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ON CONTROLLABILITY OF LINEAR AND NONLINEAR FRACTIONAL INTEGRODIFFERENTIAL SYSTEMS

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Abstract. In this article we investigate the controllability problem of linear and nonlinear fractional integrodifferential systems. We justify the controllability concepts on a fractional integrodifferential linear system, and use results, as well as Schauder's fixed point theorem, to obtain the controllability of the corresponding nonlinear system. Some applications are introduced to explain the theoretic parts.

1. Introduction

Controllability problem has attracted a lot of mathematicians and engineers attention since it plays a great role in control theory and engineering and has very important applications in these fields. Therefore, the contributions on exact and approximate controllability have been appeared in the recent years (see [6], [7] and the references therein).

However, by the recent developments on the theory of fractional differential equations, the controllability has new trends in studying the fractional control systems as more accurate models than the corresponding classical systems. These new models, motivate the researchers to investigate the controllability problems of such linear control systems (see [1], [3], [4], [9] and references therein).

On the other hand, these researches open the gate of nonlinear investigations of controllability problem for some fractional control systems (see [8], [11], [13], [14], [16]). The existence of solution for nonlinear fractional system is the main tool to solve the problem, hence, the researchers used fixed point theorems for solving controllability problems [10].

The kernel of classical controllability operator e^{At} is uniformly convergent and possess the semigroup property whereas in the correspond fractional system ($t^{\alpha-1} E_{\alpha,\alpha}(At^{\alpha})$, $0<\alpha<1$) is singular at $t=0$, and does not satisfy the semigroup property. This causes some difficulties to generalize the theory to all fractional systems, hence many constrains and restrictions must be imposed to guarantee the solvability of these problems. One of these problems is due to the term $t^{\alpha-1}$, therefore if one avoids it in the kernel of controllability operator, then the problem will be easier. For instance, the author in [16], used a fractional integrator $I_0^{1-\alpha}$ in the control and nonlinear terms

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of the given fractional system in order to avoid any such problems. Therefore, by using the analytic resolvent method and the continuity of a resolvent, the approximate controllability problem has been solved.

Motivated by the cited works, we investigate the controllability of the fractional system

$$\begin{cases} {}^{C}D_{0}^{\alpha}x(t) = Ax(t) + I_{0}^{1-\alpha}Bu(t) + f(t,x(t),u(t)), & t \in J = (0,T], \\ x(0) = x_{0}, \end{cases} \tag{1}$$

where $0 < \alpha < 1$, $x(t) \in \mathbb{R}^{n}$, and $u(t) \in \mathbb{R}^{m}$, for $t \in J$, are vector-valued functions, $A \in M(n,n)$ and $B \in M(n,m)$ are respectively $n \times n$ and $n \times m$ matrices, and $f\colon J \times \mathbb{R}^{n} \times \mathbb{R}^{m} \to \mathbb{R}^{n}$ is a given function. We prove that this system is controllable using Schauder's fixed point theorem.

2. Preliminaries

Some facts and preliminaries about fractional calculus are recalled in this section (for more details see [2], and [5]).

DEFINITION 1. The Riemann-Liouville (left-sided) fractional integral of a continuous real valued function f is defined by

$$I_{0}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-s)^{\alpha-1}f(s)\,ds, \quad t \in J, \quad n-1 < \alpha \leqslant n.$$

The inverse operator of the Riemann-Liouville integral of a function may be defined in the Caputo sense as in the next definition.

DEFINITION 2. The Caputo derivative of $f : J \to \mathbb{R}$ is defined as

$${}^{C}D_{0}^{\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}\frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds, & n-1 < \alpha < n, \\ \frac{d^{n}}{dt^{n}}f(t), & \alpha = n \in \mathbb{N}. \end{cases}$$

The two parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(k\alpha+\beta)}, \quad \alpha,\beta > 0, \quad z \in \mathbb{C}.$$

We notice that $E_{\alpha,1} = E_{\alpha}$ is the one parameter Mittag-Leffler function. If A is an $n \times n$-matrix, we infer to use in the sequel the following notation

$$E_{\alpha,\beta}(At^{\alpha}) = \sum_{k=0}^{\infty}\frac{A^{k}t^{k\alpha}}{\Gamma(k\alpha+\beta)}.$$

We introduce next some basic facts about Laplace transform which is an effective tool in control theory.

The Laplace transform of a real function f defined for all real numbers $t \geqslant 0$, is given by

$$\mathscr{L}(f)(\lambda) = \int_0^{\infty} e^{-\lambda t} f(t)\, dt, \ \ \lambda \in \mathbb{C}.$$

The Laplace transform of the fractional integral and Caputo derivative are given by

$$\begin{cases} \mathscr{L}\left\{{}^{C}D_{t_0}^{\alpha} f(t)\right\}(\lambda) = \lambda^{\alpha} F(\lambda) - \sum_{k=0}^{n-1} f^{(k)}(0)\, \lambda^{\alpha-1-k}, \\ \mathscr{L}\left\{I_0^{1-\alpha} f(t)\right\}(\lambda) = \lambda^{\alpha-1} F(\lambda). \end{cases} \tag{2}$$

Moreover, if $Re\lambda > \|A\|^{\frac{1}{\alpha}}$, the inverse Laplace transform of the resolvent operator in terms of Mittag-Leffler function is given by

$$\begin{cases} \mathscr{L}^{-1}\left\{\lambda^{\alpha-1}(\lambda^{\alpha} I - A)^{-1}\right\}(t) = E_{\alpha}(At^{\alpha}), \\ \mathscr{L}^{-1}\left\{(\lambda^{\alpha} I - A)^{-1}\right\}(t) = t^{\alpha-1} E_{\alpha,\alpha}(At^{\alpha}). \end{cases} \tag{3}$$

3. Controllability of linear systems

We establish in this section some relevant characteristics of the linear fractional control system

$$\begin{cases} {}^{C}D_0^{\alpha} x(t) = Ax(t) + I_0^{1-\alpha} Bu(t) + f(t), \ \ t \in J, \\ x(0) = x_0, \end{cases} \tag{4}$$

that may needed next in the sequel. Here $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $f \in \mathbb{R}^n$ are all continuous vector-valued functions (or in $L_1(0,T)$). Moreover, we assume that $A \in M(n,n)$, and $B \in M(n,m)$. The first result is to obtain the integral solution of the linear system (4) by using Laplace transformation technique. Hereafter, we assume that $Re\lambda > \|A\|^{\frac{1}{\alpha}}$.

LEMMA 1. *The solution of the differential equation(4) is given by*

$$\begin{aligned} x(t) &= E_{\alpha}(At^{\alpha}) x_0 + \int_0^t E_{\alpha}\left(A(t-s)^{\alpha}\right) Bu(s)\, ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(A(t-s)^{\alpha}\right) f(s) ds. \end{aligned} \tag{5}$$

Proof. Applying the Laplace transform to (5), we have

$$\mathscr{L}\left\{{}^{C}D_0^{\alpha} x(t)\right\} = \mathscr{L}\left\{Ax(t)\right\} + \mathscr{L}\left\{I_0^{1-\alpha} Bu(t)\right\} + \mathscr{L}\left\{f(t)\right\},$$

which implies, by (2), that

$$\lambda^{\alpha} X(\lambda) - \lambda^{\alpha-1} x(0) = AX(\lambda) + \lambda^{\alpha-1} BU(\lambda) + F(\lambda),$$

where $\mathscr{L}\{u(t)\} = U(\lambda)$, and $\mathscr{L}\{f(t)\} = F(\lambda)$. Therefore

$$X(\lambda) = \lambda^{\alpha-1}(\lambda^{\alpha}I - A)^{-1}x_0 + \lambda^{\alpha-1}(\lambda^{\alpha}I - A)^{-1}BU(\lambda) + (\lambda^{\alpha}I - A)^{-1}F(\lambda).$$

Applying the property (3), we have the result. □

The Mittag-Leffler matrix $E_\alpha \in M(n,n)$, hence we define the controllability operator $\mathscr{C}_t : \mathbb{R}^m \to \mathbb{R}^n$ by

$$\mathscr{C}_t u = \int_0^t E_\alpha((t-s)^\alpha A)Bu(s)ds. \tag{6}$$

Then, $\mathscr{C}_t$ is bounded for any $t \in J$, and $u \in \mathbb{R}^m$. The adjoint operator $\mathscr{C}_T^* : \mathbb{R}^n \to \mathbb{R}^m$ of $\mathscr{C}_T$ is given by

$$\mathscr{C}_T^* = B^* E_\alpha((T-\cdot)^\alpha A^*),$$

where B^*, A^* are the transposes of B, and A respectively. The controllability Gramnian $\mathscr{W} = \mathscr{C}_T \mathscr{C}_T^*$ is given by

$$\mathscr{W} = \int_0^T E_\alpha(A(T-t)^\alpha)BB^* E_\alpha(A^*(T-t)^\alpha)dt.$$

DEFINITION 3. The system (4) is said to be (complete or exact) controllable in J if given any state $y \in \mathbb{R}^n$, there exists a control function $u \in \mathbb{R}^m$ such that $x(T;u) = y$.

In accordance with equation (5), we have

$$\mathscr{C}_T u = x(T;u) - E_\alpha(AT^\alpha)x_0 - \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(A(T-s)^\alpha))f(s)ds.$$

The right term always transmits the state $x(T;u) \in \mathbb{R}^n$ to another state in $\mathbb{R}^n$ by subtracting the constant vector function

$$E_\alpha(AT^\alpha)x_0 + \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(A(T-s)^\alpha))f(s)ds.$$

Hence, the problem is devoting on the surjectivity of the operator $\mathscr{C}_T$, i.e. $Im\mathscr{C}_T = \mathbb{R}^n$. Therefore, many equivalence statements can be established for the controllability criteria

In fact, the complete controllability is the strongest definition of controllability that may not be satisfied in many applications, hence, some other weaker definition of controllability is required. For instance, the closure of the image of $\mathscr{C}_T$ is the whole state space $\mathbb{R}^n$, symbolically, $\overline{Im\mathscr{C}_T} = \mathbb{R}^n$. This is called the approximate controllability of the system (for more details see [6]). The following result is a consequence of this definition.

If every non-zero state $x_0 \in \mathbb{R}^n$ can be steered to the null state $0 \in \mathbb{R}^n$ by a steering control then the system is said to be null controllable. The next result will be established by posing the existence of Mittag-Leffler inverse function E_α^{-1} [12]. One can assume that zero is not an eigenvalue of the matrix $E_\alpha(AT^\alpha)$.

THEOREM 1. *If E_α^{-1} exists then, the linear system (4) is completely controllable if and only if it is null controllable.*

Proof. It is obvious that complete controllability implies null controllability. We now show that null controllability implies complete controllability. Suppose that the system is null-controllable and let $x_1 = x_0 - E_\alpha^{-1}(AT^\alpha)y$. Thus there exists a control u such that

$$\begin{aligned}0 &= E_\alpha(AT^\alpha)x_1 + \int_0^T E_\alpha\left(A(T-s)^\alpha\right)Bu(s)\,ds \\ &\quad + \int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(A(T-s)^\alpha))f(s)ds \\ &= E_\alpha(AT^\alpha)\left(x_0 - E_\alpha^{-1}(AT^\alpha)y\right) + \int_0^T E_\alpha\left(A(T-s)^\alpha\right)Bu(s)\,ds \\ &\quad + \int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(A(T-s)^\alpha))f(s)ds.\end{aligned}$$

Then

$$\begin{aligned}y &= E_\alpha(AT^\alpha)x_0 + \int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(T-s)^\alpha\right)Bu(s)\,ds \\ &\quad + \int_0^T E_\alpha(A(T-s)^\alpha))f(s)ds \\ &= x(T).\end{aligned}$$

Hence, the completely controllability of the system (4) is satisfied. This finishes the proof. □

PROPOSITION 1. *Assume that* $\theta > 0$, *then*

$$\langle \mathscr{W}z, z\rangle = \|\mathscr{C}_T^* z\|^2 = \int_0^T \left\|B^* E_\alpha(A^*(T-t)^\alpha)z\right\|^2 dt \geqslant \theta\,\|z\|^2 > 0,\;\; z \neq 0, \qquad (7)$$

if and only if all eigenvalues of $\mathscr{W}$ *are positive.*

Proof. The condition (7) implies that $\mathscr{W}$ is coercive, that leads to the positivity of $\mathscr{W}$. Again, (7) and Cauchy-Schwartz inequality imply that

$$\|\mathscr{W}z\|\,\|z\| \geqslant \langle \mathscr{W}z, z\rangle \geqslant \theta\,\|z\|^2,$$

hence $\|\mathscr{W}z\| \geqslant \theta\,\|z\|$, that is $\ker\mathscr{W} = \{0\}$ is injective and then it is nonsingular linear operator on $\mathbb{R}^n$. Let $\mathscr{W}^{-1}$ be an inverse of the operator $\mathscr{W}$, then it is obvious that $\mathscr{W}^{-1}$ is bounded on its image subspace. Then $Im\mathscr{W}$ is equivalent to domain of $\mathscr{W}^{-1} = \mathbb{R}^n$ which implies that $Im\mathscr{C}_T \supseteq Im\mathscr{W} = \mathbb{R}^n$. Using (7) again, we deduce the same properties for the operator $\mathscr{C}_T^*$. If the fractional system (4) is controllable on J, and $\mathscr{W}$ is not positive definite. There exists a nonzero vector function $z \in \mathbb{R}^n$ such that $z^* \mathscr{W} z = 0$ and satisfies

$$\int_0^T z^* E_\alpha(A(T-t)^\alpha) BB^* E_\alpha(A^*(T-t)^\alpha) z dt = 0.$$

We deduce that

$$z^* E_\alpha(A(T-s)^\alpha) B = 0.$$

The controllability of the system (4) implies that there exists a nonzero control u such that

$$0 = z + \int_0^T E_\alpha\left(A(T-s)^\alpha\right) Bu(s)\, ds,$$

which implies that

$$0 = z^* z + \int_0^T z^* E_\alpha\left(A(T-s)^\alpha\right) Bu(s)\, ds.$$

Then $z^* z = 0$ that implies $z = 0$. This contradicts the assumption $z \neq 0$. Thus $\mathscr{W}$ is positive definite. The positive definiteness of $\mathscr{W}$ implies that all eigenvalues of $\mathscr{W}$ are positive which exclude the zero eigenvalue. Hence, (7) is valid for any $\theta > 0$. □

It may be happened that many controls steers the system from initial state to final state at time T, but one of them is more efficient than others. The optimal control u that has minimum energy functional $\|u\|^2 = \int_0^T \|u(s)\|^2 ds$ is one of the most popular approach.

LEMMA 2. *Let $\mathscr{W}$ be nonsingular, then the control $\widetilde{u} \in \mathbb{R}^m$ defined by*

$$\widetilde{u}(t) = B^* E_\alpha\left(A^*(T-t)^\alpha\right) \times \mathscr{W}^{-1}\left(y - E_\alpha(AT^\alpha)x_0 - \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(A(T-s)^\alpha)) f(s) ds\right), \quad (8)$$

for $t \in J$, is optimal and steers the system (4) from initial state x_0 to final state y at time T.

Proof. Let $u \in \mathbb{R}^m$ be any control function that steers the system from initial state x_0 to final state y at time T. Then

$$\|u\|^2 = \|\widetilde{u}\|^2 + \|u - \widetilde{u}\|^2 + 2Re\langle \widetilde{u}, u - \widetilde{u}\rangle.$$

In virtue of (8), we have

$$\begin{aligned}
&\langle \widetilde{u}, u - \widetilde{u}\rangle \\
&= \int_0^T \langle \widetilde{u}(t), u(t) - \widetilde{u}(t)\rangle\, dt \\
&= \Big\langle \mathscr{W}^{-1}\left(y - E_\alpha(AT^\alpha)x_0 - \int_0^T (T-t)^{\alpha-1} E_{\alpha,\alpha}(A(T-t)^\alpha)) f(t) dt\right), \\
&\qquad \int_0^T E_\alpha\left(A(T-t)^\alpha\right) B[u(t) - \widetilde{u}(t)]\, dt \Big\rangle
\end{aligned}$$

$$= \left\langle \mathscr{W}^{-1} \left(y - E_\alpha (AT^\alpha) x_0 - \int_0^T (T-t)^{\alpha-1} E_{\alpha,\alpha}(A(T-t)^\alpha)) f(t) dt \right), \mathscr{C}_T [u(t) - \widetilde{u}(t)] \right\rangle.$$

Since both of u, and $\widetilde{u}$ are steering the state x_0 to y at time T, then $\mathscr{C}_T [u(t) - \widetilde{u}(t)] = 0$. Hence

$$\|u\|^2 - \|\widetilde{u}\|^2 = \|u - \widetilde{u}\|^2 \geqslant 0,$$

which shows that the norm of $\widetilde{u}$ is less than or equal the norm of any other control u. It remains to show that $\widetilde{u}$ steers the state x_0 to y at time T. For this, we have

$$\begin{aligned} x(T) &= E_\alpha (AT^\alpha) x_0 + \int_0^T E_\alpha \left(A(T-t)^\alpha \right) BB^* E_\alpha \left(A^*(T-t)^\alpha \right) \\ &\quad \times \mathscr{W}^{-1} \left(y - E_\alpha (AT^\alpha) x_0 - \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(A(T-s)^\alpha)) f(s) ds \right) dt \\ &\quad + \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(A(T-s)^\alpha)) f(s) ds \\ &= E_\alpha (AT^\alpha) x_0 + \mathscr{W} \mathscr{W}^{-1} \\ &\quad \times \left(y - E_\alpha (AT^\alpha) x_0 - \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(A(T-s)^\alpha)) f(s) ds \right) \\ &\quad + \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(A(T-s)^\alpha)) f(s) ds \\ &= y. \end{aligned}$$

This finishes the proof. □

The above arguments can be used to prove many tools for obtaining the complete and approximate controllability of the system (4), we mention only the following basic result.

THEOREM 2. *The system (4) is controllable if any only if $\mathscr{W}$ (or $\mathscr{C}_T^*$) satisfies any one of the following:*

1. *$\mathscr{W}$ is coercive;*
2. *$\mathscr{W}$ is positive definite;*
3. *$\mathscr{W}$ is nonsingular;*
4. *$Im\mathscr{W} = \mathbb{R}^n$, and $\ker \mathscr{W} = \{0\}$.*

The rank condition is an effective tool to determine whether the system is controllable. The rank condition is given by

$$\operatorname{rank} \left[B | AB | \ldots | A^{n-1} B \right] = n, \tag{9}$$

where $A^n = 0$.

THEOREM 3. *The linear system (4) is controllable if and only if the rank condition (9) is hold.*

Proof. The solution of the system (4) satisfies the equation

$$\mathscr{C}_T u = x(T;u) - E_\alpha(AT^\alpha)x_0 - \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(A(T-s)^\alpha))f(s) \in \mathbb{R}^n, \quad (10)$$

at time T, where

$$\mathscr{C}_T u = \int_0^T E_\alpha\left(A(T-s)^\alpha\right)Bu(s)\,ds.$$

If $A^n = 0$, then the Mittag-Leffler function is reduced as

$$E_\alpha\left(A(T-s)^\alpha\right) = \sum_{k=0}^{n-1}\frac{(T-s)^{\alpha k}A^k}{\Gamma(k\alpha+1)}.$$

Therefore

$$\begin{aligned}\mathscr{C}_T u &= \int_0^T \sum_{k-0}^{n-1}\frac{(T-s)^{\alpha k}}{\Gamma(k\alpha+1)}A^k Bu(s)\,ds\\ &= \sum_{k=0}^{n-1}A^k B\left(\frac{1}{\Gamma(k\alpha+1)}\int_0^T (T-s)^{\alpha k}u(s)\,ds\right)\\ &= \sum_{k=0}^{n-1}A^k B v_k^\alpha.\end{aligned}$$

where $v_k^\alpha = I_0^{k\alpha+1}u(T) \in \mathbb{R}^m$. Symbolically, we have

$$\mathscr{C}_T u = \left[B\,AB\,\ldots A^{n-1}B\right]\begin{bmatrix} v_0^\alpha \\ v_1^\alpha \\ \vdots \\ v_{n-1}^\alpha \end{bmatrix}.$$

Therefore, equation (10) has a unique solution v_k^α if and only if the condition (9) is hold that implies $\det\left[B\,AB\,\ldots A^{n-1}B\right] \neq 0$. Hence, it would always be possible to find at least one function $u \in \mathbb{R}^m$ to ensure the existence of the vectors v_k^α. □

4. Controllability of nonlinear systems

Consider the nonlinear fractional control system (1). Using Lemma 1, we can write the integral solution of (1) as

$$\begin{aligned}x(t) = E_\alpha(At^\alpha)x_0 &+ \int_0^t E_\alpha\left(A(t-s)^\alpha\right)Bu(s)\,ds\\ &+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha))f(s,x(s),u(s))ds. \quad (11)\end{aligned}$$

The analogous optimal control for non linear system of the corresponding control (8) is given by

$$u(t) = B^* E_\alpha\left(A^*(T-t)^\alpha\right)\mathscr{W}^{-1} \times\left[y - E_\alpha(AT^\alpha)x_0 - \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}\left(A(T-s)^\alpha\right) f(s,x(s),u(s))\,ds\right], \tag{12}$$

provided that the linear system is controllable. Let $\mathbf{Q}$ be the Banach space of all continuous $\mathbb{R}^n \times \mathbb{R}^m$-valued functions defined on the interval J equipped with be norm $\|(x,u)\| = \sup\{\|x\|, \|u\|\}$, where $\|x\| = \sup\{\|x(t)\|, t \in J\}$ and $\|u\| = \sup\{\|u(t)\|, t \in J\}$, that is, $\mathbf{Q} = C(J,\mathbb{R}^n) \times C(J,\mathbb{R}^m)$ where $C(J,\mathbb{R}^n)$ is the Banach space of continuous $\mathbb{R}^n$-valued function defined on the interval J with the supremum norm. The norm of a vector function $x(t) = [x_1(t) \;\; x_2(t) \cdots x_n(t)]^* \in \mathbb{R}^n$ is defined as $\|x(t)\| = \sum_{i=1}^{n} |x_i(t)|$, $t \in J$. For our convenience, let us introduce the following notations:

$a_1 = \sup_{\mathrm{t \in J}} \left\|E_\alpha(A(T-t)^\alpha\right\|$,
$a_2 = \sup_{\mathrm{t \in J}} \|E_\alpha(At^\alpha)x_0\|$,
$a_3 = \sup_{\mathrm{t \in J}} \left\|E_{\alpha,\alpha}(A(T-t)^\alpha\right\|$,
$\gamma_i = \frac{a_1 a_3 T^\alpha}{\alpha} \|\mu_i\| \|B^*\| \left\|\mathscr{W}^{-1}\right\|$, where $\mu_i : J \to \mathbb{R}$ is L^1-function for $i = 1,2,\ldots,n$.
$b_i = 2a_3 \frac{T^\alpha}{\alpha} \|\mu_i\|$,
$d_1 = a_1 \|B^*\| \left\|\mathscr{W}^{-1}\right\| \{\|y\| + a_2\}$,
$d_2 = 2a_2$,
$c_i = \max\{\gamma_i, b_i\}$, and
$d = \max\{d_1, d_2\}$.

THEOREM 4. *Let $f : J \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be given by*

$$f(t,x,u) = [\mu_1(t)\varphi_1(x,u) \;\; \mu_2(t)\varphi_2(x,u) \cdots \mu_n(t)\varphi_n(x,u)]^*$$

where $\varphi_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is measurable functions for $i = 1,2,\ldots,n$. Suppose that the linear system (4) is controllable. Then the nonlinear system (1) is controllable on J if $2a_1 T \|B\| \leqslant 1$, and

$$\lim_{r\to\infty}\left(r - \sum_{i=1}^{n} c_i \sup\{|\varphi_i(x,u)| : \|(x,u)\| \leqslant r\}\right) = +\infty. \tag{13}$$

Proof. Define the operator $\rho : \mathbf{Q} \to \mathbf{Q}$ by

$$\rho(z,v) = (x,u),$$

where x and u are given by (11) and (12) respectively. By Lebesgue dominated convergence theorem, it is obvious that ρ is continuous on $\mathbf{Q}$. Let

$$\omega_i(r) = \sup\{|\varphi_i(x,u)|; \|(x,u)\| = r\},\; i = 1,2,\ldots,n,$$

then by using the given condition (13), for any constant $d>0$ there exists $r_0>0$ such that $r_0-\sum_{i=1}^{n}c_i\omega_i(r_0)\geqslant d$ which implies that $\sum_{i=1}^{n}c_i\omega_i(r_0)+d\leqslant r_0$. Let $\|z\|\leqslant r_0$ and $\|v\|\leqslant r_0$, then by (11) and (12), we have

$$\begin{aligned}
&\|u(t)\|\\
&\leqslant \|B^*\|\left\|E_\alpha\left(A^*(T-t)^\alpha\right)\right\|\left\|\mathscr{W}^{-1}\right\|\\
&\quad\times\left(\|y\|+\|E_\alpha(AT^\alpha)x_0\|+\int_0^T(T-t)^{\alpha-1}\left\|E_{\alpha,\alpha}\left(A(T-s)^\alpha\right)\right\|\|f(s,z(s),v(s))\|\,ds\right)\\
&\leqslant a_1\|B^*\|\left\|\mathscr{W}^{-1}\right\|\{\|y\|+a_2\}+\frac{a_1a_3T^\alpha}{\alpha}\|B^*\|\left\|\mathscr{W}^{-1}\right\|\sum_{i=1}^{n}\|\mu_i\|\,|\varphi_i(z,v)|\\
&\leqslant d_1+\frac{a_1a_3T^\alpha}{\alpha}\|B^*\|\left\|\mathscr{W}^{-1}\right\|\sum_{i=1}^{n}\|\mu_i\|\,\omega_i(r_0)\\
&= d_1+\sum_{i=1}^{n}\gamma_i\omega_i(r_0)\\
&\leqslant d+\sum_{i=1}^{n}c_i\omega_i(r_0).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|x(t)\|&\leqslant \|E_\alpha(At^\alpha)x_0\|+\int_0^t\left\|E_\alpha\left(A(t-s)^\alpha\right)\right\|\|B\|\|v(s)\|\,ds\\
&\quad+\int_0^t(t-s)^{\alpha-1}\left\|E_{\alpha,\alpha}(A(t-s)^\alpha)\right\|\|f(s,z(s),v(s))\|\,ds\\
&\leqslant a_2+a_1\|B\|r_0\int_0^t ds+a_3\sum_{i=1}^{n}\|\mu_i\|\,\omega_i(r_0)\int_0^t(t-s)^{\alpha-1}ds\\
&\leqslant \frac{1}{2}\left(d_2+\sum_{i=1}^{n}c_i\omega_i(r_0)\right)+Ta_1\|B\|r_0\\
&\leqslant \frac{1}{2}\left(d_2+\sum_{i=1}^{n}c_i\omega_i(r_0)\right)+\frac{1}{2}r_0.
\end{aligned}$$

Therefore $\|u\|\leqslant r_0$, and $\|x\|\leqslant r_0$. Thus we have proved that, if

$$\mathbf{Q}(r_0)=\{(z,v)\in\mathbf{Q}:\ \|z\|\leqslant r_0 \text{ and } \|v\|\leqslant r_0\},$$

then ρ maps $\mathbf{Q}(r_0)$ into itself and clearly $\mathbf{Q}(r_0)$ is closed and bounded.

Next, we prove $\rho(\mathbf{Q}(r))$ is equicontinuous for all $r>0$. Let $t_1,t_2\in J$ with $t_1<t_2$ and for all $(x,u)\in\mathbf{Q}(r)$, we have

$$\begin{aligned}
&\|u(t_1)-u(t_2)\|\\
&=\left\|\left(B^*E_\alpha\left(A^*(T-t_1)^\alpha\right)-B^*E_\alpha\left(A^*(T-t_2)^\alpha\right)\right)\right.\\
&\quad\left.\times\mathscr{W}^{-1}\left[y-E_\alpha(AT^\alpha)x_0-\int_0^T E_\alpha\left(A(T-s)^\alpha\right)f(s,z(s),v(s))\,ds\right]\right\|
\end{aligned}$$

$$\leqslant \|B^*\| \sum_{k=0}^{\infty} \frac{\|A^*\|^k \left|(T-t_1)^{k\alpha}-(T-t_2)^{k\alpha}\right|}{\Gamma(k\alpha+1)} \left\|\mathscr{W}^{-1}\right\|$$
$$\times\left(\|y\|+\|E_\alpha(AT^\alpha)x_0\|+\int_0^T (T-s)^{\alpha-1}\left\|E_{\alpha,\alpha}\left(A(T-s)^\alpha\right)\right\|\|f(s,z(s),v(s))\|\,ds\right)$$
$$\leqslant \sum_{k=0}^{\infty} \frac{\|A^*\|^k \left|(T-t_1)^{k\alpha}-(T-t_2)^{k\alpha}\right|}{\Gamma(k\alpha+1)}\left(\|y\|+a_2+a_3T^\alpha\sum_{i=1}^{n}c_i\omega_i(r)\right)\|B^*\|\left\|\mathscr{W}^{-1}\right\|.$$

On the other hand, we have

$$\begin{aligned}
&\|x(t_1)-x(t_2)\| \\
&= \Bigg\|E_\alpha(At_1^\alpha)x_0+\int_0^{t_1}E_\alpha\left(A(t_1-s)^\alpha\right)Bv(s)\,ds \\
&\quad+\int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(A(t_1-s)^\alpha)f(s,z(s),v(s))ds-E_\alpha(At_2^\alpha)x_0 \\
&\quad-\int_0^{t_2}E_\alpha\left(A(t_2-s)^\alpha\right)Bv(s)\,ds-\int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(A(t_2-s)^\alpha)f(s,z(s),v(s))ds\Bigg\| \\
&\leqslant \|x_0\|\sum_{k=0}^{\infty}\frac{\|A\|^k\left|(T-t_1)^{k\alpha}-(T-t_2)^{k\alpha}\right|}{\Gamma(k\alpha+1)} \\
&\quad+r\|B\|\sum_{k=0}^{\infty}\frac{\|A\|^k}{\Gamma(k\alpha+1)}\left|\int_0^{t_1}\left((t_1-s)^{k\alpha}-(t_2-s)^{k\alpha}\right)ds\right| \\
&\quad+\sum_{k=0}^{\infty}\sum_{i=1}^{n}\frac{c_i\omega_i(r)\|A\|^k}{\Gamma(k\alpha+\alpha)}\left|\int_0^{t_1}\left((t_1-s)^{k\alpha+\alpha-1}-(t_2-s)^{k\alpha+\alpha-1}\right)ds\right| \\
&\quad+r\|B\|\sum_{k=0}^{\infty}\frac{\|A\|^k}{\Gamma(k\alpha+1)}\left|\int_{t_1}^{t_2}(t_2-s)^{k\alpha}ds\right| \\
&\quad+\sum_{k=0}^{\infty}\sum_{i=1}^{n}\frac{c_i\omega_i(r)\|A\|^k}{\Gamma(k\alpha+\alpha)}\left|\int_{t_1}^{t_2}(t_2-s)^{k\alpha+\alpha-1}ds\right| \\
&\leqslant \|x_0\|\sum_{k=0}^{\infty}\frac{\|A\|^k\left|(T-t_1)^{k\alpha}-(T-t_2)^{k\alpha}\right|}{\Gamma(k\alpha+1)} \\
&\quad+r\|B\|\sum_{k=0}^{\infty}\frac{\|A\|^k}{\Gamma(k\alpha+2)}\left(\left|t_1^{k\alpha+1}-t_2^{k\alpha+1}\right|+(t_2-t_1)^{k\alpha+1}\right) \\
&\quad+\sum_{k=0}^{\infty}\sum_{i=1}^{n}\frac{c_i\omega_i(r)\|A\|^k}{\Gamma(k\alpha+\alpha+1)}\left(\left|t_1^{k\alpha+\alpha}-t_2^{k\alpha+\alpha}\right|+(t_2-t_1)^{k\alpha+\alpha}\right) \\
&\quad+r\|B\|\sum_{k=0}^{\infty}\frac{\|A\|^k}{\Gamma(k\alpha+2)}(t_2-t_1)^{k\alpha+1}+\sum_{k=0}^{\infty}\sum_{i=1}^{n}\frac{c_i\omega_i(r)\|A\|^k}{\Gamma(k\alpha+\alpha+1)}(t_2-t_1)^{k\alpha+\alpha}.
\end{aligned}$$

Thus the right-hand side of the above two inequalities is independent of $(x,u)\in\mathbf{Q}(r)$ and tends to zero as $|t_1-t_2|\to 0$, hence $\rho(\mathbf{Q}(r))$ is equicontinuous for all finite $r>0$

so ρ is compact by Arzela-Ascoli Theorem and by regularity assumption on f the operator ρ is completely continuous. Now the Schauder fixed point Theorem guarantees that ρ has a fixed point $(z,v) \in \mathbf{Q}(r_0)$ such that $\rho(z,v) = (z,v)$. Thus, indeed the solution of the system (1) is $x(t)$ that given by (11) and as the proof of Lemma 2, it is easy to verify that $x(T) = y$, hence the system (1) is controllable on J. This finishes the proof. □

5. Applications

We introduce in this section a couple of examples to illustrate the applicability of the obtained results.

EXAMPLE 1. Consider the following nonlinear fractional dynamical system represented by the scalar fractional differential equation:

$$\begin{cases} {}^C D_0^{\frac{1}{2}} x(t) = x(t) + I_0^{1-\alpha} u(t) + tu(t)\sin x(t), \ \ t \in [0,1], \\ x(0) = 0, \end{cases} \tag{14}$$

where $A = B = 1, \alpha = \frac{1}{2}$, and, $f(t,x(t),u(t)) = tu(t)\sin x(t)$. The Mittag-Leffler function is given by

$$E_{\frac{1}{2}}((1-s)^{1/2}) = \sum_{k=0}^{\infty} \frac{(1-s)^{k/2}}{\Gamma\left(\frac{k}{2}+1\right)} = \frac{2 - erfc\sqrt{1-s}}{e^{s-1}}.$$

By numerical calculations, one can see that the controllability Gramnian is approximated as

$$\mathscr{W} = \int_0^1 \left(\frac{2 - erfc\sqrt{1-s}}{e^{s-1}} \right)^2 ds \approx 9.4774.$$

Therefore, the linear system of (14) is controllable using the control function

$$u(t) = \frac{2 - erfc\sqrt{1-t}}{9.4774e^{t-1}} \times \left[x(1) - \int_0^1 (1-s)^{\alpha-1} \left(\frac{1}{\sqrt{\pi}} + \frac{2\sqrt{1-s}}{e^{s-1}} - \frac{2\sqrt{1-s}erfc\sqrt{1-s}}{e^{s-1}} \right) su(s)\sin x(s)\, ds \right].$$

Another numerical calculations lead to

$$\lim_{r\to\infty} (r - 12\sup\{|u(t)|\,|\sin x(t)| : \|(x,u)\| = r\}) = \lim_{r\to\infty}(r - 12r) = -\infty,$$

and $2a_1T\|B\| \approx 10.4$, hence the conditions of Theorem 4 are not satisfied, hence no sufficient evidence to ensure that the system (14) is controllable.

EXAMPLE 2. Consider the fractional differential system

$$\begin{cases} {}^C D_0^{0.2} x(t) = Ax(t) + I_0^{1-\alpha} Bu(t) + f(t,x(t),\ u(t)),\ t \in [0,0.2], \\ x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \end{cases} \tag{15}$$

with $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, and $f(t,x,u) = \begin{bmatrix} 0 \\ \frac{x_2(t)}{1+x_1^2(t)+u_1^2(t)} \end{bmatrix}$. Since $A^2 = 0$, then, the Mittag-Leffler matrices are given by

$$E_{0.2}\left(A(t-s)^{\alpha}\right) = \begin{bmatrix} 1 & \frac{(t-s)^{0.2}}{\Gamma(1.2)} \\ 0 & 1 \end{bmatrix},$$

and

$$E_{0.2,0.2}\left(A(t-s)^{\alpha}\right) = \begin{bmatrix} 1 & \frac{(t-s)^{0.2}}{\Gamma(0.4)} \\ 0 & 1 \end{bmatrix}.$$

By simple matrix calculations one can see that the controllability matrix

$$\mathscr{W} = \begin{bmatrix} 0.552 & 0.332 \\ 0.332 & 1 \end{bmatrix},$$

which has an inverse

$$\mathscr{W}^{-1} = \begin{bmatrix} 2.261 & -0.7498 \\ -0.7498 & 1.2486 \end{bmatrix}.$$

We can ensure that the corresponding linear system of (15) is controllable. Observe that the control function is defined by. $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ such that

$$\begin{aligned} u_1(t) = & \left(1.511 + 2.463(0.2-t)^{0.2}\right) \\ & \times \left(x_1(1) - 1.789 - 0.451 \int_0^1 \frac{(0.2-s)^{-0.6} \|x_2(s)\|}{1+x_1^2(t)+u_1^2(t)} ds\right), \end{aligned}$$

and

$$u_2(t) = \left(0.4988 - 0.8166(0.2-t)^{0.2}\right)\left(x_2(1) - 1 - \int_0^1 \frac{(0.2-s)^{-0.8} \|x_2(s)\|}{1+x_1^2(t)+u_1^2(t)} ds\right).$$

Let us now check the conditions of Theorem 4. Since, $a_1 = 1.7894$, and $\|B\| = 1$, then $2a_1 T \|B\| \approx 0.72 < 1$. Furthermore,

$$\lim_{r\to\infty}\left(r - c_2 \sup\left\{\left|\frac{\|x_2\|}{1+x_1^2+u_1^2}\right| : \|(x,u)\| = r\right\}\right) \geqslant \lim_{r\to\infty}\left(r - \frac{c_2}{2r}\right) = +\infty.$$

Therefore, all conditions of Theorem 4 are satisfied, then the system (15) is controllable.

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