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ADDENDUM TO “SOME DISCRETE FRACTIONAL LYAPUNOV–TYPE INEQUALITIES” [FRACT. DIFFER. CALC. 5 (2015), NO. 1, 87–92]

RUI A. C. FERREIRA

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Abstract. We provide here some clarifications regarding the main results obtained in [R. A. C. Ferreira, Some Discrete Fractional Lyapunov-type inequalities, Fract. Differ. Calc. **5** (2015), no. 1, 87–92.].

1. The addendum

In 2015 the author presented a Lyapunov-type inequality, considering a boundary value problem depending on the discrete fractional Riemann–Liouville derivative. It reads as follows:

THEOREM 1. [1, Theorem 3.1] *If the following discrete fractional boundary value problem*

$$(\Delta^{\alpha}y)(t)+q(t+\alpha-1)y(t+\alpha-1)=0,\quad t\in[0,b+1]_{\mathbb{N}_0},\ 1<\alpha\leqslant 2 \tag{1}$$

$$y(\alpha-2)=0=y(\alpha+b+1), \tag{2}$$

has a nontrivial solution, then

$$\sum_{s=0}^{b+1}|q(s+\alpha-1)|>4\Gamma(\alpha)\frac{\Gamma(b+\alpha+2)\Gamma^2(\frac{b}{2}+2)}{(b+2\alpha)(b+2)\Gamma^2(\frac{b}{2}+\alpha)\Gamma(b+3)},\ \textit{if } b \textit{ is even}, \tag{3}$$

or

$$\sum_{s=0}^{b+1}|q(s+\alpha-1)|>\Gamma(\alpha)\frac{\Gamma(b+\alpha+2)\Gamma^2(\frac{b+3}{2})}{\Gamma^2(\frac{b+1}{2}+\alpha)\Gamma(b+3)},\ \textit{if } b \textit{ is odd}. \tag{4}$$

It has been noticed though that (3) and (4) might be equalities. In fact we may prove the following:

PROPOSITION 1. *Suppose that the BVP* (1)–(2) *has a nontrivial solution. Let* $M=\max_{(t,s)\in[\alpha-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}\times[0,b+1]_{\mathbb{N}_0}}G(t,s)$, *where* G *is the Green function for* (1)–(2) *(see the details in* [1]*). Then,*

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1. *If b is even, $q(s+\alpha-1)=0$ $\forall s\in[0,b+1]_{\mathbb{N}_0}$ except in only one of the two points $\{\frac{b}{2},\frac{b}{2}+1\}$ if and only if*
$$1=M\sum_{s=0}^{b+1}|q(s+\alpha-1)|;$$
2. *If b is odd, $q(s+\alpha-1)=0$ $\forall s\in[0,b+1]_{\mathbb{N}_0}\backslash\{\frac{b+1}{2}\}$ if and only if*
$$1=M\sum_{s=0}^{b+1}|q(s+\alpha-1)|.$$

REMARK 1. We note that the condition $1=M\sum_{s=0}^{b+1}|q(s+\alpha-1)|$ in Proposition 1 means that the conditions in (3) and (4) are actually equalities, respectively. That is due to the analysis done in [1, Section 3.1] where it is shown that

1. $M=G\left(\frac{b}{2}+\alpha-1,\frac{b}{2}\right)=\frac{1}{4}\frac{(b+2\alpha)(b+2)\Gamma^2(\frac{b}{2}+\alpha)\Gamma(b+3)}{\Gamma(\alpha)\Gamma(b+\alpha+2)\Gamma^2(\frac{b}{2}+2)}$, if b is even;
2. $M=G\left(\frac{b+1}{2}+\alpha-1,\frac{b+1}{2}\right)=\frac{1}{\Gamma(\alpha)}\frac{\Gamma^2(\frac{b+1}{2}+\alpha)\Gamma(b+3)}{\Gamma(b+\alpha+2)\Gamma^2(\frac{b+3}{2})}$, if b is odd.

Proof of Proposition 1. We will prove 1., being the proof of 2. analogous. We know from the previous Remark that $M=G\left(\frac{b}{2}+\alpha-1,\frac{b}{2}\right)$. Moreover, we also know from [1] that $M=G\left(\frac{b}{2}+\alpha,\frac{b}{2}+1\right)$ and there are no other points where G reaches its maximum.

Suppose firstly that $q(s+\alpha-1)=0$ $\forall s\in[0,b+1]_{\mathbb{N}_0}$ except in only one of the two points $\{\frac{b}{2},\frac{b}{2}+1\}$. We assume that $q(\frac{b}{2}+\alpha-1)\neq 0$ (the other case is analogous). Then (see the proof of [1, Theorem 3.1]),
$$y(t)=\sum_{s=0}^{b+1}G(t,s)q(s+\alpha-1)y(s+\alpha-1)=G\left(t,\frac{b}{2}\right)q\left(\frac{b}{2}+\alpha-1\right)y\left(\frac{b}{2}+\alpha-1\right),$$
for all t. Certainly $y\left(\frac{b}{2}+\alpha-1\right)\neq 0$ otherwise y would be the trivial solution. Hence,
$$1=G\left(\frac{b}{2}+\alpha-1,\frac{b}{2}\right)\left|q\left(\frac{b}{2}+\alpha-1\right)\right|=M\left|q\left(\frac{b}{2}+\alpha-1\right)\right|.$$

Suppose now that there exists a $s^\star$ different from $\frac{b}{2}$ and $\frac{b}{2}+1$ such that $q(s^\star+\alpha-1)\neq 0$. Then,
$$|y(t)|\leqslant M\sum_{s\in[0,b+1]_{\mathbb{N}_0}\backslash\{s^\star\}}|q(s+\alpha-1)|\|y\|+G(t,s^\star)|q(s^\star+\alpha-1)||y(s^\star+\alpha-1)|.$$
Since $0<G(t,s^\star)<M$ for all t different from $\alpha-2$ and $\alpha+b+1$ (this is just to assure that G is not zero) we finally conclude that
$$1<M\sum_{s=0}^{b+1}|q(s+\alpha-1)|.\quad\square$$

The correct version of the discrete fractional Lyapunov inequality is the following:

THEOREM 2. *If the following discrete fractional boundary value problem*

$$\begin{aligned}(\Delta^{\alpha} y)(t)+q(t+\alpha-1)y(t+\alpha-1)=0, \quad t\in[0,b+1]_{\mathbb{N}_0},\ 1<\alpha\leqslant 2\\ y(\alpha-2)=0=y(\alpha+b+1),\end{aligned}$$

has a nontrivial solution, then

$$\sum_{s=0}^{b+1}|q(s+\alpha-1)|\geqslant 4\Gamma(\alpha)\frac{\Gamma(b+\alpha+2)\Gamma^2(\frac{b}{2}+2)}{(b+2\alpha)(b+2)\Gamma^2(\frac{b}{2}+\alpha)\Gamma(b+3)},\ \textit{if } b \textit{ is even,}$$

or

$$\sum_{s=0}^{b+1}|q(s+\alpha-1)|\geqslant \Gamma(\alpha)\frac{\Gamma(b+\alpha+2)\Gamma^2(\frac{b+3}{2})}{\Gamma^2(\frac{b+1}{2}+\alpha)\Gamma(b+3)},\ \textit{if } b \textit{ is odd.}$$

Equality happens within the conditions provided in Proposition 1.

REMARK 2. Similar results were obtained for a right-focal fractional BVP in [1]. The same remarks made here for the conjugate BVP also applies to the right-focal BVP with the corresponding necessary changes.

REFERENCES

[1] R. A. C. FERREIRA, *Some Discrete Fractional Lyapunov-type inequalities*, Fract. Differ. Calc. **5** (2015), no. 1, 87–92.

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Rui A. C. Ferreira
Grupo Física-Matemática
Faculdade de Ciências, Universidade de Lisboa
Av. Prof. Gama Pinto, 2, 1649-003 Lisboa, Portugal
e-mail: raferreira@fc.ul.pt