+(1-u)a)^s f(ux+(1-u)a)}{\left[x^{s+1}-(ux+(1-u)a)^{s+1}\right]^{1-\frac{\alpha}{k}}}\, du \tag{8}$$

and

$$ {}^{s}_{k}\mathscr{J}^{\alpha}_{b^-} f(x) = (b-x)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_0^1 \frac{(ux+(1-u)b)^s f(ux+(1-u)b)}{\left[(ux+(1-u)b)^{s+1}-x^{s+1}\right]^{1-\frac{\alpha}{k}}}\, du. \tag{9}$$

Noting that $\tilde{f}((1-u)a+ub) = f(ua+(1-u)b)$, we also obtain

$$ {}^{s}_{k}\mathscr{J}^{\alpha}_{a^+} \tilde{f}(x) = (x-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_0^1 \frac{(ux+(1-u)a)^s f((1-u)x+ua)}{\left[x^{s+1}-(ux+(1-u)a)^{s+1}\right]^{1-\frac{\alpha}{k}}}\, du \tag{10}$$

and

$$ {}^{s}_{k}\mathscr{J}^{\alpha}_{b^-} \tilde{f}(x) = (b-x)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_0^1 \frac{(ux+(1-u)b)^s f((1-u)x+ub)}{\left[(ux+(1-u)b)^{s+1}-x^{s+1}\right]^{1-\frac{\alpha}{k}}}\, du. \tag{11}$$

By substituting $x = b$ and $x = a$ in (10) and (11) respectively, one obtains

$$ {}^{s}_{k}\mathscr{J}^{\alpha}_{a^+} \tilde{f}(b) = (b-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_0^1 \frac{(ub+(1-u)a)^s f((1-u)b+ua)}{\left[b^{s+1}-(ub+(1-u)a)^{s+1}\right]^{1-\frac{\alpha}{k}}}\, du, \tag{12}$$

and

$$ {}^{s}_{k}\mathscr{J}^{\alpha}_{b^-} \tilde{f}(a) = (b-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_0^1 \frac{(ua+(1-u)b)^s f((1-u)a+ub)}{\left[(ua+(1-u)b)^{s+1}-a^{s+1}\right]^{1-\frac{\alpha}{k}}}\, du. \tag{13}$$

Similar substitutions can be done in (8) and (9). We are now in position to state and prove our first result.

THEOREM 7. *Let $\alpha, k > 0$, $s \in \mathbb{R} \setminus \{-1\}$, $f : I \to \mathbb{R}$ be a positive function on $[a,b] \subset I^\circ$, and $f \in L_1[a,b]$ with $a < b$. If, in addition, f is quasi-convex on $[a,b]$, then we have the following (k,s)-fractional integral inequality:*

$$\frac{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}\left[{}^{s}_{k}\mathscr{J}^{\alpha}_{a^+} F(b) + {}^{s}_{k}\mathscr{J}^{\alpha}_{b^-} F(a)\right] \leqslant \max\{f(a), f(b)\}. \tag{14}$$

Proof. Using the fact that f is quasi-convex on $[a,b]$, we have that for $t \in [0,1]$,

$$f(ta+(1-t)b) \leqslant \max\{f(a),f(b)\} \tag{15}$$

and

$$f((1-t)a+tb) \leqslant \max\{f(a),f(b)\}. \tag{16}$$

Adding Inequalities (15) and (16), one gets

$$f(ta+(1-t)b)+f((1-t)a+tb) \leqslant 2\max\{f(a),f(b)\}. \tag{17}$$

Multiplying both sides of (17) by

$$(b-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\frac{(tb+(1-t)a)^s}{\left[b^{s+1}-(tb+(1-t)a)^{s+1}\right]^{1-\frac{\alpha}{k}}}$$

and integrating over $[0,1]$ with respect to t, we get

$$\begin{aligned}
&(b-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^1 \frac{(tb+(1-t)a)^s f((1-t)b+ta)}{\left[b^{s+1}-(tb+(1-t)a)^{s+1}\right]^{1-\frac{\alpha}{k}}}dt\\
&\quad+(b-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^1 \frac{(tb+(1-t)a)^s f(tb+(1-t)a)}{\left[b^{s+1}-(tb+(1-t)a)^{s+1}\right]^{1-\frac{\alpha}{k}}}dt\\
&\leqslant 2\max\{f(a),f(b)\}(b-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^1 \frac{(tb+(1-t)a)^s}{\left[b^{s+1}-(tb+(1-t)a)^{s+1}\right]^{1-\frac{\alpha}{k}}}dt.
\end{aligned} \tag{18}$$

Using (8) and (12), we get

$${}_k^s\mathscr{J}_{a^+}^{\alpha}\tilde{f}(b)+{}_k^s\mathscr{J}_{a^+}^{\alpha}f(b) \leqslant \frac{2(s+1)^{1-\frac{\alpha}{k}}(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}{(s+1)\alpha\Gamma_k(\alpha)}\max\{f(a),f(b)\}.$$

That is,

$${}_k^s\mathscr{J}_{a^+}^{\alpha}F(b) \leqslant \frac{2(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\max\{f(a),f(b)\}. \tag{19}$$

Similarly, multiplying again both sides of (17) by

$$(b-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\frac{(tb+(1-t)a)^s}{\left[(tb+(1-t)a)^{s+1}-a^{s+1}\right]^{1-\frac{\alpha}{k}}}$$

and integrating with respect to t over $[0,1]$ to arrive at

$${}_k^s\mathscr{J}_{b^-}^{\alpha}F(a) \leqslant \frac{2(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\max\{f(a),f(b)\}. \tag{20}$$

The desired inequality follows from adding (19) and (20). □

REMARK 1. Theorem 7 becomes Theorem 2 by choosing $s=0$.

For the rest of our results, we will need the following lemmas.

LEMMA 1. ([1]) *Let* $\alpha, k>0$ *and* $s\in\mathbb{R}\setminus\{-1\}$. *If* $f:I\to\mathbb{R}$ *is differentiable on* I° *with* $a,b\in I^\circ$ *such that* $f'\in L[a,b]$ *with* $a<b$, *then the following identity holds*

$$\frac{f(a)+f(b)}{2}-\frac{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}\left[{}_k^s\mathscr{J}_{a^+}^{\alpha}F(b)+{}_k^s\mathscr{J}_{b^-}^{\alpha}F(a)\right]$$
$$=\frac{b-a}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}\int_0^1\nabla_{\alpha,s}(t)f'(ta+(1-t)b)\,dt, \tag{21}$$

where $\nabla_{\alpha,s}:[0,1]\to\mathbb{R}$ *is defined by*

$$\nabla_{\alpha,s}(t):=\left[(ta+(1-t)b)^{s+1}-a^{s+1}\right]^{\frac{\alpha}{k}}-\left[(tb+(1-t)a)^{s+1}-a^{s+1}\right]^{\frac{\alpha}{k}}$$
$$+\left[b^{s+1}-(tb+(1-t)a)^{s+1}\right]^{\frac{\alpha}{k}}-\left[b^{s+1}-(ta+(1-t)b)^{s+1}\right]^{\frac{\alpha}{k}}.$$

LEMMA 2. *Under the conditions of Lemma 1, we have that*

$$\int_0^1|\nabla_{\alpha,s}(t)|\,dt=\frac{1}{b-a}\Big(\mathfrak{R}_1+\mathfrak{R}_2+\mathfrak{R}_3+\mathfrak{R}_4\Big),$$

where

$$\mathfrak{R}_1=\int_{\frac{a+b}{2}}^{b}\left(u^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}}du-\int_a^{\frac{a+b}{2}}\left(u^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}}du,$$

$$\mathfrak{R}_2=\int_{\frac{a+b}{2}}^{b}\left[b^{s+1}-(b+a-u)^{s+1}\right]^{\frac{\alpha}{k}}du-\int_a^{\frac{a+b}{2}}\left[b^{s+1}-(b+a-u)^{s+1}\right]^{\frac{\alpha}{k}}du,$$

$$\mathfrak{R}_3=\int_a^{\frac{a+b}{2}}\left(b^{s+1}-u^{s+1}\right)^{\frac{\alpha}{k}}du-\int_{\frac{a+b}{2}}^{b}\left(b^{s+1}-u^{s+1}\right)^{\frac{\alpha}{k}}du,$$

and

$$\mathfrak{R}_4=\int_a^{\frac{a+b}{2}}\left[(b+a-u)^{s+1}-a^{s+1}\right]^{\frac{\alpha}{k}}du-\int_{\frac{a+b}{2}}^{b}\left[(b+a-u)^{s+1}-a^{s+1}\right]^{\frac{\alpha}{k}}du.$$

Proof. The proof of this lemma is also given in [10]. For completeness, we outline the proof here. Using the substitution $u=ta+(1-t)b$, we get

$$\int_0^1|\nabla_{\alpha,s}(t)|\,dt=\frac{1}{b-a}\int_a^b|\wp(u)|\,du, \tag{22}$$

where

$$\wp(u)=\left(u^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}}-\left[(b+a-u)^{s+1}-a^{s+1}\right]^{\frac{\alpha}{k}}$$
$$+\left[b^{s+1}-(b+a-u)^{s+1}\right]^{\frac{\alpha}{k}}-\left(b^{s+1}-u^{s+1}\right)^{\frac{\alpha}{k}}.$$

The required result follows from (22) and by observing that the function $\wp$ is a non-decreasing function on $[a,b]$, $\wp(a) = -2(b^{s+1} - a^{s+1})^{\frac{\alpha}{k}} < 0$, $\wp\left(\frac{a+b}{2}\right) = 0$, and thus

$$\begin{cases} \wp(u) \leqslant 0 & \text{if} \quad a \leqslant u \leqslant \frac{a+b}{2}, \\ \wp(u) > 0 & \text{if} \quad \frac{a+b}{2} < u \leqslant b. \end{cases} \tag{23}$$

□

THEOREM 8. *Let $\alpha, k > 0$, $s \in \mathbb{R} \setminus \{-1\}$, $f : I \to \mathbb{R}$ be a differentiable function on I° and $a,b \in I^\circ$ with $a < b$. Suppose $f \in L_1[a,b]$ and $|f'|$ is quasi-convex on $[a,b]$, then we have the following (k,s)-fractional integral inequality:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{(s+1)^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}} \left[{}^s_k\mathscr{J}^{\alpha}_{a^+} F(b) + {}^s_k\mathscr{J}^{\alpha}_{b^-} F(a) \right] \right| \leqslant \frac{\mathfrak{R}_1 + \mathfrak{R}_2 + \mathfrak{R}_3 + \mathfrak{R}_4}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}} \max\{|f'(a)|, |f'(b)|\}. \tag{24}$$

Proof. The function $|f'|$ is quasi-convex implies that for $t \in [0,1]$, we have

$$|f'(ta + (1-t)b)| \leqslant \max\{|f'(a)|, |f'(b)|\}. \tag{25}$$

Using Lemmas 1 and 2, Inequality (25) and the properties of modulus, we obtain

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{(s+1)^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}} \left[{}^s_k\mathscr{J}^{\alpha}_{a^+} F(b) + {}^s_k\mathscr{J}^{\alpha}_{b^-} F(a) \right] \right| \\
&\leqslant \frac{b-a}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}} \int_0^1 |\nabla_{\alpha,s}(t)| |f'(ta+(1-t)b)| \, dt \\
&\leqslant \frac{b-a}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}} \int_0^1 |\nabla_{\alpha,s}(t)| \max\{|f'(a)|, |f'(b)|\} \, dt \\
&= \frac{b-a}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}} \max\{|f'(a)|, |f'(b)|\} \int_0^1 |\nabla_{\alpha,s}(t)| \, dt \\
&= \frac{b-a}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}} \max\{|f'(a)|, |f'(b)|\} \frac{1}{b-a} \left(\mathfrak{R}_1 + \mathfrak{R}_2 + \mathfrak{R}_3 + \mathfrak{R}_4 \right) \\
&= \frac{\mathfrak{R}_1 + \mathfrak{R}_2 + \mathfrak{R}_3 + \mathfrak{R}_4}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}} \max\{|f'(a)|, |f'(b)|\}.
\end{aligned}$$

The desired result follows. □

REMARK 2. If we set $s = 0$ in Theorem 8, then we recapture Theorem 3 by observing (for this case) that

$$\mathfrak{R}_1 = \mathfrak{R}_2 = \mathfrak{R}_3 = \mathfrak{R}_4 = \frac{1}{1+\frac{\alpha}{k}} \left[(b-a)^{\frac{\alpha}{k}+1} - 2\left(\frac{b-a}{2}\right)^{\frac{\alpha}{k}+1} \right].$$

THEOREM 9. *Let f be differentiable on I° with $a,b \in I^\circ$ such that $f' \in L_1[a,b]$. If $|f'|^q$ is quasi-convex on $[a,b]$ and $q>1$, then the following (k,s)-fractional integral inequality holds:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}} \left[{}^s_k\mathscr{J}^{\alpha}_{a^+}F(b) + {}^s_k\mathscr{J}^{\alpha}_{b^-}F(a) \right] \right|$$
$$\leqslant \frac{b-a}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}} \left(\int_0^1 |\nabla_{\alpha,s}(t)|^p\,dt \right)^{\frac{1}{p}} \left(\max\{|f'(a)|^q, |f'(b)|^q\} \right)^{\frac{1}{q}}, \tag{26}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. The function $|f'|^q$ is quasi-convex implies that for $t \in [0,1]$, we have

$$|f'(ta+(1-t)b)| \leqslant \max\{|f'(a)|^q, |f'(b)|^q\}. \tag{27}$$

Using Lemma 1, Inequality (27), the Hölder's inequality and the properties of modulus, we get

$$\left| \frac{f(a)+f(b)}{2} - \frac{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}} \left[{}^s_k\mathscr{J}^{\alpha}_{a^+}F(b) + {}^s_k\mathscr{J}^{\alpha}_{b^-}F(a) \right] \right|$$
$$\leqslant \frac{b-a}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}} \int_0^1 |\nabla_{\alpha,s}(t)||f'(ta+(1-t)b)|\,dt$$
$$\leqslant \frac{b-a}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}} \left(\int_0^1 |\nabla_{\alpha,s}(t)|^p\,dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(ta+(1-t)b)|^q\,dt \right)^{\frac{1}{q}}$$
$$\leqslant \frac{b-a}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}} \left(\int_0^1 |\nabla_{\alpha,s}(t)|^p\,dt \right)^{\frac{1}{p}} \left(\max\{|f'(a)|^q, |f'(b)|^q\} \right)^{\frac{1}{q}}.$$

Hence, the proof is complete. □

REMARK 3. By setting $s=0$ in Theorem 9, we recover Theorem 4. For this case

$$\nabla_{\alpha,0}(t) = 2(b-a)^{\frac{\alpha}{k}} \left[(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}} \right].$$

THEOREM 10. *Let f be differentiable on I° with $a,b \in I^\circ$ such that $f' \in L_1[a,b]$. If $|f'|^q$ is quasi-convex on $[a,b]$ and $q>1$, then the following (k,s)-fractional integral inequality holds:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}} \left[{}^s_k\mathscr{J}^{\alpha}_{a^+}F(b) + {}^s_k\mathscr{J}^{\alpha}_{b^-}F(a) \right] \right|$$
$$\leqslant \frac{\mathfrak{R}_1+\mathfrak{R}_2+\mathfrak{R}_3+\mathfrak{R}_4}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}} \left(\max\{|f'(a)|^q, |f'(b)|^q\} \right)^{\frac{1}{q}}. \tag{28}$$

Proof. Following a similar approach as in the proof of the above theorem, we have [by using Lemmas 1 and 2 combined with the power mean inequality plus Inequality (27)].

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}\left[{}^s_k\mathscr{J}^{\alpha}_{a^+}F(b)+{}^s_k\mathscr{J}^{\alpha}_{b^-}F(a)\right]\right|\\
&\leqslant \frac{b-a}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}\int_0^1|\nabla_{\alpha,s}(t)||f'(ta+(1-t)b)|\,dt\\
&\leqslant \frac{b-a}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}\left(\int_0^1|\nabla_{\alpha,s}(t)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|\nabla_{\alpha,s}(t)||f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\\
&\leqslant \frac{b-a}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}\left(\max\{|f'(a)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}}\int_0^1|\nabla_{\alpha,s}(t)|\,dt\\
&\leqslant \frac{\mathfrak{R}_1+\mathfrak{R}_2+\mathfrak{R}_3+\mathfrak{R}_4}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}\left(\max\{|f'(a)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}}.
\end{aligned}$$

That gives the intended inequality. □

REMARK 4. Theorem 10 boils down to Theorem 5 by setting $s=0$.

3. Conclusion

Four results of the Hermite–Hadamard type for quasi-convex functions have been established. For $s=0$, we get results obtained in [5]. More results can be derived by making appropriate choice of the parameters α, k and s. For example, if we choose $k=1$ and thereafter take limit $s\to-1^+$, then we obtain results involving the Hadamard fractional integral operator [8].

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