

ON SOME HARDY-TYPE INEQUALITIES FOR GENERALIZED FRACTIONAL INTEGRALS

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*Dedicated to Professor Josip Pečarić
on his 70th birthday*

(Communicated by S. Varošanec)

Abstract. In this article we establish the variant of Hardy-type and refined Hardy-type inequalities for a generalized Riemann-Liouville fractional integral operator and Riemann-Liouville k -fractional integral operator using convex and monotone convex functions. We also discuss one dimensional cases of our related results. As special cases of our general results we obtain the consequences of Iqbal et al. [11]. We also obtained exponentially convex linear functionals for the generalized fractional integral operators. Moreover, it includes Cauchy means for the above mentioned operators.

1. Introduction

The subject of fractional calculus achieve a significant popularity during last few decades due to its demonstrated applications in the fields of science and engineering. It provide several potentially useful tools for solving differential and integral equations. Now a days the applications of fractional calculus include fluid flow, rheology, dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, electrical networks, probability and statistics, control theory of dynamical systems, Optics and signal processing, and so on. Many mathematician originate Hardy-type inequalities for different fractional order integrals and derivatives.

The general theory for the Hardy-type inequalities attracted the scientists a long time, see e.g., the books ([19], [17]). One reason is that such inequalities has many useful applications like to stable the degenerate stationary waves (see [15]). It catches the attention of many mathematicians and they furnish interesting generalizations and improvements of such inequalities. Čižmešija, Krulić Himmelreich, Pečarić, Iqbal, Samraiz and Persson ([5], [2], [18], [1], [12], [8]) has studied a lot of Hardy-type inequalities which is an incredible contribution in theory of inequalities. But our purpose is to present such type of inequalities for generalized Riemann-Loiouville fractional integral operators via convex and monotone convex functions.

The first definition is presented in [22].

Mathematics subject classification (2010): 26D15, 26D10, 26A33, 34B27.

Keywords and phrases: Hardy-type inequalities, fractional integral operators, convex function, mean value theorem, exponential convexity.

DEFINITION 1. Let I be an interval in \mathbb{R} . A function $\Phi : I \rightarrow \mathbb{R}$ is called convex if

$$\Phi(\lambda x + (1 - \lambda)y) \leq \lambda \Phi(x) + (1 - \lambda)\Phi(y), \quad (1)$$

for all points $x, y \in I$ and all $\lambda \in [0, 1]$. The function Φ is strictly convex if inequality (1) holds strictly for all distinct points in I and $\lambda \in (0, 1)$.

The generalized L_p space given in [21] is defined as follows:

DEFINITION 2. A space $L_{p,r}[a, b]$ is defined as a space of continuous real valued functions $h(y)$ on $[a, b]$, such that

$$\left(\int_a^b |h(y)|^p y^r dy \right)^{\frac{1}{p}} < \infty,$$

where $1 \leq p < \infty$, and $r \geq 0$. Specially for $r = 0$, $p = 1$, $L_{p,r}[a, b] = L_1[a, b]$.

Next we give the well known definition of Riemann-Liouville fractional integrals, (see [16]).

DEFINITION 3. Let $[a, b]$ be a finite interval on \mathbb{R} . The left and right sided Riemann-Liouville fractional integrals $I_{a^+}^\alpha f$ and $I_{b^-}^\alpha f$ of order $\alpha > 0$ are defined as:

$$I_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy, \quad x > a,$$

and

$$I_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) dy, \quad x < b,$$

respectively. Here Γ represents Gamma function.

DEFINITION 4. Let $\Phi : I \rightarrow \mathbb{R}$ be a convex function, then the sub-differential of Φ in x is denoted by $\partial\Phi(x)$ and is defined as:

$$\partial\Phi(x) = \{y \in \mathbb{R} : y \text{ is the slope of a support line at } x\}.$$

Let $(\Sigma_1, \Omega_1, \mu_1)$ and $(\Sigma_2, \Omega_2, \mu_2)$ be measure spaces with positive σ -finite measures and $U(f, k)$ denote the class of functions $g : \Omega_1 \rightarrow \mathbb{R}$ with the representation

$$g(x) = \int_{\Omega_2} k(x, t) f(t) d\mu_2(t)$$

and A_k be an integral operator defined by

$$A_k f(x) := \frac{g(x)}{K(x)} = \frac{1}{K(x)} \int_{\Omega_2} k(x, t) f(t) d\mu_2(t), \quad (2)$$

where $k : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}$ is a measurable and non-negative kernel, $f : \Omega_2 \rightarrow \mathbb{R}$ is a measurable function and

$$0 < K(x) := \int_{\Omega_2} k(x,t) d\mu_2(t), \quad x \in \Omega_1. \quad (3)$$

The following theorem is given in [18].

THEOREM 1. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ -finite measures, u be a weight function on Ω_1 , k a non-negative measurable kernel on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by (3). Suppose $K(x) > 0$ for all $x \in \Omega_1$, that the function $x \mapsto u(x) \frac{k(x,t)}{K(x)}$ is integrable on Ω_1 for each $t \in \Omega_2$ and that v is defined on Ω_2 by*

$$v(t) := \int_{\Omega_1} u(x) \frac{k(x,t)}{K(x)} d\mu_1(x) < \infty. \quad (4)$$

If Φ is a convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\Omega_1} u(x) \Phi(A_k f(x)) d\mu_1(x) \leq \int_{\Omega_2} v(t) \Phi(f(t)) d\mu_2(t) \quad (5)$$

holds for all measurable function $f : \Omega_2 \rightarrow \mathbb{R}$, such that $Imf \subseteq I$, where A_k is defined by (2).

Substitute $k(x,t)$ by $k(x,t)f_2(t)$ and f by $\frac{f_1}{f_2}$, where $f_i : \Omega_2 \rightarrow \mathbb{R}$, ($i = 1, 2$) are measurable functions in Theorem 1, we obtain the following result presented in [9].

THEOREM 2. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ -finite measures, u be a weight function on Ω_1 , k a non-negative measurable kernel on $\Omega_1 \times \Omega_2$. Assume that the function $x \mapsto u(x) \frac{k(x,t)}{g_2(x)}$ is integrable on Ω_1 for each fixed $t \in \Omega_2$. Define p on Ω_2 by*

$$p(t) := f_2(t) \int_{\Omega_1} u(x) \frac{k(x,t)}{g_2(x)} d\mu_1(x) < \infty.$$

If $\Phi : I \rightarrow \mathbb{R}$ is a convex function and $\frac{g_1(x)}{g_2(x)}, \frac{f_1(t)}{f_2(t)} \in I$, then the inequality

$$\int_{\Omega_1} u(x) \Phi\left(\frac{g_1(x)}{g_2(x)}\right) d\mu_1(x) \leq \int_{\Omega_2} p(t) \Phi\left(\frac{f_1(t)}{f_2(t)}\right) d\mu_2(t) \quad (6)$$

holds for all $g_i \in U(f_i, k)$, ($i = 1, 2$) and for all measurable function $f_i : \Omega_2 \rightarrow \mathbb{R}$, ($i = 1, 2$).

REMARK 1. If Φ is strictly convex on I and $\frac{f_1(x)}{f_2(x)}$ is non-constant, then the inequality given in (6) is strict.

New refined general weighted Hardy-type inequality with a non-negative kernel and related to an arbitrary convex function is given in the following theorem (see [4]).

THEOREM 3. *Let the assumptions of Theorem 1 be satisfied. Moreover, if Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \rightarrow \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \text{Int}I$, then the inequality*

$$\begin{aligned} \int_{\Omega_2} v(t)\Phi(f(t))d\mu_2(t) - \int_{\Omega_1} u(x)\Phi(A_k f(x))d\mu_1(x) \\ \geq \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} k(x,t) \left[|\Phi(f(t)) - \Phi(A_k f(x))| \right. \\ \left. - |\varphi(A_k f(x))| \cdot |f(t) - A_k f(x)| \right] d\mu_2(t)d\mu_1(x) \end{aligned}$$

holds for all measurable function $f : \Omega_2 \rightarrow \mathbb{R}$.

If Φ is a monotone convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\begin{aligned} \int_{\Omega_2} v(t)\Phi(f(t))d\mu_2(t) - \int_{\Omega_1} u(x)\Phi(A_k f(x))d\mu_1(x) \\ \geq \left| \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} \text{sgn}(f(t) - A_k f(x))k(x,t) \left[\Phi(f(t)) - \Phi(A_k f(x)) \right. \right. \\ \left. \left. - |\varphi(A_k f(x))| \cdot (f(t) - A_k f(x)) \right] d\mu_2(t)d\mu_1(x) \right| \quad (7) \end{aligned}$$

holds for all measurable function $f : \Omega_2 \rightarrow \mathbb{R}$, where $A_k f$ is defined by (2).

If ϕ is a non-negative monotone concave function, then the order of the terms on left hand side of (7) is reversed.

In the following theorem, we give a refinement of a Hardy-type inequality obtained by S. Kaijser et al. in [13].

THEOREM 4. *Let $u : (0, b) \rightarrow \mathbb{R}$ be a weight function such that the functions $x \mapsto \frac{u(x)}{x} \cdot \frac{k(x,t)}{K(x)}$ are integrable on (t, b) for each fixed $t \in (0, b)$, and let the function $w : (0, b) \rightarrow \mathbb{R}$ be defined by*

$$w(t) = t \int_t^b \frac{k(x,t)}{K(x)} u(x) \frac{dx}{x},$$

where $0 < b \leq \infty$ and $k : (0, b) \times (0, b) \rightarrow \mathbb{R}$ be a non-negative measurable function, such that

$$K(x) = \int_0^x k(x,t) dt > 0, \quad x \in (0, b).$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \rightarrow \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \text{Int}I$, then the inequality

$$\begin{aligned} & \int_0^b w(t)\Phi(f(t)) \frac{dt}{t} - \int_0^b u(x)\Phi(A_k f(x)) \frac{dx}{x} \\ & \geq \int_0^b \frac{u(x)}{K(x)} \int_0^x k(x,t) \cdot \left[|\Phi(f(t)) - \Phi(A_k f(x))| - |\varphi(A_k f(x))| \right. \\ & \qquad \qquad \qquad \left. \times |f(t) - A_k f(x)| \right] dt \frac{dx}{x} \end{aligned} \quad (8)$$

holds for all measurable function $f : (0, b) \rightarrow \mathbb{R}$ with values in I , where $A_k f$ is defined by

$$A_k f(x) = \frac{1}{K(x)} \int_0^x k(x,t) f(t) dt, \quad x \in (0, b).$$

If the function Φ is concave, the order of integrals on the left-hand side of (8) is reversed. If Φ is a monotone convex on the interval $I \subseteq \mathbb{R}$, then the following inequality

$$\begin{aligned} & \int_0^b w(t)\Phi(f(t)) \frac{dt}{t} - \int_0^b u(x)\Phi(A_k f(x)) \frac{dx}{x} \\ & \geq \left| \int_0^b \frac{u(x)}{K(x)} \int_0^x \text{sgn}(f(t) - A_k f(x)) k(x,t) \left[\Phi(f(t)) - \Phi(A_k f(x)) \right. \right. \\ & \qquad \qquad \qquad \left. \left. - |\varphi(A_k f(x))| \cdot (f(t) - A_k f(x)) \right] dt \frac{dx}{x} \right| \end{aligned}$$

holds for all measurable function $f : (0, b) \rightarrow \mathbb{R}$ with values in I .

Next mean value theorem is given in [6] which involve functions of the space $C^2(I)$ i.e., the functions having continuous derivatives up to order 2 over the set I .

THEOREM 5. Let $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ -finite measures and $u : \Omega_1 \rightarrow \mathbb{R}$ be a weight function. Let I be compact interval of \mathbb{R} , $\tilde{h} \in C^2(I)$, and $f : \Omega_2 \rightarrow \mathbb{R}$ a measurable function. Then there exists $\eta \in I$ such that

$$\begin{aligned} & \int_{\Omega_2} v(t) \tilde{h}(f(t)) d\mu_2(t) - \int_{\Omega_1} u(x) \tilde{h}(A_k f(x)) d\mu_1(x) \\ & = \frac{\tilde{h}''(\eta)}{2} \left[\int_{\Omega_2} v(t) f^2(t) d\mu_2(t) - \int_{\Omega_1} u(x) (A_k f(x))^2 d\mu_1(x) \right], \end{aligned}$$

where $A_k f$ and v are defined by (2) and (4) respectively.

2. Exponential convexity

We continue with the definition of exponentially convex function as originally given by Bernstein in [3].

DEFINITION 5. A function $\Phi: (a, b) \rightarrow \mathbb{R}$ is *exponentially convex* if it is continuous and

$$\sum_{i,j=1}^n t_i t_j \Phi(x_i + x_j) \geq 0,$$

for all $n \in \mathbb{N}$ and all sequences $(t_n)_{n \in \mathbb{N}}$ and $(x_n)_{n \in \mathbb{N}}$ of real numbers, such that $x_i + x_j \in (a, b)$, $1 \leq i, j \leq n$.

LEMMA 1. Let $s \in \mathbb{R}$ and let the function $\varphi_s: (0, \infty) \rightarrow \mathbb{R}$ be defined by

$$\varphi_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0, 1, \\ -\log x, & s = 0, \\ x \log x, & s = 1. \end{cases} \quad (9)$$

Then $\varphi_s''(x) = x^{s-2}$, that is φ_s is a convex function.

The upcoming theorem is presented in [6].

THEOREM 6. Let the conditions of Theorem 1 be satisfied and φ_s be defined by (9). Let f be a positive function. Then the function $\xi: \mathbb{R} \rightarrow [0, \infty)$ defined by

$$\xi(s) = \int_{\Omega_2} v(t) \varphi_s(f(t)) d\mu_2(t) - \int_{\Omega_1} u(x) \varphi_s(A_k f(x)) d\mu_1(x),$$

is exponentially convex.

THEOREM 7. Let the conditions of Theorem 5 be satisfied. Moreover, $\zeta, \tilde{h} \in C^2(I)$ such that $\tilde{h}''(x) \neq 0$ for every $x \in I$ and

$$\int_{\Omega_2} v(t) \tilde{h}(f(t)) d\mu_2(t) - \int_{\Omega_1} u(x) \tilde{h}(A_k f(x)) d\mu_1(x) \neq 0.$$

Then there exists $\eta \in I$ such that

$$\frac{\zeta''(\eta)}{\tilde{h}''(\eta)} = \frac{\int_{\Omega_2} v(t) \zeta(f(t)) d\mu_2(t) - \int_{\Omega_1} u(x) \zeta(A_k f(x)) d\mu_1(x)}{\int_{\Omega_2} v(t) \tilde{h}(f(t)) d\mu_2(t) - \int_{\Omega_1} u(x) \tilde{h}(A_k f(x)) d\mu_1(x)}$$

holds.

By considering the positive difference of inequality (5), we define the following positive linear functional:

$$\Delta_1(\Phi) = \int_{\Omega_2} v(t)\Phi(f(t)) d\mu_2(t) - \int_{\Omega_1} u(x)\Phi(A_k f(x)) d\mu_1(x). \quad (10)$$

We also define a linear functional by taking the positive difference of the left-hand side and right-hand side of the inequality (6) given in Theorem 2 as:

$$\Delta_2(\Phi) = \int_{\Omega_2} p(t)\Phi\left(\frac{f_1(t)}{f_2(t)}\right) d\mu_2(t) - \int_{\Omega_1} u(x)\Phi\left(\frac{g_1(x)}{g_2(x)}\right) d\mu_1(x). \quad (11)$$

First, we give some necessary details about the divided differences. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \rightarrow \mathbb{R}$ be a function. Then for distinct points $z_i \in I$, $i = 0, 1, 2$, the divided differences of first and second order are defined by:

$$\begin{aligned} [z_i, z_{i+1}; f] &= \frac{f(z_{i+1}) - f(z_i)}{z_{i+1} - z_i} \quad (i = 0, 1), \\ [z_0, z_1, z_2; f] &= \frac{[z_1, z_2; f] - [z_0, z_1; f]}{z_2 - z_0}. \end{aligned} \quad (12)$$

The values of the divided differences are independent of the order of points z_0, z_1, z_2 and may be extended to include the cases when some or all points are equal, that is

$$[z_0, z_0; f] = \lim_{z_1 \rightarrow z_0} [z_0, z_1; f] = f'(z_0),$$

provided that f' exists.

Now passing through the limit $z_1 \rightarrow z_0$ and replacing z_2 by z in (12), we have

$$[z_0, z_0, z; f] = \lim_{z_1 \rightarrow z_0} [z_0, z_1, z; f] = \frac{f(z) - f(z_0) - (z - z_0)f'(z_0)}{(z - z_0)^2} \quad z \neq z_0,$$

provided that f' exists. Also passing to the limit $z_i \rightarrow z$ ($i = 0, 1, 2$) in (12), we have

$$[z, z, z; f] = \lim_{z_i \rightarrow z} [z_0, z_1, z_2; f] = \frac{f''(z)}{2},$$

provided that f'' exists.

One can observe that if for all $z_0, z_1 \in I$, $[z_0, z_1; f] \geq 0$, then f is increasing on I and if for all $z_0, z_1, z_2 \in I$, $[z_0, z_1, z_2; f] \geq 0$, then f is convex on I .

Next, we recall the notion of n -exponential convexity given in [24].

DEFINITION 6. For any open interval I of \mathbb{R} the function $\Phi : I \rightarrow \mathbb{R}$ is n -exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^n t_i t_j \Phi\left(\frac{\zeta_i + \zeta_j}{2}\right) \geq 0$$

holds for all choices of $t_i \in \mathbb{R}$, $\zeta_i \in I$, $i = 1, \dots, n$.

A function $\Phi : I \rightarrow \mathbb{R}$ is n -exponentially convex on I if it is n -exponentially convex in the Jensen sense and continuous on I .

The following theorem is given in [10].

THEOREM 8. *Let $\Gamma = \{\Phi_p : p \in J\}$ be a family of functions defined on I , such that the function $p \mapsto [z_0, z_1, z_2; \Phi_p]$ is n -exponentially convex in the Jensen sense on J for every three distinct points $z_0, z_1, z_2 \in I$. Let Δ_i ($i = 1, 2$) be linear functionals defined by (10), (11). Then the function $p \mapsto \Delta_i(\Phi_p)$ ($i = 1, 2$) is n -exponentially convex in the Jensen sense on J , if it is continuous on J .*

The rest of the paper is planned in the following way: In Section 3, we prove new Hardy-type inequalities and their refinements involving generalized Riemann-Liouville fractional integral operator. Section 4 deals with Hardy-type, refined Hardy-type inequalities for generalized k -Riemann-Liouville fractional integral operator. In each section, we originate the results regarding Cauchy means and exponentially convex linear functionals.

3. Results for generalized Riemann-Liouville fractional integral operator

In this section, first we give the definition of generalized Riemann-Liouville fractional integral operator presented in [14].

DEFINITION 7. Let $\alpha > 0$, $a \geq 0$ and $r \neq -1$, be real numbers and let $f \in L_{1,r}[a, b]$. Then the generalized Riemann-Liouville fractional integral $I_a^{\alpha,r}$ is defined by

$$I_a^{\alpha,r} f(x) = \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt, \quad x \in (a, b). \quad (13)$$

We note that if $r \rightarrow -1^+$ the integral operator (13) reduces to the famous Hadamard fractional integral:

$$I_a^{\alpha,-1^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t} dt. \quad (14)$$

THEOREM 9. *Let $f \in L_{1,r}[a, b]$ such that $r \neq -1$, $a \geq 0$. Suppose u is a weight function on (a, b) and that a function $x \mapsto \alpha(r+1) \frac{(x^{r+1} - t^{r+1})^{\alpha-1}}{(x^{r+1} - a^{r+1})^\alpha} t^r u(x)$ is integrable on (a, b) for each $t \in (a, b)$ the weight function s is defined by*

$$s(t) := \alpha(r+1) t^r \int_t^b u(x) \frac{(x^{r+1} - t^{r+1})^{\alpha-1}}{(x^{r+1} - a^{r+1})^\alpha} dx < \infty. \quad (15)$$

If Φ is a convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_a^b u(x)\Phi\left(\frac{\alpha(r+1)}{(x^{r+1}-a^{r+1})^\alpha}\int_a^x(x^{r+1}-t^{r+1})^{\alpha-1}t^r f(t)dt\right)dx \leq \int_a^b s(t)\Phi(f(t))dt \quad (16)$$

holds for all measurable function $f : (a, b) \rightarrow \mathbb{R}$.

Proof. Applying Theorem 1 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$,

$$\tilde{k}(x, t) = \begin{cases} \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}(x^{r+1}-t^{r+1})^{\alpha-1}t^r, & a \leq t \leq x; \\ 0, & x < t \leq b, \end{cases} \quad (17)$$

we get

$$\tilde{K}(x) = \frac{1}{\Gamma(\alpha+1)(r+1)^\alpha}(x^{r+1}-a^{r+1})^\alpha \quad (18)$$

and the integral operator $A_k f(x)$ takes the form

$$\tilde{A}_k f(x) = \frac{\alpha(r+1)}{(x^{r+1}-a^{r+1})^\alpha}\int_a^x(x^{r+1}-t^{r+1})^{\alpha-1}t^r f(t)dt, \quad (19)$$

we get inequality (16). \square

COROLLARY 1. In particular if $r \rightarrow -1$ and $a > 0$ in Theorem 9, we get

$$\tilde{s}(t) := \frac{\alpha}{t}\int_t^b u(x)\frac{(\log \frac{x}{t})^{\alpha-1}}{(\log \frac{x}{a})^\alpha}dx$$

and the inequality (16) reduces to

$$\int_a^b u(x)\Phi\left(\alpha\int_a^x\frac{(\log \frac{x}{t})^{\alpha-1}}{(\log \frac{x}{a})^\alpha}\frac{f(t)}{t}dt\right)dx \leq \int_a^b \tilde{s}(t)\Phi(f(t))dt.$$

THEOREM 10. Let u be a weight function defined on (a, b) , $I_a^{\alpha, r}$ be the generalized Riemann-Liouville fractional integral operator of order $\alpha > 0$, $a \geq 0$ and $r \neq -1$. Assume that the function $x \mapsto \frac{(x^{r+1}-t^{r+1})^{\alpha-1}}{I_a^{\alpha, r} f_2(x)}u(x)$ is integrable on (a, b) , then for each $t \in (a, b)$ the weight function $q(t)$ is defined by

$$q(t) := \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}t^r f_2(t)\int_t^b u(x)\frac{(x^{r+1}-t^{r+1})^{\alpha-1}}{I_a^{\alpha, r} f_2(x)}dx < \infty.$$

If $\Phi : I \rightarrow \mathbb{R}$ is a convex function and $\frac{I_a^{\alpha,r} f_1(x)}{I_a^{\alpha,r} f_2(x)}, \frac{f_1(t)}{f_2(t)} \in I$, then the inequality

$$\int_a^b u(x) \Phi \left(\frac{I_a^{\alpha,r} f_1(x)}{I_a^{\alpha,r} f_2(x)} \right) dx \leq \int_a^b q(t) \Phi \left(\frac{f_1(t)}{f_2(t)} \right) dt \quad (20)$$

holds for all measurable function $f_i : (a, b) \rightarrow \mathbb{R}$, ($i = 1, 2$).

Proof. Applying Theorem 2 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, $g_i(x) = I_a^{\alpha,r} f_i(x)$, $i = 1, 2$ and $\tilde{k}(x, t)$ given in (17), we get inequality (20). \square

COROLLARY 2. In particular if we choose $r \rightarrow -1$ in Theorem 10, we get

$$\tilde{q}(t) = \frac{1}{\Gamma(\alpha)} \frac{f_2(t)}{t} \int_t^b \frac{u(x)}{I_a^{\alpha, -1+} f_2(x)} \left(\log \frac{x}{t} \right)^{\alpha-1} dx$$

and the inequality (20) takes the form

$$\int_a^b u(x) \Phi \left(\frac{I_a^{\alpha, -1+} f_1(x)}{I_a^{\alpha, -1+} f_2(x)} \right) dx \leq \int_a^b \tilde{q}(t) \Phi \left(\frac{f_1(t)}{f_2(t)} \right) dt.$$

In next theorem we give the refinement of Theorem 10.

THEOREM 11. Let the assumptions of Theorem 9 be satisfied. Moreover, if Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \rightarrow \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \text{Int } I$, then the inequality

$$\begin{aligned} & \int_a^b s(t) \Phi(f(t)) dt - \int_a^b u(x) \Phi \left(\frac{\alpha(r+1)}{(x^{r+1} - a^{r+1})^\alpha} \int_a^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right) dx \\ & \geq \int_a^b u(x) \frac{\alpha(r+1)}{(x^{r+1} - a^{r+1})^\alpha} \int_a^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r \\ & \times \left| \Phi(f(t)) - \Phi \left(\frac{\alpha(r+1)}{(x^{r+1} - a^{r+1})^\alpha} \int_a^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right) \right| \\ & - \left| \varphi \left(\frac{\alpha(r+1)}{(x^{r+1} - a^{r+1})^\alpha} \int_a^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right) \right| \\ & \times \left| f(t) - \frac{\alpha(r+1)}{(x^{r+1} - a^{r+1})^\alpha} \int_a^b (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right| dt dx \quad (21) \end{aligned}$$

holds for all measurable function $f : (a, b) \rightarrow \mathbb{R}$.

If Φ is a monotone convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\begin{aligned} & \int_a^b s(t)\Phi(f(t)) dt - \int_a^b u(x)\Phi\left(\frac{\alpha(r+1)}{(x^{r+1}-a^{r+1})^\alpha} \int_a^x (x^{r+1}-t^{r+1})^{\alpha-1} t^r f(t) dt\right) dx \\ & \geq \left| \int_a^b u(x) \frac{\alpha(r+1)}{(x^{r+1}-a^{r+1})^\alpha} \int_a^x \operatorname{sgn}\left(f(t) - \frac{\alpha(r+1)}{(x^{r+1}-a^{r+1})^\alpha} \int_a^x (x^{r+1}-t^{r+1})^{\alpha-1} t^r f(t) dt\right) \right. \\ & \quad \times (x^{r+1}-t^{r+1})^{\alpha-1} t^r \left[\Phi(f(t)) - \Phi\left(\frac{\alpha(r+1)}{(x^{r+1}-a^{r+1})^\alpha} \int_a^x (x^{r+1}-t^{r+1})^{\alpha-1} t^r f(t) dt\right) \right. \\ & \quad \left. \left. - \left| \varphi\left(\frac{\alpha(r+1)}{(x^{r+1}-a^{r+1})^\alpha} \int_a^x (x^{r+1}-t^{r+1})^{\alpha-1} t^r f(t) dt\right) \right| \right. \right. \\ & \quad \left. \left. \times \left(f(t) - \frac{\alpha(r+1)}{(x^{r+1}-a^{r+1})^\alpha} \int_a^x (x^{r+1}-t^{r+1})^{\alpha-1} t^r f(t) dt\right) \right] dt dx \right| \end{aligned} \tag{22}$$

holds for all measurable function $f : (a, b) \rightarrow \mathbb{R}$.

Proof. Applying Theorem 3 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, $\tilde{k}(x, t)$, $\tilde{K}(x)$, and $\tilde{A}_k f(x)$ are given by (17), (18) and (19) respectively, we get inequalities (21) and (22) respectively. \square

REMARK 2. Choose a particular convex function $\Phi(x) = x^\nu$, $\nu \geq 1$ and weight function $u(x) = \frac{1}{\alpha\Gamma(\alpha)(r+1)^\alpha} (x^{r+1} - a^{r+1})^\alpha$ in Theorem 9, we obtain

$$\tilde{p}(t) = \int_t^b \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)} (x^{r+1} - t^{r+1})^{\alpha-1} t^r dx,$$

which can be written as:

$$\tilde{p}(t) = \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)} t^r \int_t^b x^{(r+1)(\alpha-1)} \left(1 - \left(\frac{t}{x}\right)^{r+1}\right)^{\alpha-1} dx.$$

Substituting $y = 1 - \left(\frac{t}{x}\right)^{r+1}$ and after a little calculation, we get

$$\tilde{p}(t) = \frac{t^{\alpha(r+1)}}{\Gamma(\alpha)(r+1)^\alpha} \int_0^{1-\left(\frac{t}{b}\right)^{r+1}} x^{\alpha-1} (1-x)^{\frac{r}{r+1}\alpha-1} dx,$$

which involve incomplete Beta function $B_x(p, q) = \int_0^x t^{p-1} (1-t)^{q-1} dt$, (see [7, page 910]) i.e.,

$$\tilde{p}(t) = \frac{t^{\alpha(r+1)}}{(r+1)^\alpha \Gamma(\alpha)} B_{1-\left(\frac{t}{b}\right)^{r+1}}\left(\alpha, \frac{r}{r+1} - \alpha\right) := K_1(t).$$

Using the above calculated weight function in inequality (16), we obtain

$$\tilde{K}^{1-\nu}(b) \int_a^b I_a^{\alpha,r} f^\nu(x) dx \leq \frac{t^{\alpha(r+1)}}{(r+1)^\alpha \Gamma(\alpha)} B_{1-\left(\frac{a}{b}\right)^{(r+1)}} \left(\alpha, \frac{r}{r+1} - \alpha \right) \int_a^b f^\nu(t) dt.$$

Consequently, we have the inequality of G. H. Hardy as follows:

$$\|I_a^{\alpha,r} f\|_\nu(a,b) \leq \left(\frac{t^{\alpha(r+1)} B_{1-\left(\frac{a}{b}\right)^{(r+1)}} \left(\alpha, \frac{r}{r+1} - \alpha \right)}{(r+1)^\alpha \Gamma(\alpha) \tilde{K}^{1-\nu}(b)} \right)^{\frac{1}{\nu}} \|f\|_\nu(a,b).$$

Result for one dimensional setting involving generalized Riemann-Liouville fractional integral is as follows:

THEOREM 12. *Let $u : (0, b) \rightarrow \mathbb{R}$ be a weight function, such that the function $x \mapsto \frac{(x^{r+1} - t^{r+1})^{\alpha-1} u(x)}{x^{(r+1)\alpha}}$, $r \neq -1$ is integrable on (t, b) , then the function $j : (0, b) \rightarrow \mathbb{R}$ be defined by*

$$j(t) := t^{r+1} \alpha(r+1) \int_t^b \frac{(x^{r+1} - t^{r+1})^{\alpha-1}}{x^{(r+1)\alpha}} u(x) \frac{dx}{x},$$

where $0 < b \leq \infty$ and $\hat{k} : (0, b) \times (0, b) \rightarrow \mathbb{R}$ be a non-negative measurable kernel, such that

$$\hat{K}(x) = \frac{x^{(r+1)\alpha}}{\Gamma(\alpha+1)(r+1)^\alpha} > 0, \quad x \in (0, b).$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \rightarrow \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \text{Int } I$, then the inequality

$$\begin{aligned} & \int_0^b j(t) \Phi(f(t)) \frac{dt}{t} - \int_0^b u(x) \Phi \left(\frac{\alpha(r+1)}{x^{(r+1)\alpha}} \int_0^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right) \frac{dx}{x} \\ & \geq \int_0^b u(x) \frac{(r+1)\alpha}{x^{(r+1)\alpha}} \int_0^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r \\ & \quad \times \left| \Phi(f(t)) - \Phi \left(\frac{\alpha(r+1)}{x^{(r+1)\alpha}} \int_0^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right) \right| \\ & \quad - \left| \varphi \left(\frac{\alpha(r+1)}{x^{(r+1)\alpha}} \int_0^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right) \right| \\ & \quad \times \left| f(t) - \frac{\alpha(r+1)}{x^{(r+1)\alpha}} \int_0^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right| \left| dt \frac{dx}{x} \right. \end{aligned} \quad (23)$$

holds for all measurable function $f : (0, b) \rightarrow \mathbb{R}$ and the integral operator $A_{k,f}$ takes the form

$$\hat{A}_k f(x) := \frac{\alpha(r+1)}{x^{(r+1)\alpha}} \int_0^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt, \quad x \in (0, b).$$

If the function Φ is a concave, the order of integrals on the left-hand side of (23) is reversed. If Φ is a monotone convex on the interval $I \subseteq \mathbb{R}$, then the following inequality

$$\begin{aligned} & \int_0^b j(t) \Phi(f(t)) \frac{dt}{t} - \int_0^b u(x) \Phi \left(\frac{\alpha(r+1)}{x^{(r+1)\alpha}} \int_0^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right) \frac{dx}{x} \\ & \geq \left| \int_0^b u(x) \frac{\alpha(r+1)}{x^{(r+1)\alpha}} \int_0^x \operatorname{sgn} \left(f(t) - \frac{\alpha(r+1)}{x^{(r+1)\alpha}} \int_0^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right) \right. \\ & \quad \times (x^{r+1} - t^{r+1})^{\alpha-1} t^r \cdot \left[\Phi(f(t)) - \Phi \left(\frac{\alpha(r+1)}{x^{(r+1)\alpha}} \int_0^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right) \right. \\ & \quad \left. \left. - \left| \varphi \left(\frac{\alpha(r+1)}{x^{(r+1)\alpha}} \int_0^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right) \right| \right. \right. \\ & \quad \left. \left. \times \left(f(t) - \frac{\alpha(r+1)}{x^{(r+1)\alpha}} \int_0^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right) \right] dt \frac{dx}{x} \right| \end{aligned} \tag{24}$$

holds for all measurable function $f : (0, b) \rightarrow \mathbb{R}$.

Proof. Applying Theorem 4 with $\Omega_1 = \Omega_2 = (0, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$,

$$\hat{k}(x, t) = \begin{cases} \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)} (x^{r+1} - t^{r+1})^{\alpha-1} t^r, & 0 \leq t \leq x; \\ 0, & x < t \leq b, \end{cases}$$

we get inequalities (23) and (24) respectively. \square

THEOREM 13. Let $f \in L_{1,r}[a, b]$, $I_a^{\alpha,r}$ be the generalized fractional integral of order $\alpha > 0$ and $r \neq -1$ with $u : (a, b) \rightarrow \mathbb{R}$ a weight function. Let I be a compact interval of \mathbb{R} , $\tilde{h} \in C^2(I)$, and $f : (a, b) \rightarrow \mathbb{R}$ a measurable function. Then there exists $\eta \in I$ such that the equation

$$\begin{aligned} & \int_a^b s(t) \tilde{h}(f(t)) dt - \int_a^b u(x) \tilde{h} \left(\frac{\alpha(r+1)}{(x^{r+1} - a^{r+1})^\alpha} \int_a^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right) dx \\ & = \frac{\tilde{h}''(\eta)}{2} \left[\int_a^b s(t) f^2(t) dt - \int_a^b u(x) \left(\frac{\alpha(r+1)}{(x^{r+1} - a^{r+1})^\alpha} \int_a^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right)^2 dx \right], \end{aligned} \tag{25}$$

holds true.

Proof. Applying Theorem 5 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, $s(t)$ and $\tilde{A}_k f$ are defined by (15) and (19) respectively, we get the required Cauchy mean presented in (25). \square

Next theorem provide the exponential convexity of the linear functional by taking the non-negative difference of Hardy-type inequality given in (16).

THEOREM 14. *Let the conditions of Theorem 9 be satisfied and φ_s be defined by Lemma 1. Let f be a positive function, then the function $\Psi : \mathbb{R} \rightarrow [0, \infty)$ defined by*

$$\Psi(s) = \int_a^b s(t) \varphi_s(f(t)) dt - \int_a^b u(x) \varphi_s \left(\frac{\alpha(r+1)}{(x^{r+1} - a^{r+1})^\alpha} \int_a^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right) dx$$

is exponentially convex.

Proof. Applying Theorem 6 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and $\tilde{k}(x, t)$ given by (17), we obtain the required result. \square

THEOREM 15. *Let the conditions of Theorem 13 be satisfied. Moreover, $g, \tilde{h} \in C^2(I)$ such that $\tilde{h}''(x) \neq 0$ for every $x \in I$ and*

$$\int_a^b s(t) \tilde{h}(f(t)) dt - \int_a^b u(x) \tilde{h} \left(\frac{\alpha(r+1)}{(x^{r+1} - a^{r+1})^\alpha} \int_a^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right) dx \neq 0.$$

Then there exists $\eta \in I$ such that it holds

$$\frac{g''(\eta)}{\tilde{h}''(\eta)} = \frac{\int_a^b s(t) g(f(t)) dt - \int_a^b u(x) g \left(\frac{\alpha(r+1)}{(x^{r+1} - a^{r+1})^\alpha} \int_a^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right) dx}{\int_a^b s(t) \tilde{h}(f(t)) dt - \int_a^b u(x) \tilde{h} \left(\frac{\alpha(r+1)}{(x^{r+1} - a^{r+1})^\alpha} \int_a^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right) dx} \quad (26)$$

Proof. Applying Theorem 7 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, $s(t)$ and $\tilde{A}_k f$ are defined by (15) and (19) respectively, we get (26). \square

Under the assumptions of Theorem 9, we define a linear functional by taking the positive difference of the inequality stated in (16) as:

$$\eta_1(\Phi) = \int_a^b s(t) \Phi(f(t)) dt - \int_a^b u(x) \Phi \left(\frac{\alpha(r+1)}{(x^{r+1} - a^{r+1})^\alpha} \int_a^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt \right) dx. \quad (27)$$

We also define a linear functional by taking the positive difference of the left-hand side and right-hand side of the inequality (20) given in Theorem 10 as:

$$\eta_2(\Phi) = \int_a^b q(t) \Phi \left(\frac{f_1(t)}{f_2(t)} \right) dt - \int_a^b u(x) \Phi \left(\frac{I_a^{\alpha,r} f_1(x)}{I_a^{\alpha,r} f_2(x)} \right) dx. \quad (28)$$

THEOREM 16. Let $\Gamma = \{\Phi_p : p \in J\}$ be a family of functions defined on I , such that the function $p \mapsto [z_0, z_1, z_2; \Phi_p]$ is n -exponentially convex in the Jensen sense on J for every three distinct points $z_0, z_1, z_2 \in I$. Let η_i ($i = 1, 2$) be linear functionals defined by (27), (28). Then the function $p \mapsto \eta_i(\Phi_p)$ ($i = 1, 2$) is n -exponentially convex in the Jensen sense on J , if it is continuous on J .

Proof. Applying Theorem 8 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, we get the desired outcome. \square

REMARK 3. If we choose $r = 0$, in inequalities (16), (20), (21), (22), (23), (24) and equations (25), (26), we acquire [11, Corollary 3].

REMARK 4. If we choose $r \rightarrow -1^+$, in inequalities (21), (22), (23), (24) and equations (25), (26), we get the results for the famous Hadamard fractional integral operator presented in (14).

4. Consequences for the Riemann-Liouville k -fractional integral

In this section, we derive results for the Riemann-Liouville k -fractional integral presented in [20] and is defined as:

DEFINITION 8. Let $f \in L_1[a, b]$, then the Riemann-Liouville k -fractional integral $I_{a,k}^\alpha$ of order $\alpha > 0$ and $k > 0$, is given by

$$I_{a,k}^\alpha f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt, \quad t \in (a, b), \quad (29)$$

where Γ_k is defined by

$$\Gamma_k(t) = \int_0^\infty x^{t-1} e^{-\frac{x^k}{k}} dx, \quad \operatorname{Re}(x) > 0.$$

Moreover, if we choose $k = 1$ the integral operator (29) represents the left sided Riemann-Liouville fractional integral.

THEOREM 17. Let $f \in L_1[a, b]$ and u be a weight function on (a, b) . Suppose $x \mapsto \frac{\alpha}{k} \frac{(x-t)^{\frac{\alpha}{k}-1}}{(x-a)^{\frac{\alpha}{k}}} u(x)$ is integrable on (a, b) for each $t \in (a, b)$ and that the function β is defined by

$$\beta(t) := \frac{\alpha}{k} \int_t^b u(x) \frac{(x-t)^{\frac{\alpha}{k}-1}}{(x-a)^{\frac{\alpha}{k}}} dx < \infty. \quad (30)$$

If Φ is a convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_a^b u(x) \Phi \left(\frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) dx \leq \int_a^b \beta(t) \Phi(f(t)) dt \quad (31)$$

holds for all measurable function $f : (a, b) \rightarrow \mathbb{R}$.

Proof. Applying Theorem 1 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$,

$$\check{k}(x, t) = \begin{cases} \frac{1}{k\Gamma_k(\alpha)} (x-t)^{\frac{\alpha}{k}-1}, & a \leq t \leq x; \\ 0, & x < t \leq b, \end{cases} \quad (32)$$

$$\check{K}(x) = \frac{1}{\alpha\Gamma_k(\alpha)} (x-a)^{\frac{\alpha}{k}}, \quad (33)$$

and

$$\check{A}_k f(x) = \frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt, \quad (34)$$

we get inequality (31). \square

In next result we give the quotient form of Theorem 17.

THEOREM 18. Let u be a weight function, $I_{a,k}^\alpha$ be the generalized Riemann-Liouville k -fractional integral operator of order $\alpha > 0$ and $k > 0$. Assume that the function $x \mapsto \frac{1}{k\Gamma_k(\alpha)} \frac{(x-t)^{\frac{\alpha}{k}-1}}{I_{a,k}^\alpha f_2(x)} u(x)$ is integrable on (a, b) , then for each $t \in (a, b)$, define a function γ by

$$\gamma(t) := \frac{1}{k\Gamma_k(\alpha)} f_2(t) \int_t^b u(x) \frac{(x-t)^{\frac{\alpha}{k}-1}}{I_{a,k}^\alpha f_2(x)} dx < \infty.$$

If $\Phi : I \rightarrow \mathbb{R}$ is a convex function and $\frac{I_{a,k}^\alpha f_1(x)}{I_{a,k}^\alpha f_2(x)}, \frac{f_1(t)}{f_2(t)} \in I$, then the inequality

$$\int_a^b u(x) \Phi \left(\frac{I_{a,k}^\alpha f_1(x)}{I_{a,k}^\alpha f_2(x)} \right) dx \leq \int_a^b \gamma(t) \Phi \left(\frac{f_1(t)}{f_2(t)} \right) dt \quad (35)$$

holds for all measurable function $f_i : (a, b) \rightarrow \mathbb{R}$, ($i = 1, 2$).

Proof. Applying Theorem 2 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, and $\check{k}(x, t)$ given by (32), we arrive at inequality (35). \square

THEOREM 19. *Let the assumptions of Theorem 17 be satisfied. Moreover, if Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \rightarrow \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \text{Int} I$, then the inequality*

$$\begin{aligned} & \int_a^b \beta(t) \Phi(f(t)) dt - \int_a^b u(x) \Phi \left(\frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) dx \\ & \geq \int_a^b u(x) \frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} \left| \Phi(f(t)) - \Phi \left(\frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) \right| \\ & \quad - \left| \varphi \left(\frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) \right| \left| \left| f(t) - \frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right| \right| dt dx \end{aligned} \quad (36)$$

holds for all measurable function $f : (a, b) \rightarrow \mathbb{R}$.

If Φ is a monotone convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\begin{aligned} & \int_a^b \beta(t) \Phi(f(t)) dt - \int_a^b u(x) \Phi \left(\frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) dx \\ & \geq \left| \int_a^b u(x) \frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x \text{sgn} \left(f(t) - \frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) (x-t)^{\frac{\alpha}{k}-1} \right. \\ & \quad \times \left[\Phi(f(t)) - \Phi \left(\frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) \right. \\ & \quad \left. \left. - \left| \varphi \left(\frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) \right| \left| \left(f(t) - \frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) \right| \right] dt dx \end{aligned} \quad (37)$$

holds for all measurable function $f : (a, b) \rightarrow \mathbb{R}$ and for all fixed $t \in (a, b)$.

Proof. Applying Theorem 3 by with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and $\check{k}(x, t)$ is given by (32), we get inequalities (36) and (37) respectively. \square

REMARK 5. Choose the particular convex function $\Phi(x) = x^\nu$, $\nu \geq 1$ and weight function $u(x) = \frac{1}{\alpha \Gamma_k(\alpha)} (x-a)^{\frac{\alpha}{k}}$ in Theorem 17, we obtain

$$\tilde{\beta}(t) = \frac{1}{\Gamma_k(\alpha+1)} (b-t)^{\frac{\alpha}{k}} =: K_2(t).$$

The inequality (31) can be written as:

$$\int_a^b \left(\frac{1}{\alpha \Gamma_k(\alpha)} (x-a)^{\frac{\alpha}{k}} \right) \left(\frac{1}{\frac{1}{\alpha \Gamma_k(\alpha)} (x-a)^{\frac{\alpha}{k}}} \int_a^x \frac{1}{k \Gamma_k(\alpha)} (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right)^v dx \leq \int_a^b \beta(t) f^v(t) dt,$$

it can also be written as

$$\int_a^b \left(\frac{1}{\alpha \Gamma_k(\alpha)} (x-a)^{\frac{\alpha}{k}} \right)^{1-v} \left(\int_a^x \frac{1}{k \Gamma_k(\alpha)} (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right)^v dx \leq \int_a^b \beta(t) f^v(t) dt$$

this implies that

$$\int_a^b \check{K}^{1-v}(x) \left(\int_a^x \frac{1}{k \Gamma_k(\alpha)} (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right)^v dx \leq \frac{1}{\Gamma_k(\alpha+1)} \int_a^b (b-t)^{\frac{\alpha}{k}} f^v(t) dt$$

$$\check{K}^{1-v}(b) \int_a^b I_{a,k}^\alpha f^v(x) dx \leq \frac{1}{\Gamma_k(\alpha+1)} (b-a)^{\frac{\alpha}{k}} \int_a^b f^v(t) dt.$$

After some calculation, we turn up the inequality

$$\|I_{a,k}^\alpha f\|_v(a,b) \leq \left(\frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+1) \check{K}^{1-v}(b)} \right)^{\frac{1}{v}} \|f\|_v(a,b),$$

which is an inequality of G. H. Hardy.

THEOREM 20. *Let $u : (0, b) \rightarrow \mathbb{R}$ be a weight function such that the function $x \mapsto \frac{\alpha}{k} \frac{(x-t)^{\frac{\alpha}{k}-1}}{(x-a)^{\frac{\alpha}{k}}} \frac{u(x)}{x}$ is integrable on (t, b) for each $t \in (0, b)$, and let the function $\lambda : (0, b) \rightarrow \mathbb{R}$ be defined by*

$$\lambda(t) := t \frac{\alpha}{k} \int_t^b \frac{(x-t)^{\frac{\alpha}{k}-1}}{x^{\frac{\alpha}{k}}} u(x) dx,$$

where $0 < b \leq \infty$ and $\bar{k} : (0, b) \times (0, b) \rightarrow \mathbb{R}$ be a non-negative measurable kernel, such that

$$\bar{K}(x) = \frac{1}{\alpha \Gamma(\alpha)} x^{\frac{\alpha}{k}} > 0, \quad x \in (0, b).$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \rightarrow \mathbb{R}$ is such that $\varphi(x) \in \partial \Phi(x)$ for all $x \in \text{Int } I$, then the inequality

$$\int_0^b \lambda(t) \Phi(f(t)) \frac{dt}{t} - \int_0^b u(x) \Phi \left(\frac{\alpha}{k x^{\frac{\alpha}{k}}} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) \frac{dx}{x}$$

$$\geq \int_0^b u(x) \frac{\alpha}{k x^{\frac{\alpha}{k}}} \int_0^x (x-t)^{\frac{\alpha}{k}-1} \left\| \Phi(f(t)) - \Phi \left(\frac{\alpha}{k x^{\frac{\alpha}{k}}} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) \right\|$$

$$- \left| \varphi \left(\frac{\alpha}{kx^{\frac{\alpha}{k}}} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) \right| \left| f(t) - \left(\frac{\alpha}{kx^{\frac{\alpha}{k}}} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) \right| dt \frac{dx}{x} \quad (38)$$

holds for all measurable function $f : (0, b) \rightarrow \mathbb{R}$ and

$$\bar{A}_k f(x) := \frac{\alpha}{kx^{\frac{\alpha}{k}}} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt, \quad x \in (0, b).$$

If the function Φ is concave, the order of integrals on the left-hand side of (38) is reversed. If Φ is monotone convex on the interval $I \subseteq \mathbb{R}$, then the following inequality

$$\begin{aligned} & \int_0^b \lambda(t) \Phi(f(t)) \frac{dt}{t} - \int_0^b u(x) \Phi \left(\frac{\alpha}{kx^{\frac{\alpha}{k}}} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) \frac{dx}{x} \\ & \geq \left| \int_0^b u(x) \frac{\alpha}{kx^{\frac{\alpha}{k}}} \int_0^x \operatorname{sgn} \left(f(t) - \frac{\alpha}{kx^{\frac{\alpha}{k}}} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) \right. \\ & \quad \times (x-t)^{\frac{\alpha}{k}-1} \left[\Phi(f(t)) - \Phi \left(\frac{\alpha}{kx^{\frac{\alpha}{k}}} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) \right. \\ & \quad \left. \left. - \left| \varphi \left(\frac{\alpha}{kx^{\frac{\alpha}{k}}} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) \right| \left(f(t) - \frac{\alpha}{kx^{\frac{\alpha}{k}}} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) \right] dt \frac{dx}{x} \right| \quad (39) \end{aligned}$$

holds for all measurable function $f : (0, b) \rightarrow \mathbb{R}$.

Proof. Applying Theorem 4 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$,

$$\bar{k}(x, t) = \begin{cases} \frac{1}{k\Gamma_k(\alpha)} (x-t)^{\frac{\alpha}{k}-1}, & 0 \leq t \leq x; \\ 0, & x < t \leq b, \end{cases}$$

we get inequalities (38) and (39) respectively. \square

Next we give the mean value theorems for the Riemann-Liouville k -fractional integral operator.

THEOREM 21. Let $f \in L_1[a, b]$, $I_{a,k}^\alpha$ be the generalized fractional integral of order $\alpha \geq 0$ and $k > 0$ and let $u : (a, b) \rightarrow \mathbb{R}$ be a weight function. Moreover, I a compact interval of \mathbb{R} , $\tilde{h} \in C^2(I)$, and $f : (a, b) \rightarrow \mathbb{R}$ a measurable function such that $\operatorname{Im} f \subseteq I$. Then there exists $\eta \in I$ such that the equation

$$\int_a^b \beta(t) \tilde{h}(f(t)) dt - \int_a^b u(x) \tilde{h} \left(\frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) dx$$

$$= \frac{\tilde{h}''(\eta)}{2} \left[\int_a^b \beta(t) f^2(t) dt - \int_a^b u(x) \left(\frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right)^2 dx \right] \quad (40)$$

holds true, where β and $\check{A}_k f$ are defined by (30) and (34) respectively.

Proof. Applying Theorem 5 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, we get equation (40). \square

THEOREM 22. *Let the conditions of Theorem 17 be satisfied and φ_s be defined by (9). Let f be a positive function. Then the function $\Upsilon : \mathbb{R} \rightarrow [0, \infty)$ defined by*

$$\Upsilon(s) = \int_a^b \beta(t) \varphi_s(f(t)) dt - \int_a^b u(x) \varphi_s \left(\frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) dx \quad (41)$$

is exponentially convex.

Proof. Applying Theorem 6 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and the value of $\check{A}_k f$ is defined by (34), we get (41). \square

THEOREM 23. *Let the conditions of Theorem 22 be satisfied. Moreover, $g, \tilde{h} \in C^2(I)$ such that $\tilde{h}''(x) \neq 0$ for every $x \in I$ and*

$$\int_a^b \beta(t) \tilde{h}(f(t)) dt - \int_a^b u(x) \tilde{h} \left(\frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) dx \neq 0.$$

Then there exists $\eta \in I$ such that

$$\frac{g''(\eta)}{\tilde{h}''(\eta)} = \frac{\int_a^b \beta(t) g(f(t)) dt - \int_a^b u(x) g \left(\frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) dx}{\int_a^b \beta(t) \tilde{h}(f(t)) dt - \int_a^b u(x) \tilde{h} \left(\frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) dx}. \quad (42)$$

Proof. Applying Theorem 7 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, $\check{k}(x, t)$ and $\check{K}(x)$ are given by (32) and (33) respectively, we get (42). \square

Under the assumptions of the Theorem 17, we define a linear functional by taking the positive difference of the inequality stated in (31) as:

$$\Omega_1(\Phi) = \int_a^b \beta(t) \Phi(f(t)) dt - \int_a^b u(x) \Phi \left(\frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt \right) dx. \quad (43)$$

We also define a linear functional by taking the positive difference of the left-hand side and right-hand side of the inequality (35) given in Theorem 18 as:

$$\Omega_2(\Phi) = \int_a^b \gamma(t) \Phi \left(\frac{f_1(x)}{f_2(x)} \right) dt - \int_a^b u(x) \Phi \left(\frac{I_{a,k}^\alpha f_1(x)}{I_{a,k}^\alpha f_2(x)} \right) dx. \quad (44)$$

THEOREM 24. *Let $\Gamma = \{\Phi_p : p \in J\}$ be a family of functions defined on I , such that the function $p \mapsto [z_0, z_1, z_2; \Phi_p]$ is n -exponentially convex in the Jensen sense on J for every three distinct points $z_0, z_1, z_2 \in I$. Let Ω_i ($i = 1, 2$) be linear functionals defined by (43) and (44). Then the function $p \mapsto \Omega_i(\Phi_p)$ ($i = 1, 2$) is n -exponentially convex in the Jensen sense on J , if it is continuous on J .*

Proof. Applying Theorem 8 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, we get the desired outcome. \square

REMARK 6. If we choose $k = 1$, in inequalities (31), (35), (36), (37), (38), (39) and equations (40), (42), we acquire [11, Corollary 3].

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(Received March 31, 2018)

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