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ON THE WEAK SOLUTION $u \in C_{1-\alpha}(I,E)$ OF A FRACTIONAL–ORDER WEIGHTED CAUCHY TYPE PROBLEM IN REFLEXIVE BANACH SPACES

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Abstract. In this paper, we study the existence of a weak solution $u \in C_{1-\alpha}(I,E)$ of the nonlinear weighted Cauchy type problem of fractional-order.

1. Introduction

In this paper, we study the existence of solutions, in the Banach space $C_{1-\alpha}[I,E]$, for the nonlinear weighted Cauchy-type problem of the following type

$$\begin{cases} D^{\alpha}\, u(t) = f(t,u(t)), \;\; t>0, \;\; \alpha \in (0,1) \\ t^{1-\alpha}\, u(t)|_{t=0} = b, \;\; b>0. \end{cases} \tag{1}$$

This problem has been studied by many authors for example in ([4]), the author supposed that the function $f(t,u)$ is continuous on $R^{+}\times R$, $|f(t,u)| \leqslant t^{\mu}e^{-\sigma t}\psi(t)|u|^{m}$, $\mu \geqslant 0$, $m>1$, $\sigma>0$, $\psi(t)$ is a continuous function on R^{+}. Also; In ([2]–[3]) the author proved the existence of L_1 and L_p solution of the same problem respectively.

2. Preliminaries

Let $L_1(I)$ be the space of Lebesgue integrable functions on the interval $I=[0,1]$. Unless otherwise stated, E is a reflexive Banach space with norm $||.||$ and dual E^{*}. We will denote by E_w the space E endowed with the weak topology $\sigma(E,E^{*})$ and denote by $C(I,E)$ the space of continuous functions defined on $I=[0,1]$ with norm

$$||u||_{C} = \sup_{t\in[0,1]} ||u(t)||.$$

Also; define the space $C_{1-\alpha}(I,E)$ by

$$C_{1-\alpha}(I,E) = \left\{u : t^{1-\alpha}u(t) \text{ is continuous on } I=[0,1]\right\},$$

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with norm

$$||u||_{C_{1-\alpha}} = ||t^{1-\alpha}\, u||_C.$$

We recall that the fractional integral operator of order $\alpha > 0$ with left-hand point a is defined by (see [9], [14], [15] and [20])

$$I_a^\alpha\, u(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\, ds.$$

DEFINITIONS. Let E be a Banach space and let $u : I \to E$. Then

(1) $u(.)$ is said to be weakly continuous (measurable) at $t_0 \in I$ if for every $\varphi \in E^*$ we have $\varphi(u(.))$ continuous (measurable) at t_0.

(2) A function $h : E \to E$ is said to be weakly sequentially continuous if h takes weakly convergent sequences in E to weakly convergent sequences in E.

Note that:

(1) If u is weakly continuous on I, then u is strongly measurable (see [7]), hence weakly measurable.

(2) In reflexive Banach spaces weakly measurable functions are Pettis integrable (see [1], [7] and [13] for the definition) if and only if $\varphi(u(.))$ is Lebesgue integrable on I for every $\varphi \in E^*$.

Now, we present some auxiliary results that will be needed in this paper. Firstly, we state O'Regan fixed point theorem ([12]).

THEOREM 2.1. *Let E be a Banach space with Q a nonempty, bounded, closed, convex, equicontinuous subset of $C[I,E]$. Suppose $T : Q \to Q$ is weakly sequentially continuous and assume $TQ(t)$ is weakly relatively compact in E for each $t \in I$, holds. Then the operator T has a fixed point in Q.*

The following theorems can be found in [5], [22] and [10] respectively:

THEOREM 2.2. (Dominated convergence theorem for Pettis integral) *Let $u : I \to E$. Suppose there is a sequence (u_n) of Pettis integrable functions from I into E such that $\lim_{n\to\infty} \varphi(u_n) = \varphi(u)$ a.e. for $\varphi \in E^*$. If there is a scalar function $\psi \in L_1(I)$ with $||u_n(\cdot)|| < \psi(\cdot)$ a.e. for all n, then u is Pettis integrable and*

$$\int_J u_n(s)\, ds \;\to\; \int_J u(s)\, ds \;\; \textit{weakly} \;\; \forall\, t \in I.$$

THEOREM 2.3. *A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

THEOREM 2.4. *Let Q be a weakly compact subset of $C[I,E]$. Then $Q(t)$ is weakly compact subset of E for each $t \in I$.*

Finally, we state some results which is an immediate consequence of the Hahn-Banach theorem.

THEOREM 2.5. *Let E be a normed space with $u_0 \neq 0$. then there exists a $\varphi \in E^*$ with $||\varphi|| = 1$ and $\varphi(u_0) = ||u_0||$.*

THEOREM 2.6. *If $u_0 \in E$ is such that $\varphi(u_0) = 0$ for every $\varphi \in E^*$, then $u_0 = 0$.*

Now consider the fractional-order integral equation

$$u(t) = b\, t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\, ds,\ \ t \in [0,1]. \tag{2}$$

In [12] the author studied the integral equation

$$y(t) = x_0 + \int_0^t f(s,y(s))\, ds,\ \ t \in [0,T],\ \ x_0 \in E$$

where $E = (E,|.|)$ is a real Banach space, under the assumptions that $f(t,.)$ is weakly sequentially continuous for each $t \in [0,T]$ and $f(.,y(.))$ is Pettis integrable on $[0,T]$ for each continuous function $y : [0,T] \to E$ and $|f(t,y)| \leqslant h_r(t)$ for a.e. $t \in [0,T]$ and all $y \in E$ with $|y| \leqslant r, r > 0, h_r \in L_1[0,T]$.
Also, in [11] the author studied the Volterra-Hammerstein integral equation

$$y(t) = h(t) + \int_0^t k(t,s)\, f(s,y(s))\, ds,\ \ t \in [0,T],\ \ T > 0,$$

under the assumptions that $f : [0,T] \times B \to B$ is weakly-weakly continuous and $h : [0,T] \to B$ is weakly continuous, where B is a reflexive Banach space.

Here we study the existence of weak solution of the fractional-order integral equation (2) such that the function $f : I \times B_r \to E$ satisfies the following conditions:

(1) For each $t \in I, f_t = f(t,.)$ is weakly sequentially continuous.

(2) For each $u \in E_r, f(.,u(.))$ is weakly measurable on I.

(3) for any $r > 0$, the weak closure of the range of $f(I \times B_r)$ is weakly compact in E (or equivalently; there exists an M_r such that $||f(t,u)|| \leqslant M_r$ for all $(t,u) \in I \times B_r$).

EXAMPLE 2.1. Let T be the interval $[0,1]$ and define $f : T \to L^\infty(T)$ by $f(t) = \chi_{[0,t]}$. This function is weakly measurable and for each $\phi \in L^*_\infty$, we have $\phi f \in L_1$ (each ϕf is a function of bounded variation). Thus, according to Lemma 3.2 , $I^\alpha f$ exists. Also, the fractional order Pettis integral of f exists see [6, 16, 18].

DEFINITION 2.1. By a weak solution of (2) we mean a function $u \in C_{1-\alpha}[I,E]$ such that for all $\varphi \in E^*$

$$\varphi(u(t)) = b\, t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(f(s,u(s)))\, ds,\ \ t \in [0,1].$$

3. Fractional-order integrals in reflexive Banach spaces

Here, we define the fractional-order integral operator in reflexive Banach spaces. Definition given below is an extension of such a notion for real-valued functions.

DEFINITION 3.1. Let $u: I \to E$ be a weakly measurable function, such that $\varphi(u(.)) \in L_1(I)$, and let $\alpha > 0$. Then the fractional (arbitrary) order Pettis integral (shortly FPI) $I^{\alpha}u(t)$ is defined by

$$I^{\alpha}\, u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, u(s)\, ds.$$

In the above definition the sign "$\int$" denotes the Pettis integral.

LEMMA 3.1. [16] *Let $u: I \to E$ be a weakly measurable function, such that $\varphi(u(.)) \in L_1(I)$, and let $\alpha > 0$. The fractional (arbitrary) order Pettis integral*

$$I^{\alpha}\, u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, u(s)\, ds$$

exists for almost every $t \in I$ as a function from I into E and $\varphi(I^{\alpha}u(t)) = I^{\alpha}\varphi(u(t))$.

LEMMA 3.2. [17] *Let $u: I \to E$ be weakly continuous function on $[0,1]$. Then, FPI of u exists for almost every $t \in [0,1]$ as a weakly continuous function from $[0,1]$ to E. Moreover,*

$$\varphi(I^{\alpha}u(t)) = I^{\alpha}\varphi(u(t)), \text{ for all } \varphi \in E^*.$$

DEFINITION 3.2. [13] Let $u: I \to E$. We define the fractional-Pseudo derivative (shortly FPD) of u of order $\alpha \in (n-1,n)$, $n \in N$ by

$$\frac{d^{\alpha}}{dt^{\alpha}}\, u(t) = D^n I^{n-\alpha} u(t).$$

In the above definition the sign "D" denotes the Pseudo differential operator.

LEMMA 3.3. [21] *Let $u: [0,1] \to E$ be weakly continuous function on $[0,1]$ such that the real-valued function $I^{n-\alpha}\varphi u$ is n-times differentiable. Then, the FPD of u of order $\alpha \in (n-1,n)$ exists.*

DEFINITION 3.3. A function $u: I \to E$ is called Pseudo solution of (1) if $u \in C_{1-\alpha}[I,E]$ has FPD of order $\alpha \in (0,1)$, $t^{1-\alpha}u(t)|_{t=0} = b$, $b > 0$ and satisfies

$$\frac{d}{dt}\varphi(I^{1-\alpha}\, u(t)) = \varphi(f(t,u(t))), \text{ a.e. on } [0,1], \text{ for each } \varphi \in E^*.$$

Now, for the properties of the integrals of fractional-orders in reflexive spaces we have the following lemma (see [16]):

LEMMA 3.4. *Let $u: I \to E$ be weakly measurable and $\varphi(u(.)) \in L_1(I)$. If α, $\beta \in (0,1)$, we have:*

(1) $I^\alpha I^\beta u(t) = I^{\alpha+\beta} u(t)$ *for a.e.* $t \in I$.

(2) $\lim_{\alpha\to 1} I^\alpha u(t) = I^1 u(t)$ *weakly uniformly on* I *if only these integrals exist on* I.

(3) $\lim_{\alpha\to 0} I^\alpha u(t) = u(t)$ *weakly in* E *for a.e.* $t \in I$.

(4) If, for a fixed $t \in I$, $\varphi(u(t))$ *is bounded for each* $\varphi \in E^*$, *then* $lim_{t\to 0} I^\alpha u(t) = 0$.

4. Main result

In this section we present our main result by proving the existence of solution of equation (2) in $C_{1-\alpha}[I,E]$.

Let E be a reflexive Banach space. And let

$$E_r = \left\{ u \in C_{1-\alpha}[I,E] : ||u||_{C_{1-\alpha}} < b + \frac{M_r}{\Gamma(1+\alpha)} \right\}.$$

We will consider the set

$$B_r = \{u(t) \in E : \ u \in E_r, \ t \in I\}.$$

Now, we are in a position to formulate and prove our main result.

THEOREM 4.1. *Let the assumptions* (1)–(3) *are satisfied, then equation* (2) *has at least one weak solution* $u \in C_{1-\alpha}[I,E]$.

Proof. Let us define the operator T as

$$Tu(t) = b\, t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\, ds, \ \ t \in [0,1].$$

We will solve equation (2) by finding a fixed point of the operator T.

We will prove that

$$T: C_{1-\alpha}[I,E] \to C_{1-\alpha}[I,E].$$

First note that from assumption (2), we get that for each $u \in C_{1-\alpha}[I,E]$, $f(.,u(.))$ is weakly measurable on I. Since f has weakly compact range, then $\varphi(f(.,u(.)))$ is Lebesgue integrable on I for every $\varphi \in E^*$ and thus the operator T is well defined. Now, we show that if $u \in C_{1-\alpha}[I,E]$, then $Tu \in C_{1-\alpha}[I,E]$. Note that there exists $r > 0$ with $||u||_{C_{1-\alpha}} = \sup_{t\in I} ||t^{1-\alpha} u(t)|| < b + \frac{M_r}{\Gamma(1+\alpha)}$.

Now assumption (3) implies that

$$||f(t,u(t))|| \leqslant M_r \ \text{ for } \ t \in [0,1].$$

Let $t, \tau \in [0,1]$ with $t > \tau$. Without loss of generality, assume $t^{1-\alpha} Tu(t) - \tau^{1-\alpha} Tu(\tau) \neq 0$. Then there exists (a consequence of Theorem 2.5) $\varphi \in E^*$ with $||\varphi|| = 1$ and

$$||t^{1-\alpha} Tu(t) - \tau^{1-\alpha} Tu(\tau)|| = \varphi(t^{1-\alpha} Tu(t) - \tau^{1-\alpha} Tu(\tau)).$$

Thus

$$\begin{aligned}
||t^{1-\alpha}Tu(t)-\tau^{1-\alpha}Tu(\tau)|| &\leqslant \left| t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(f(s,u(s)))\,ds \right.\\
&\quad \left. -\tau^{1-\alpha}\int_0^\tau \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(f(s,u(s)))\,ds \right|\\
&\leqslant \left| \int_0^\tau \frac{t^{1-\alpha}(t-s)^{\alpha-1}-\tau^{1-\alpha}(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(f(s,u(s)))\,ds \right|\\
&\quad + \left| \int_\tau^t \frac{t^{1-\alpha}(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(f(s,u(s)))\,ds \right|\\
&\leqslant \frac{M_r}{\Gamma(\alpha)}\left(\int_0^\tau |t^{1-\alpha}(t-s)^{\alpha-1}-\tau^{1-\alpha}(\tau-s)^{\alpha-1}|\,ds \right.\\
&\quad \left. + \int_\tau^t |t^{1-\alpha}(t-s)^{\alpha-1}|\,ds \right)\\
&\leqslant \frac{M_r}{\Gamma(1+\alpha)}\Big(2(t-\tau)^\alpha + |t-\tau| \Big). \qquad (3)
\end{aligned}$$

which proves that $Tu \in C_{1-\alpha}[I,E]$.

Now, let

$$Q=\left\{ u\in E_r : (\forall t,\tau \in I)||t^{1-\alpha}u(t)-\tau^{1-\alpha}u(\tau)|| \leqslant \frac{M_r}{\Gamma(1+\alpha)}\Big(2(t-\tau)^\alpha+|t-\tau|\Big)\right\},$$

Note that Q is nonempty, closed, bounded, convex and equicontinuous subset of $C_{1-\alpha}[I,E]$. Now, we claim that $T:Q\to Q$ and is weakly sequentially continuous. If this is true then according to Theorem 2.3, TQ is bounded in $C_{1-\alpha}[I,E]$ (hence, Theorem 2.4, implies $TQ(t)$ is weakly relatively compact in E for each $t\in I$) and the result follows immediately from Theorem 2.1. It remains to prove our claim. First we show that T maps Q into Q. To see this, note that the inequality (3) shows that TQ is norm continuous. Now, take $u\in Q$; without loss of generality, we may assume that $t^{1-\alpha}I^\alpha f(t,u(t)) \neq 0$, then, by Theorem 2.5, there exists $\varphi \in E^*$ with $||\varphi||=1$ and $||t^{1-\alpha}I^\alpha f(t,u(t))|| = \varphi(t^{1-\alpha}I^\alpha f(t,u(t)))$. Thus

$$||t^{1-\alpha}Tu(t)|| \leqslant b+\frac{M_r}{\Gamma(1+\alpha)}, \qquad (4)$$

therefore

$$||Tu||_{C_{1-\alpha}} < b+\frac{M_r}{\Gamma(1+\alpha)}.$$

Thus $T:Q\to Q$. Finally, we will show that T is weakly sequentially continuous. To see this, let $\{u_n\}_{n=1}^\infty$ be a sequence in Q and let $u_n(t)\to u(t)$ in E_w for each $t\in[0,1]$. Recall [10] that a sequence $\{u_n\}_{n=1}^\infty$ is weakly convergent in $C[I,E]$ if and only if it is weakly pointwise convergent in E. Fix $t\in I$. From the weak sequential continuity of $f(t,.)$, the Lebsegue dominated convergence theorem (see assumption (3)) for the

Pettis integral [5] implies for each $\varphi \in E^*$ that $\varphi(Tu_n(t)) \to \varphi(Tu(t))$ a.e. on I, $Tu_n(t) \to Tu(t)$ in E_w. So $T : Q \to Q$ is weakly sequentially continuous. The proof is complete. □

Now, we are looking for sufficient conditions to ensure the existence of Pseudo solution to the nonlinear weighted Cauchy-type problem (1).

Note that, the following theorem is a generalization of the results of §3.3 in [8]:

THEOREM 4.2. *If $f : I \times B_r \to E$ satisfies the assumptions of Theorem 4.1, then the nonlinear weighted Cauchy-type problem (1) has a fractional-Pseudo derivative (FPD) $u \in C_{1-\alpha}[I,E]$.*

Proof. Let us remark, that by assumptions (2), (3) the FPI of f of order $\alpha > 0$ exists and

$$\varphi(I^\alpha f(t,u(t))) = I^\alpha \varphi(f(t,u(t))), \text{ for all } \varphi \in E^*.$$

Let u be a solution of equation (2), then

$$u(t) = bt^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\, ds, \ t \in [0,1].$$

It is clear that

$$t^{1-\alpha} u(t)|_{t=0} = b.$$

Furthermore, we have

$$u(t) = b\, t^{\alpha-1} + I^\alpha\, f(t,u(t)) \tag{5}$$

since $u \in C_{1-\alpha}[I,E]$, then $\varphi(I^{1-\alpha}u(t)) = I^{1-\alpha}\varphi(u(t))$, for all $\varphi \in E^*$ (see Lemma 3.2). From equation (5), we deduce that

$$\varphi(u(t)) = bt^{\alpha-1} + \varphi(I^\alpha\, f(t,u(t))). \tag{6}$$

Operating by $I^{1-\alpha}$ on both sides of the equation (6) and using the properties of fractional calculus in the space $L_1[0,1]$ (see [19] and [20]) result in

$$I^{1-\alpha}\varphi(u(t)) = b_1 + I\varphi(f(t,u(t))).$$

Therefore,

$$\varphi(I^{1-\alpha}u(t)) = b_1 + I\varphi(f(t,u(t))).$$

Thus

$$\frac{d}{dt}\varphi(I^{1-\alpha}u(t)) = \varphi(f(t,u(t))) \text{ a.e. on } [0,1].$$

That is u has the FPD of order $\alpha \in (0,1)$ and u is a solution of the differential equation (1). Conversely, let $u(t)$ be a solution of (1), integrate both sides, then

$$I^{1-\alpha}\varphi(u(t)) - I^{1-\alpha}\varphi(u(t))|_{t=0} = I\varphi(f(t,u(t))),$$

operating by I^{α} on both sides of the last equation, then

$$I\varphi(u(t)) - I^{\alpha}C = I^{1+\alpha}\ \varphi(f(t,u(t))),$$

differentiate both sides, then

$$\varphi(u(t)) - C_1 t^{\alpha-1} = I^{\alpha}\varphi(f(t,u(t))),$$

from the initial condition, we find that $C_1 = b$, then we obtain (2), i.e. Problem (1) and equation (2) are equivalent to each other. □

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