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SOME NEW HERMITE–HADAMARD TYPE INEQUALITIES VIA k–FRACTIONAL INTEGRALS CONCERNING DIFFERENTIABLE GENERALIZED–m–$((h_1^p, h_2^q);(\eta_1,\eta_2))$–CONVEX MAPPINGS

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Abstract. The authors discovered a new identity concerning differentiable mappings defined on $\mathbf{m}$-invex set via k-fractional integrals. By using the obtained identity as an auxiliary result, some new estimates with respect to Hermite-Hadamard type inequalities via k-fractional integrals for generalized-$\mathbf{m}$-$((h_1^p, h_2^q);(\eta_1,\eta_2))$-convex mappings are presented. It is pointed out that some new special cases can be deduced from main results. At the end, some applications to special means for different positive real numbers are provided as well.

1. Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$. For any subset $K \subseteq \mathbb{R}^n$, K° is the interior of K. The set of integrable functions on the interval $[a,b]$ is denoted by $L[a,b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

THEOREM 1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a,b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

This inequality (1) is also known as trapezium inequality.

The trapezium type inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. For other recent results which generalize, improve and extend the inequality (1) through various classes of convex functions interested readers are referred to [1],[3]-[20],[22],[23],[25]-[28],[30],[31],[34],[37],[38].

Let us recall some special functions and evoke some basic definitions as follows.

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DEFINITION 1. The Euler beta function is defined for $a,b>0$ as

$$\beta(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt. \tag{2}$$

DEFINITION 2. [22] Let $f\in L[a,b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha}f$ and $J_{b^-}^{\alpha}f$ of order $\alpha>0$ with $a\geqslant 0$ are defined by

$$J_{a^+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\quad x>a$$

and

$$J_{b^-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\quad b>x,$$

where $\Gamma(\alpha)=\int_0^{+\infty}e^{-u}u^{\alpha-1}du$. Here $J_{a^+}^{0}f(x)=J_{b^-}^{0}f(x)=f(x)$.
Note that $\alpha=1$, the fractional integral reduces to the classical integral.

DEFINITION 3. For $k\in\mathbb{R}^+$ and $x\in\mathbb{C}$, the k-gamma function is defined by

$$\Gamma_k(x)=\lim_{n\longrightarrow\infty}\frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}. \tag{3}$$

Its integral representation is given by

$$\Gamma_k(\alpha)=\int_0^{\infty}t^{\alpha-1}e^{-\frac{t^k}{k}}dt. \tag{4}$$

One can note that

$$\Gamma_k(\alpha+k)=\alpha\Gamma_k(\alpha). \tag{5}$$

For $k=1$, (4) gives integral representation of gamma function.

DEFINITION 4. [25] Let $f\in L[a,b]$. Then k-fractional integrals of order $\alpha,k>0$ with $a\geqslant 0$ are defined as

$$I_{a^+}^{\alpha,k}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1}f(t)dt,\quad x>a$$

and

$$I_{b^-}^{\alpha,k}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1}f(t)dt,\quad b>x. \tag{6}$$

For $k=1$, k-fractional integrals give Riemann-Liouville integrals.

DEFINITION 5. [36] A set $S\subseteq\mathbb{R}^n$ is said to be invex set with respect to the mapping $\eta:S\times S\longrightarrow\mathbb{R}^n$, if $x+t\eta(y,x)\in S$ for every $x,y\in S$ and $t\in[0,1]$.

The invex set S is also termed an η-connected set.

DEFINITION 6. [24] Let $h:[0,1]\longrightarrow\mathbb{R}$ be a non-negative function and $h\neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η, if

$$f\big(x+t\eta(y,x)\big)\leqslant h(1-t)f(x)+h(t)f(y) \tag{7}$$

for each $x,y\in K$ and $t\in[0,1]$ where $f(\cdot)>0$.

Clearly, when putting $h(t)=t$ in Definition 6, f becomes a preinvex function, see [29]. If the mapping $\eta(y,x)=y-x$ in Definition 6, then the non-negative function f reduces to h-convex mappings, see [33].

DEFINITION 7. [35] Let $S\subseteq\mathbb{R}^n$ be an invex set with respect to $\eta:S\times S\longrightarrow\mathbb{R}^n$. A function $f:S\longrightarrow[0,+\infty)$ is said to be s-preinvex (or s-Breckner-preinvex) with respect to η and $s\in(0,1]$, if for every $x,y\in S$ and $t\in[0,1]$,

$$f\big(x+t\eta(y,x)\big)\leqslant(1-t)^s f(x)+t^s f(y). \tag{8}$$

DEFINITION 8. [26] A function $f:K\longrightarrow\mathbb{R}$ is said to be s-Godunova-Levin--Dragomir-preinvex of second kind, if

$$f\big(x+t\eta(y,x)\big)\leqslant(1-t)^{-s}f(x)+t^{-s}f(y), \tag{9}$$

for each $x,y\in K, t\in(0,1)$ and $s\in(0,1]$.

DEFINITION 9. [32] A non-negative function $f:K\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ is said to be tgs--convex on K if the inequality

$$f\big((1-t)x+ty\big)\leqslant t(1-t)[f(x)+f(y)] \tag{10}$$

grips for all $x,y\in K$ and $t\in(0,1)$.

DEFINITION 10. [21] A function $f:I\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ is said to MT-convex, if it is non-negative and $\forall x,y\in I$ and $t\in(0,1)$ satisfies the subsequent inequality

$$f(tx+(1-t)y)\leqslant\frac{\sqrt{t}}{2\sqrt{1-t}}f(x)+\frac{\sqrt{1-t}}{2\sqrt{t}}f(y). \tag{11}$$

DEFINITION 11. [28] Let $K\subseteq\mathbb{R}$ be an open m-invex set respecting $\eta:K\times K\longrightarrow\mathbb{R}$ and $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$. A function $f:K\longrightarrow\mathbb{R}$ is said to be generalized (m,h_1,h_2)-preinvex, if

$$f\big(mx+t\eta(y,mx)\big)\leqslant mh_1(t)f(x)+h_2(t)f(y) \tag{12}$$

is valid for all $x,y\in K$ and $t\in[0,1]$, for some fixed $m\in(0,1]$.

The concept of η-convex functions (at the beginning was named by θ-convex functions), considered in [13], has been introduced as the following.

DEFINITION 12. Consider a convex set $I \subseteq \mathbb{R}$ and a bifunction $\eta : f(I) \times f(I) \longrightarrow \mathbb{R}$. A function $f : I \longrightarrow \mathbb{R}$ is called convex with respect to η (briefly η-convex), if

$$f\big(\lambda x + (1-\lambda)y\big) \leqslant f(y) + \lambda \eta(f(x), f(y)), \tag{13}$$

is valid for all $x, y \in I$ and $\lambda \in [0,1]$.

Geometrically it says that if a function is η-convex on I, then for any $x, y \in I$, its graph is on or under the path starting from $(y, f(y))$ and ending at $(x, f(y) + \eta(f(x), f(y)))$. If $f(x)$ should be the end point of the path for every $x, y \in I$, then we have $\eta(x,y) = x - y$ and the function reduces to a convex one. For more results about η--convex functions, see [8],[9],[12],[13].

DEFINITION 13. [1] Let $I \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : I \times I \longrightarrow \mathbb{R}$. Consider $f : I \longrightarrow \mathbb{R}$ and $\eta_2 : f(I) \times f(I) \longrightarrow \mathbb{R}$. The function f is said to be (η_1, η_2)--convex if

$$f\big(x + \lambda \eta_1(y,x)\big) \leqslant f(x) + \lambda \eta_2(f(y), f(x)), \tag{14}$$

is valid for all $x, y \in I$ and $\lambda \in [0,1]$.

Motivated by the above literatures, the main objective of this paper is to establish in Sect. 2, some new estimates on Hermite-Hadamard type inequalities via k-fractional integrals associated with generalized-$\mathbf{m}$-$((h_1^p, h_2^q); (\eta_1, \eta_2))$-convex mappings. It is pointed out that some new special cases will be deduced from main results. In Sect. 3, some applications to special means for different positive real numbers will be obtain. In Sect. 4, some conclusion and future research are given.

2. Main results

The following definitions will be used in this section.

DEFINITION 14. Let $\mathbf{m} : [0,1] \longrightarrow (0,1]$ be a function. A set $K \subseteq \mathbb{R}^n$ is named as $\mathbf{m}$-invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $\mathbf{m}(t)x + \xi \eta(y, \mathbf{m}(t)x) \in K$ holds for each $x, y \in K$ and any $t, \xi \in [0,1]$.

REMARK 1. In Definition 14, under certain conditions, the mapping $\eta(y, \mathbf{m}(t)x)$ for any $t, \xi \in [0,1]$ could reduce to $\eta(y, mx)$. For example when $\mathbf{m}(t) = m$ for all $t \in [0,1]$, then the $\mathbf{m}$-invex set degenerates an m-invex set on K.

We next introduce the concept of generalized-$\mathbf{m}$-$((h_1^p, h_2^q); (\eta_1, \eta_2))$-convex mappings.

DEFINITION 15. Let $K \subseteq \mathbb{R}$ be an open $\mathbf{m}$-invex set with respect to the mapping $\eta_1 : K \times K \longrightarrow \mathbb{R}$ and $\mathbf{m} : [0,1] \longrightarrow (0,1]$. Suppose $h_1, h_2 : [0,1] \longrightarrow [0, +\infty)$ and $\theta : I \longrightarrow K$ are continuous. Consider $f : K \longrightarrow (0, +\infty)$ and $\eta_2 : f(K) \times f(K) \longrightarrow \mathbb{R}$. The mapping f is said to be generalized-$\mathbf{m}$-$((h_1^p, h_2^q); (\eta_1, \eta_2))$-convex if

$$f\big(\mathbf{m}(t)\theta(x) + \xi \eta_1(\theta(y), \mathbf{m}(t)\theta(x))\big) \leqslant \big[\mathbf{m}(\xi) h_1^p(\xi) f^r(x) + h_2^q(\xi) \eta_2(f^r(y), f^r(x))\big]^{\frac{1}{r}}, \tag{15}$$

holds for all $x,y \in I$, $r \neq 0$, $t,\xi \in [0,1]$ and any fixed $p,q > -1$.

REMARK 2. In Definition 15, if we choose $\mathbf{m} = p = q = r = 1$ and $\theta(x) = x$, then we get Definition 13.

REMARK 3. In Definition 15, if we choose $\mathbf{m} = p = q = r = 1$, $h_1(t) = 1$, $h_2(t) = t$, $\eta_1(\theta(y),\mathbf{m}(t)\theta(x)) = \theta(y) - \mathbf{m}(t)\theta(x)$, $\eta_2(f^r(y), f^r(x)) = \eta(f^r(y), f^r(x))$ and $\theta(x) = x$, $\forall x \in I$, then we get Definition 12. Also, in Definition 15, if we choose $\mathbf{m} = p = q = r = 1$, $h_1(t) = 1$, $h_2(t) = t$ and $\theta(x) = x$, $\forall x \in I$, then we get Definition 13. Under some suitable choices as we done above, we can get also the Definitions 7 and 8.

REMARK 4. Let us discuss some special cases in Definition 15 as follows.

(i) Taking $h_1(t) = h(1-t)$ and $h_2(t) = h(t)$, then we get generalized-$\mathbf{m}$-$((h^p(1-t), h^q(t)); (\eta_1,\eta_2))$-convex mappings.

(ii) Taking $h_1(t) = (1-t)^s$ and $h_2(t) = t^s$ for $s \in (0,1]$, then we get generalized-$\mathbf{m}$-$(((1-t)^{sp}, t^{sq}); (\eta_1,\eta_2))$-Breckner-convex mappings.

(iii) Taking $h_1(t) = (1-t)^{-s}$ and $h_2(t) = t^{-s}$ for $s \in (0,1]$, then we get generalized-$\mathbf{m}$-$(((1-t)^{-sp}, t^{-sq}); (\eta_1,\eta_2))$-Godunova-Levin-Dragomir-convex mappings.

(iv) Taking $h_1(t) = h_2(t) = t(1-t)$, then we get generalized-$\mathbf{m}$-$((t(1-t))^{sp}, (t(1-t))^{sq}); (\eta_1,\eta_2))$-convex mappings.

(v) Taking $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$ and $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized-$\mathbf{m}$-$\left(\left(\left(\frac{\sqrt{1-t}}{2\sqrt{t}}\right)^p, \left(\frac{\sqrt{t}}{2\sqrt{1-t}}\right)^q\right); (\eta_1,\eta_2)\right)$-convex mappings.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

Let see the following example of a generalized-$\mathbf{m}$-$((h_1^p, h_2^q); (\eta_1,\eta_2))$-convex mapping which is not convex.

EXAMPLE 1. Let take $\mathbf{m} = r = \frac{1}{2}$, $h_1(t) = t^l$, $h_2(t) = t^s$ for all $l,s \in [0,1]$, any fixed $p,q \geqslant 1$ and θ an identity function. Consider the function $f : [0,+\infty) \longrightarrow [0,+\infty)$ by

$$f(x) = \begin{cases} x, & 0 \leqslant x \leqslant 1; \\ 2, & x > 1. \end{cases}$$

Define two bifunctions $\eta_1 : [0,+\infty) \times [0,+\infty) \longrightarrow \mathbb{R}$ and $\eta_2 : [0,+\infty) \times [0,+\infty) \longrightarrow [0,+\infty)$ by

$$\eta_1(x,y) = \begin{cases} -y, & 0 \leqslant y \leqslant 1; \\ x+y, & y > 1, \end{cases}$$

and

$$\eta_2(x,y)=\begin{cases}x+y, & x\leqslant y;\\ 2(x+y), & x>y.\end{cases}$$

Then f is generalized $\frac{1}{2}$-$((t^{lp},t^{sq});(\eta_1,\eta_2))$-convex mapping. But f is not preinvex with respect to η_1 and also it is not convex (consider $x=0, y=2$ and $t\in(0,1]$).

For establishing our main results we need to prove the following lemma.

LEMMA 1. *Let $\theta: I\longrightarrow K$ be a continuous function and $\boldsymbol{m}:[0,1]\longrightarrow(0,1]$. Suppose $K=[\boldsymbol{m}(t)\theta(a),\boldsymbol{m}(t)\theta(a)+\Psi(\theta(b),\boldsymbol{m}(t)\theta(a))]\subseteq\mathbb{R}$ be an open $\boldsymbol{m}$-invex subset with respect to $\Psi: K\times K\longrightarrow\mathbb{R}$ for $\Psi(\theta(b),\boldsymbol{m}(t)\theta(a))>0$ and $\forall t\in[0,1]$. Assume that $f: K\longrightarrow\mathbb{R}$ be a differentiable mapping on K° such that $f'\in L(K)$. Then for $\alpha,k>0$ and $\lambda\in[0,1]$, the following equality for k-fractional integrals holds:*

$$\begin{aligned}&\frac{\left(1-\frac{\alpha}{k}(1-\lambda)\right)f(\boldsymbol{m}(t)\theta(a))+\left(1+\frac{\alpha}{k}(1-\lambda)\right)f(\boldsymbol{m}(t)\theta(a)+\Psi(\theta(b),\boldsymbol{m}(t)\theta(a)))}{2}\\&-\frac{\Gamma_k(\alpha+k)}{2\Psi^{\frac{\alpha}{k}}(\theta(b),\boldsymbol{m}(t)\theta(a))}\times\Big[I^{\alpha,k}_{(\boldsymbol{m}(t)\theta(a))^+}f(\boldsymbol{m}(t)\theta(a)+\Psi(\theta(b),\boldsymbol{m}(t)\theta(a)))\\&+I^{\alpha,k}_{(\boldsymbol{m}(t)\theta(a)+\Psi(\theta(b),\boldsymbol{m}(t)\theta(a)))^-}f(\boldsymbol{m}(t)\theta(a))\Big]\\&=\frac{\Psi(\theta(b),\boldsymbol{m}(t)\theta(a))}{2}\\&\times\int_0^1\left(\xi^{\frac{\alpha}{k}}+\frac{\alpha}{k}(1-\lambda)-(1-\xi)^{\frac{\alpha}{k}}\right)f'(\boldsymbol{m}(t)\theta(a)+\xi\Psi(\theta(b),\boldsymbol{m}(t)\theta(a)))d\xi.\end{aligned}\tag{16}$$

We denote

$$\begin{aligned}&T_f^{\alpha,k}(\Psi,\theta,\boldsymbol{m};\lambda,a,b):=\frac{\Psi(\theta(b),\boldsymbol{m}(t)\theta(a))}{2}\\&\times\int_0^1\left(\xi^{\frac{\alpha}{k}}+\frac{\alpha}{k}(1-\lambda)-(1-\xi)^{\frac{\alpha}{k}}\right)f'(\boldsymbol{m}(t)\theta(a)+\xi\Psi(\theta(b),\boldsymbol{m}(t)\theta(a)))d\xi.\end{aligned}\tag{17}$$

Proof. Integrating by parts, we get

$$\begin{aligned}&T_f^{\alpha,k}(\Psi,\theta,\mathbf{m};\lambda,a,b)\\&=\frac{\Psi(\theta(b),\mathbf{m}(t)\theta(a))}{2}\times\left[\int_0^1\xi^{\frac{\alpha}{k}}f'(\mathbf{m}(t)\theta(a)+\xi\Psi(\theta(b),\mathbf{m}(t)\theta(a)))d\xi\right.\\&+\frac{\alpha}{k}(1-\lambda)\int_0^1 f'(\mathbf{m}(t)\theta(a)+\xi\Psi(\theta(b),\mathbf{m}(t)\theta(a)))d\xi\\&\left.-\int_0^1(1-\xi)^{\frac{\alpha}{k}}f'(\mathbf{m}(t)\theta(a)+\xi\Psi(\theta(b),\mathbf{m}(t)\theta(a)))d\xi\right]\\&=\frac{\Psi(\theta(b),\mathbf{m}(t)\theta(a))}{2}\times\left[\frac{\xi^{\frac{\alpha}{k}}f(\mathbf{m}(t)\theta(a)+\xi\Psi(\theta(b),\mathbf{m}(t)\theta(a)))}{\Psi(\theta(b),\mathbf{m}(t)\theta(a))}\right|_0^1\end{aligned}$$

$$
\begin{aligned}
&-\frac{\alpha}{k\Psi(\theta(b),\mathbf{m}(t)\theta(a))}\times\int_0^1 \xi^{\frac{\alpha}{k}-1} f(\mathbf{m}(t)\theta(a)+\xi\Psi(\theta(b),\mathbf{m}(t)\theta(a)))d\xi\\
&+\frac{\alpha}{k}(1-\lambda)\Big[f(\mathbf{m}(t)\theta(a)+\xi\Psi(\theta(b),\mathbf{m}(t)\theta(a)))-f(\mathbf{m}(t)\theta(a))\Big]\\
&-\frac{(1-\xi)^{\frac{\alpha}{k}} f(\mathbf{m}(t)\theta(a)+\xi\Psi(\theta(b),\mathbf{m}(t)\theta(a)))}{\Psi(\theta(b),\mathbf{m}(t)\theta(a))}\Bigg|_0^1\\
&+\frac{\alpha}{k\Psi(\theta(b),\mathbf{m}(t)\theta(a))}\times\int_0^1 (1-\xi)^{\frac{\alpha}{k}-1} f(\mathbf{m}(t)\theta(a)+\xi\Psi(\theta(b),\mathbf{m}(t)\theta(a)))d\xi\Bigg]\\
=&\frac{\left(1-\frac{\alpha}{k}(1-\lambda)\right)f(\mathbf{m}(t)\theta(a))+\left(1+\frac{\alpha}{k}(1-\lambda)\right)f(\mathbf{m}(t)\theta(a)+\Psi(\theta(b),\mathbf{m}(t)\theta(a)))}{2}\\
&-\frac{\Gamma_k(\alpha+k)}{2\Psi^{\frac{\alpha}{k}}(\theta(b),\mathbf{m}(t)\theta(a))}\times\Big[I^{\alpha,k}_{(\mathbf{m}(t)\theta(a))^+} f(\mathbf{m}(t)\theta(a)+\Psi(\theta(b),\mathbf{m}(t)\theta(a)))\\
&+I^{\alpha,k}_{(\mathbf{m}(t)\theta(a)+\Psi(\theta(b),\mathbf{m}(t)\theta(a)))^-} f(\mathbf{m}(t)\theta(a))\Big].
\end{aligned}
$$

This completes the proof of our lemma. □

REMARK 5. For $\Psi(\theta(b),\mathbf{m}(t)\theta(a))=\theta(b)-\mathbf{m}(t)\theta(a)$, where $\mathbf{m}(t)\equiv 1$ for all $t\in[0,1]$ and $\lambda=1$, we get the following Hermite-Hadamard integral identity

$$
\begin{aligned}
&\frac{f(\theta(a))+f(\theta(b))}{2}-\frac{\Gamma_k(\alpha+k)}{2(\theta(b)-\theta(a))^{\frac{\alpha}{k}}}\times\Big[I^{\alpha,k}_{(\theta(a))^+} f(\theta(b))+I^{\alpha,k}_{(\theta(b))^-} f(\theta(a))\Big]\\
=&\frac{(\theta(b)-\theta(a))}{2}\times\int_0^1\left(\xi^{\frac{\alpha}{k}}-(1-\xi)^{\frac{\alpha}{k}}\right)f'(\theta(a)+\xi(\theta(b)-\theta(a)))d\xi. \qquad (18)
\end{aligned}
$$

Using Lemma 1, we now state the following theorems for the corresponding version for power of first derivative.

THEOREM 2. *Let $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$ and $\theta:I\longrightarrow K$ are continuous functions and $\boldsymbol{m}:[0,1]\longrightarrow(0,1]$. Suppose $K=[\boldsymbol{m}(t)\theta(a),\boldsymbol{m}(t)\theta(a)+\Psi_1(\theta(b),\boldsymbol{m}(t)\theta(a))]\subseteq\mathbb{R}$ be an open $\boldsymbol{m}$-invex subset with respect to $\Psi_1:K\times K\longrightarrow\mathbb{R}$ for $\Psi_1(\theta(b),\boldsymbol{m}(t)\theta(a))>0,\forall t\in[0,1]$ and $\Psi_2:f(K)\times f(K)\longrightarrow\mathbb{R}$. Assume that $f:K\longrightarrow(0,+\infty)$ be a differentiable mapping on K° such that $f'\in L(K)$. If $(f'(x))^q$ is positive generalized-$\boldsymbol{m}$-$((h_1^{p_1},h_2^{p_2});(\Psi_1,\Psi_2))$-convex mapping, $0<r\leqslant 1$, $p_1,p_2>-1$, $q>1$, $p^{-1}+q^{-1}=1$, then the following inequality for $\alpha,k>0$ and $\lambda\in[0,1]$, holds:*

$$
\begin{aligned}
\left|T^{\alpha,k}_f(\Psi_1,\theta,\boldsymbol{m};\lambda,a,b)\right|\leqslant&\frac{\Psi_1(\theta(b),\boldsymbol{m}(t)\theta(a))}{2}\times\left[\frac{\alpha}{k}(1-\lambda)+2\sqrt[p]{\frac{k}{p\alpha+k}}\right] \qquad (19)\\
&\times\sqrt[rq]{(f'(a))^{rq}I^r(h_1(\xi))+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)I^r(h_2(\xi))},
\end{aligned}
$$

where

$$
I(h_1(\xi)):=\int_0^1 \boldsymbol{m}^{\frac{1}{r}}(\xi)h_1^{\frac{p_1}{r}}(\xi)d\xi,\quad I(h_2(\xi)):=\int_0^1 h_2^{\frac{p_2}{r}}(\xi)d\xi.
$$

Proof. From Lemma 1, positive generalized-$\mathbf{m}$-$((h_1^{p_1},h_2^{p_2});(\Psi_1,\Psi_2))$-convexity of $(f'(x))^q$, Hölder inequality, Minkowski inequality, properties of the modulus and changing the variable $u=\mathbf{m}(t)\theta(a)+\xi\Psi_1(\theta(b),\mathbf{m}(t)\theta(a))$, $\forall t\in[0,1]$, we have

$$
\begin{aligned}
&\left|T_f^{\alpha,k}(\Psi_1,\theta,\mathbf{m};\lambda,a,b)\right|\\
\leqslant&\frac{|\Psi_1(\theta(b),\mathbf{m}(t)\theta(a))|}{2}\\
&\times\int_0^1\left|\xi^{\frac{\alpha}{k}}+\frac{\alpha}{k}(1-\lambda)-(1-\xi)^{\frac{\alpha}{k}}\right|\left|f'(\mathbf{m}(t)\theta(a)+\xi\Psi_1(\theta(b),\mathbf{m}(t)\theta(a)))\right|d\xi\\
\leqslant&\frac{\Psi_1(\theta(b),\mathbf{m}(t)\theta(a))}{2}\\
&\times\left[\int_0^1\left(\xi^{\frac{\alpha}{k}}+\frac{\alpha}{k}(1-\lambda)\right)f'(\mathbf{m}(t)\theta(a)+\xi\Psi_1(\theta(b),\mathbf{m}(t)\theta(a)))d\xi\right.\\
&\left.+\int_0^1(1-\xi)^{\frac{\alpha}{k}}f'(\mathbf{m}(t)\theta(a)+\xi\Psi_1(\theta(b),\mathbf{m}(t)\theta(a)))d\xi\right]\\
\leqslant&\frac{\Psi_1(\theta(b),\mathbf{m}(t)\theta(a))}{2}\\
&\times\left[\left(\int_0^1\left(\xi^{\frac{\alpha}{k}}+\frac{\alpha}{k}(1-\lambda)\right)^p d\xi\right)^{\frac{1}{p}}\right.\\
&\left(\int_0^1\left(f'(\mathbf{m}(t)\theta(a)+\xi\Psi_1(\theta(b),\mathbf{m}(t)\theta(a)))\right)^q d\xi\right)^{\frac{1}{q}}\\
&\left.+\left(\int_0^1(1-\xi)^{\frac{p\alpha}{k}}d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left(f'(\mathbf{m}(t)\theta(a)+\xi\Psi_1(\theta(b),\mathbf{m}(t)\theta(a)))\right)^q d\xi\right)^{\frac{1}{q}}\right]\\
\leqslant&\frac{\Psi_1(\theta(b),\mathbf{m}(t)\theta(a))}{2}\\
&\times\left[\left(\int_0^1\xi^{\frac{p\alpha}{k}}d\xi\right)^{\frac{1}{p}}+\left(\int_0^1\left(\frac{\alpha}{k}(1-\lambda)\right)^p d\xi\right)^{\frac{1}{p}}+\left(\int_0^1(1-\xi)^{\frac{p\alpha}{k}}d\xi\right)^{\frac{1}{p}}\right]\\
&\times\left(\int_0^1\left[\mathbf{m}(\xi)h_1^{p_1}(\xi)(f'(a))^{rq}+h_2^{p_2}(\xi)\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)\right]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}\\
\leqslant&\frac{\Psi_1(\theta(b),\mathbf{m}(t)\theta(a))}{2}\times\left[\frac{\alpha}{k}(1-\lambda)+2\sqrt[p]{\frac{k}{p\alpha+k}}\right]\\
&\times\left[\left(\int_0^1\mathbf{m}^{\frac{1}{r}}(\xi)(f'(a))^q h_1^{\frac{p_1}{r}}(\xi)d\xi\right)^r+\left(\int_0^1\Psi_2^{\frac{1}{r}}\left((f'(b))^{rq},(f'(a))^{rq}\right)h_2^{\frac{p_2}{r}}(\xi)d\xi\right)^r\right]^{\frac{1}{rq}}\\
=&\frac{\Psi_1(\theta(b),\mathbf{m}(t)\theta(a))}{2}\times\left[\frac{\alpha}{k}(1-\lambda)+2\sqrt[p]{\frac{k}{p\alpha+k}}\right]
\end{aligned}
$$

$$\times \sqrt[rq]{(f'(a))^{rq} I^r(h_1(\xi)) + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) I^r(h_2(\xi))}.$$

So, the proof of this theorem is completed. □

We point out some special cases of Theorem 2.

COROLLARY 1. *In Theorem 2 for* $p = q = 2$, *we get*

$$\left|T_f^{\alpha,k}(\Psi_1,\theta,\boldsymbol{m};\lambda,a,b)\right| \leqslant \frac{\Psi_1(\theta(b),\boldsymbol{m}(t)\theta(a))}{2} \times \left[\frac{\alpha}{k}(1-\lambda) + 2\sqrt{\frac{k}{2\alpha+k}}\right] \times \sqrt[2r]{(f'(a))^{2r} I^r(h_1(\xi)) + \Psi_2\left((f'(b))^{2r}, (f'(a))^{2r}\right) I^r(h_2(\xi))}. \tag{20}$$

COROLLARY 2. *Under the conditions of Remark 5 using Theorem 2, we get*

$$\left|\frac{f(\theta(a)) + f(\theta(b))}{2} - \frac{\Gamma_k(\alpha+k)}{2(\theta(b)-\theta(a))^{\frac{\alpha}{k}}} \times \left[I^{\alpha,k}_{(\theta(a))^+} f(\theta(b)) + I^{\alpha,k}_{(\theta(b))^-} f(\theta(a))\right]\right| \leqslant \frac{(\theta(b)-\theta(a))}{2} \times 2\sqrt[p]{\frac{k}{p\alpha+k}} \times \sqrt[rq]{(f'(a))^{rq} I^r(h_1(\xi)) + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) I^r(h_2(\xi))}. \tag{21}$$

COROLLARY 3. *In Theorem 2 for* $h_1(t) = h(1-t)$, $h_2(t) = h(t)$ *and* $\boldsymbol{m}(t) = m \in (0,1]$ *for all* $t \in [0,1]$, *we get the following inequality for generalized-*m*-*$((h^{p_1}(1-t), h^{p_2}(t)); (\Psi_1,\Psi_2))$*-convex mappings:*

$$\left|T_f^{\alpha,k}(\Psi_1,\theta,m;\lambda,a,b)\right| \leqslant \frac{\Psi_1(\theta(b),m\theta(a))}{2} \times \left[\frac{\alpha}{k}(1-\lambda) + 2\sqrt[p]{\frac{k}{p\alpha+k}}\right] \times \sqrt[rq]{m(f'(a))^{rq} I^r(h(1-\xi)) + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right) I^r(h(\xi))}. \tag{22}$$

COROLLARY 4. *In Corollary 3 for* $h_1(t) = (1-t)^s$ *and* $h_2(t) = t^s$, *we get the following inequality for generalized-*m*-*$(((1-t)^{sp_1}, t^{sp_2}); (\Psi_1,\Psi_2))$*-Breckner-convex mappings:*

$$\left|T_f^{\alpha,k}(\Psi_1,\theta,m;\lambda,a,b)\right| \leqslant \frac{\Psi_1(\theta(b),m\theta(a))}{2} \times \left[\frac{\alpha}{k}(1-\lambda) + 2\sqrt[p]{\frac{k}{p\alpha+k}}\right] \times \sqrt[rq]{m(f'(a))^{rq}\left(\frac{r}{r+sp_1}\right)^r + \Psi_2\left((f'(b))^{rq}, (f'(a))^{rq}\right)\left(\frac{r}{r+sp_2}\right)^r}. \tag{23}$$

COROLLARY 5. *In Corollary 3 for* $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ *and* $r > s \cdot \max\{p_1, p_2\}$, *we get the following inequality for generalized-*m*-*$(((1-t)^{-sp_1}, t^{-sp_2}); (\Psi_1,\Psi_2))$*-Godunova-Levin-Dragomir-convex mappings:*

$$\left|T_f^{\alpha,k}(\Psi_1,\theta,m;\lambda,a,b)\right| \leqslant \frac{\Psi_1(\theta(b),m\theta(a))}{2} \times \left[\frac{\alpha}{k}(1-\lambda) + 2\sqrt[p]{\frac{k}{p\alpha+k}}\right] \tag{24}$$

$$\times \sqrt[rq]{m(f'(a))^{rq}\left(\frac{r}{r-sp_1}\right)^r + \Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)\left(\frac{r}{r-sp_2}\right)^r}.$$

COROLLARY 6. *In Theorem 2 for $h_1(t) = h_2(t) = t(1-t)$ and $\boldsymbol{m}(t) = m \in (0,1]$ for all $t \in [0,1]$, we get the following inequality for generalized-m-$(((t(1-t))^{sp_1},(t(1-t))^{sp_2});(\Psi_1,\Psi_2))$-convex mappings:*

$$\left|T_f^{\alpha,k}(\Psi_1,\theta,m;\lambda,a,b)\right| \leqslant \frac{\Psi_1(\theta(b),m\theta(a))}{2} \times \left[\frac{\alpha}{k}(1-\lambda) + 2\sqrt[p]{\frac{k}{p\alpha+k}}\right] \tag{25}$$

$$\times \sqrt[rq]{m(f'(a))^{rq}\beta^r\left(1+\frac{p_1}{r},1+\frac{p_1}{r}\right) + \Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)\beta^r\left(1+\frac{p_2}{r},1+\frac{p_2}{r}\right)}.$$

COROLLARY 7. *In Corollary 3 for $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$ and $r > \frac{1}{2}\cdot\max\{p_1,p_2\}$, we get the following inequality for generalized-m-$\left(\left(\left(\frac{\sqrt{1-t}}{2\sqrt{t}}\right)^{p_1},\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\right)^{p_2}\right);(\Psi_1,\Psi_2)\right)$-convex mappings:*

$$\begin{aligned}\left|T_f^{\alpha,k}(\Psi_1,\theta,m;\lambda,a,b)\right| \leqslant{}& \frac{\Psi_1(\theta(b),m\theta(a))}{2} \times \left[\frac{\alpha}{k}(1-\lambda) + 2\sqrt[p]{\frac{k}{p\alpha+k}}\right] \\ &\times \left[m(f'(a))^{rq}\left(\frac{1}{2}\right)^{\frac{p_1}{r}}\beta^r\left(1-\frac{p_1}{2r},1+\frac{p_1}{2r}\right)\right. \\ &\left.+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)\left(\frac{1}{2}\right)^{\frac{p_2}{r}}\beta^r\left(1-\frac{p_2}{2r},1+\frac{p_2}{2r}\right)\right]^{\frac{1}{rq}}.\end{aligned} \tag{26}$$

THEOREM 3. *Let $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$ and $\theta : I \longrightarrow K$ are continuous functions and $\boldsymbol{m} : [0,1] \longrightarrow (0,1]$. Suppose $K = [\boldsymbol{m}(t)\theta(a), \boldsymbol{m}(t)\theta(a) + \Psi_1(\theta(b),\boldsymbol{m}(t)\theta(a))] \subseteq \mathbb{R}$ be an open $\boldsymbol{m}$-invex subset with respect to $\Psi_1 : K \times K \longrightarrow \mathbb{R}$ for $\Psi_1(\theta(b),\boldsymbol{m}(t)\theta(a)) > 0, \forall t \in [0,1]$ and $\Psi_2 : f(K)\times f(K) \longrightarrow \mathbb{R}$. Assume that $f : K \longrightarrow (0,+\infty)$ be a differentiable mapping on K° such that $f' \in L(K)$. If $(f'(x))^q$ is positive generalized-$\boldsymbol{m}$-$((h_1^{p_1},h_2^{p_2});(\Psi_1,\Psi_2))$-convex mapping, $0 < r \leqslant 1$, $p_1, p_2 > -1$, $q \geqslant 1$, then the following inequality for $\alpha, k > 0$ and $\lambda \in [0,1]$, holds:*

$$\begin{aligned}&\left|T_f^{\alpha,k}(\Psi_1,\theta,\boldsymbol{m};\lambda,a,b)\right| \\ &\leqslant \frac{\Psi_1(\theta(b),\boldsymbol{m}(t)\theta(a))}{2} \times \left\{\left(\frac{k}{\alpha+k}+\frac{\alpha}{k}(1-\lambda)\right)^{1-\frac{1}{q}}\right. \\ &\quad\times \sqrt[rq]{(f'(a))^{rq}F^r(h_1(\xi)) + \Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)F^r(h_2(\xi))} \\ &\quad\left.+\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}} \times \sqrt[rq]{(f'(a))^{rq}G^r(h_1(\xi)) + \Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)G^r(h_2(\xi))}\right\},\end{aligned} \tag{27}$$

where

$$F(h_1(\xi)) := \int_0^1 \boldsymbol{m}^{\frac{1}{r}}(\xi)\left(\xi^{\frac{\alpha}{k}} + \frac{\alpha}{k}(1-\lambda)\right) h_1^{\frac{p_1}{r}}(\xi)d\xi;$$
$$F(h_2(\xi)) := \int_0^1 \left(\xi^{\frac{\alpha}{k}} + \frac{\alpha}{k}(1-\lambda)\right) h_2^{\frac{p_2}{r}}(\xi)d\xi,$$

and

$$G(h_1(\xi)) := \int_0^1 \boldsymbol{m}^{\frac{1}{r}}(\xi)(1-\xi)^{\frac{\alpha}{k}} h_1^{\frac{p_1}{r}}(\xi)d\xi;\ G(h_2(\xi)) := \int_0^1 (1-\xi)^{\frac{\alpha}{k}} h_2^{\frac{p_2}{r}}(\xi)d\xi.$$

Proof. From Lemma 1, positive generalized-$\mathbf{m}$-$((h_1^{p_1}, h_2^{p_2}); (\Psi_1, \Psi_2))$-convexity of $(f'(x))^q$, the well-known power mean inequality, Minkowski inequality, properties of the modulus and changing the variable $u = \mathbf{m}(t)\theta(a) + \xi\Psi_1(\theta(b), \mathbf{m}(t)\theta(a))$, $\forall t \in [0,1]$, we have

$$\begin{aligned}
&\left|T_f^{\alpha,k}(\Psi_1, \theta, \mathbf{m}; \lambda, a, b)\right| \\
\leqslant &\frac{|\Psi_1(\theta(b), \mathbf{m}(t)\theta(a))|}{2} \\
&\times \int_0^1 \left|\xi^{\frac{\alpha}{k}} + \frac{\alpha}{k}(1-\lambda) - (1-\xi)^{\frac{\alpha}{k}}\right| \left|f'(\mathbf{m}(t)\theta(a) + \xi\Psi_1(\theta(b), \mathbf{m}(t)\theta(a)))\right| d\xi \\
\leqslant &\frac{\Psi_1(\theta(b), \mathbf{m}(t)\theta(a))}{2} \\
&\times \left[\int_0^1 \left(\xi^{\frac{\alpha}{k}} + \frac{\alpha}{k}(1-\lambda)\right) f'(\mathbf{m}(t)\theta(a) + \xi\Psi_1(\theta(b), \mathbf{m}(t)\theta(a)))d\xi\right. \\
&\left. + \int_0^1 (1-\xi)^{\frac{\alpha}{k}} f'(\mathbf{m}(t)\theta(a) + \xi\Psi_1(\theta(b), \mathbf{m}(t)\theta(a)))d\xi\right] \\
\leqslant &\frac{\Psi_1(\theta(b), \mathbf{m}(t)\theta(a))}{2} \times \left[\left(\int_0^1 \left(\xi^{\frac{\alpha}{k}} + \frac{\alpha}{k}(1-\lambda)\right) d\xi\right)^{1-\frac{1}{q}}\right. \\
&\times \left(\int_0^1 \left(\xi^{\frac{\alpha}{k}} + \frac{\alpha}{k}(1-\lambda)\right) \left(f'(\mathbf{m}(t)\theta(a) + \xi\Psi_1(\theta(b), \mathbf{m}(t)\theta(a)))\right)^q d\xi\right)^{\frac{1}{q}} \\
&+ \left(\int_0^1 (1-\xi)^{\frac{\alpha}{k}} d\xi\right)^{1-\frac{1}{q}} \\
&\left.\times \left(\int_0^1 (1-\xi)^{\frac{\alpha}{k}} \left(f'(\mathbf{m}(t)\theta(a) + \xi\Psi_1(\theta(b), \mathbf{m}(t)\theta(a)))\right)^q d\xi\right)^{\frac{1}{q}}\right] \\
\leqslant &\frac{\Psi_1(\theta(b), \mathbf{m}(t)\theta(a))}{2} \times \left[\left(\frac{k}{\alpha+k} + \frac{\alpha}{k}(1-\lambda)\right)^{1-\frac{1}{q}}\right. \\
&\times \left(\int_0^1 \left(\xi^{\frac{\alpha}{k}} + \frac{\alpha}{k}(1-\lambda)\right)\right.
\end{aligned}$$

$$\left[\mathbf{m}(\xi)h_1^{p_1}(\xi)(f'(a))^{rq}+h_2^{p_2}(\xi)\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)\right]^{\frac{1}{r}}d\xi\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1(1-\xi)^{\frac{\alpha}{k}}\left[\mathbf{m}(\xi)h_1^{p_1}(\xi)(f'(a))^{rq}+h_2^{p_2}(\xi)\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)\right]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}\Bigg]$$
$$\leqslant\frac{\Psi_1(\theta(b),\mathbf{m}(t)\theta(a))}{2}\times\Bigg\{\Bigg[\left(\frac{k}{\alpha+k}+\frac{\alpha}{k}(1-\lambda)\right)^{1-\frac{1}{q}}$$
$$\times\Bigg[\left(\int_0^1\mathbf{m}^{\frac{1}{r}}(\xi)(f'(a))^q\left(\xi^{\frac{\alpha}{k}}+\frac{\alpha}{k}(1-\lambda)\right)h_1^{\frac{p_1}{r}}(\xi)d\xi\right)^r$$
$$+\left(\int_0^1\Psi_2^{\frac{1}{r}}\left((f'(b))^{rq},(f'(a))^{rq}\right)\left(\xi^{\frac{\alpha}{k}}+\frac{\alpha}{k}(1-\lambda)\right)h_2^{\frac{p_2}{r}}(\xi)d\xi\right)^r\Bigg]^{\frac{1}{rq}}$$
$$+\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\times\Bigg[\left(\int_0^1\mathbf{m}^{\frac{1}{r}}(\xi)(f'(a))^q(1-\xi)^{\frac{\alpha}{k}}h_1^{\frac{p_1}{r}}(\xi)d\xi\right)^r$$
$$+\left(\int_0^1\Psi_2^{\frac{1}{r}}\left((f'(b))^{rq},(f'(a))^{rq}\right)(1-\xi)^{\frac{\alpha}{k}}h_2^{\frac{p_2}{r}}(\xi)d\xi\right)^r\Bigg]^{\frac{1}{rq}}\Bigg\}$$
$$=\frac{\Psi_1(\theta(b),\mathbf{m}(t)\theta(a))}{2}$$
$$\times\Bigg\{\left(\frac{k}{\alpha+k}+\frac{\alpha}{k}(1-\lambda)\right)^{1-\frac{1}{q}}$$
$$\times\sqrt[rq]{(f'(a))^{rq}F^r(h_1(\xi))+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)F^r(h_2(\xi))}$$
$$+\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\times\sqrt[rq]{(f'(a))^{rq}G^r(h_1(\xi))+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)G^r(h_2(\xi))}\Bigg\}.$$

So, the proof of this theorem is completed. □

We point out some special cases of Theorem 3.

COROLLARY 8. *In Theorem 3 for $q=1$, we get the following inequality:*

$$\left|T_f^{\alpha,k}(\Psi_1,\theta,\boldsymbol{m};\lambda,a,b)\right| \tag{28}$$
$$\leqslant\frac{\Psi_1(\theta(b),\boldsymbol{m}(t)\theta(a))}{2}\times\Big\{\sqrt[r]{(f'(a))^rF^r(h_1(\xi))+\Psi_2\left((f'(b))^r,(f'(a))^r\right)F^r(h_2(\xi))}$$
$$+\sqrt[r]{(f'(a))^rG^r(h_1(\xi))+\Psi_2\left((f'(b))^r,(f'(a))^r\right)G^r(h_2(\xi))}\Big\}.$$

COROLLARY 9. *Under the conditions of Remark 5 using Theorem 3, we get*

$$
\begin{aligned}
&\left|\frac{f(\theta(a))+f(\theta(b))}{2}-\frac{\Gamma_k(\alpha+k)}{2(\theta(b)-\theta(a))^{\frac{\alpha}{k}}}\times\left[I^{\alpha,k}_{(\theta(a))^+}f(\theta(b))+I^{\alpha,k}_{(\theta(b))^-}f(\theta(a))\right]\right| \\
&\leqslant\frac{(\theta(b)-\theta(a))}{2}\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}} \qquad (29)\\
&\times\Big\{\sqrt[rq]{(f'(a))^{rq}F^r(h_1(\xi))+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)F^r(h_2(\xi))} \\
&+\sqrt[rq]{(f'(a))^{rq}G^r(h_1(\xi))+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)G^r(h_2(\xi))}\Big\}.
\end{aligned}
$$

COROLLARY 10. *In Theorem 3 for $h_1(t)=h(1-t)$, $h_2(t)=h(t)$ and $\mathbf{m}(t)=m\in(0,1]$ for all $t\in[0,1]$, we get the following inequality for generalized-m-$((h^{p_1}(1-t),h^{p_2}(t));(\Psi_1,\Psi_2))$-convex mappings:*

$$
\begin{aligned}
&\left|T^{\alpha,k}_f(\Psi_1,\theta,m;\lambda,a,b)\right| \qquad (30)\\
&\leqslant\frac{\Psi_1(\theta(b),m\theta(a))}{2}\times\Big\{\left(\frac{k}{\alpha+k}+\frac{\alpha}{k}(1-\lambda)\right)^{1-\frac{1}{q}} \\
&\times\sqrt[rq]{m(f'(a))^{rq}F^r(h(1-\xi))+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)F^r(h(\xi))} \\
&+\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\times\sqrt[rq]{m(f'(a))^{rq}G^r(h(1-\xi))+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)G^r(h(\xi))}\Big\}.
\end{aligned}
$$

COROLLARY 11. *In Corollary 10 for $h_1(t)=(1-t)^s$ and $h_2(t)=t^s$, we get the following inequality for generalized-m-$(((1-t)^{sp_1},t^{sp_2});(\Psi_1,\Psi_2))$-Breckner-convex mappings:*

$$
\begin{aligned}
&\left|T^{\alpha,k}_f(\Psi_1,\theta,m;\lambda,a,b)\right| \qquad (31)\\
&\leqslant\frac{\Psi_1(\theta(b),m\theta(a))}{2}\times\Big\{\left(\frac{k}{\alpha+k}+\frac{\alpha}{k}(1-\lambda)\right)^{1-\frac{1}{q}} \\
&\times\Big[m(f'(a))^{rq}\left(\beta\left(\frac{sp_1}{r}+1,\frac{\alpha}{k}+1\right)+\frac{r\alpha}{k(r+sp_1)}(1-\lambda)\right)^r \\
&+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)\left(\frac{1}{\frac{sp_2}{r}+\frac{\alpha}{k}+1}+\frac{r\alpha}{k(r+sp_2)}(1-\lambda)\right)^r\Big]^{\frac{1}{rq}} \\
&+\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\times\Big[m(f'(a))^{rq}\left(\frac{1}{\frac{sp_1}{r}+\frac{\alpha}{k}+1}\right)^r \\
&+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)\beta^r\left(\frac{sp_2}{r}+1,\frac{\alpha}{k}+1\right)\Big]^{\frac{1}{rq}}\Big\}.
\end{aligned}
$$

COROLLARY 12. *In Corollary 10 for $h_1(t)=(1-t)^{-s}$, $h_2(t)=t^{-s}$ and $r>s\cdot\max\{p_1,p_2\}$, we get the following inequality for generalized-m-$(((1-t)^{-sp_1},t^{-sp_2});(\Psi_1,\Psi_2))$-Godunova-Levin-Dragomir-convex mappings:*

$$\begin{aligned}
&\left|T_f^{\alpha,k}(\Psi_1,\theta,m;\lambda,a,b)\right| \qquad (32)\\
&\leqslant\frac{\Psi_1(\theta(b),m\theta(a))}{2}\times\left\{\left(\frac{k}{\alpha+k}+\frac{\alpha}{k}(1-\lambda)\right)^{1-\frac{1}{q}}\right.\\
&\times\left[m(f'(a))^{rq}\left(\beta\left(1-\frac{sp_1}{r},\frac{\alpha}{k}+1\right)+\frac{r\alpha}{k(r-sp_1)}(1-\lambda)\right)^r\right.\\
&\left.+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)\left(\frac{1}{\frac{\alpha}{k}-\frac{sp_2}{r}+1}+\frac{r\alpha}{k(r-sp_2)}(1-\lambda)\right)^r\right]^{\frac{1}{rq}}\\
&+\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\times\left[m(f'(a))^{rq}\left(\frac{1}{\frac{\alpha}{k}-\frac{sp_1}{r}+1}\right)^r\right.\\
&\left.\left.+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)\beta^r\left(1-\frac{sp_2}{r},\frac{\alpha}{k}+1\right)\right]^{\frac{1}{rq}}\right\}.
\end{aligned}$$

COROLLARY 13. *In Theorem 3 for $h_1(t)=h_2(t)=t(1-t)$ and $\boldsymbol{m}(t)=m\in(0,1]$ for all $t\in[0,1]$, we get the following inequality for generalized-m-$((t(1-t))^{sp_1},(t(1-t))^{sp_2});(\Psi_1,\Psi_2))$-convex mappings:*

$$\begin{aligned}
&\left|T_f^{\alpha,k}(\Psi_1,\theta,m;\lambda,a,b)\right| \qquad (33)\\
&\leqslant\frac{\Psi_1(\theta(b),m\theta(a))}{2}\times\left\{\left(\frac{k}{\alpha+k}+\frac{\alpha}{k}(1-\lambda)\right)^{1-\frac{1}{q}}\right.\\
&\times\left[m(f'(a))^{rq}\left(\beta\left(\frac{p_1}{r}+\frac{\alpha}{k}+1,\frac{p_1}{r}+1\right)+\frac{\alpha}{k}(1-\lambda)\beta\left(\frac{p_1}{r}+1,\frac{p_1}{r}+1\right)\right)^r\right.\\
&+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)\\
&\left.\left(\beta\left(\frac{p_2}{r}+\frac{\alpha}{k}+1,\frac{p_2}{r}+1\right)+\frac{\alpha}{k}(1-\lambda)\beta\left(\frac{p_2}{r}+1,\frac{p_2}{r}+1\right)\right)^r\right]^{\frac{1}{rq}}+\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\\
&\left.\times\sqrt[rq]{m(f'(a))^{rq}\beta^r\left(\frac{p_1}{r}+\frac{\alpha}{k}+1,\frac{p_1}{r}+1\right)+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)\beta^r\left(\frac{p_2}{r}+\frac{\alpha}{k}+1,\frac{p_2}{r}+1\right)}\right\}.
\end{aligned}$$

COROLLARY 14. *In Corollary 10 for $h_1(t)=\dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t)=\dfrac{\sqrt{t}}{2\sqrt{1-t}}$ and $r>\frac{1}{2}\cdot\max\{p_1,p_2\}$, we get the following inequality for generalized-m-*

- $\left(\left(\left(\frac{\sqrt{1-t}}{2\sqrt{t}}\right)^{p_1},\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\right)^{p_2}\right);(\Psi_1,\Psi_2)\right)$*-convex mappings:*

$$\left|T_f^{\alpha,k}(\Psi_1,\theta,m;\lambda,a,b)\right| \tag{34}$$
$$\leqslant\frac{\Psi_1(\theta(b),m\theta(a))}{2}\times\Bigg\{\left(\frac{k}{\alpha+k}+\frac{\alpha}{k}(1-\lambda)\right)^{1-\frac{1}{q}}$$
$$\times\Bigg[m(f'(a))^{rq}\left(\frac{1}{2}\right)^{\frac{p_1}{r}}\left(\beta\left(\frac{\alpha}{k}-\frac{p_1}{2r}+1,\frac{p_1}{2r}+1\right)+\frac{\alpha}{k}(1-\lambda)\beta\left(1-\frac{p_1}{2r},\frac{p_1}{2r}+1\right)\right)^r$$
$$+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)\left(\frac{1}{2}\right)^{\frac{p_2}{r}}$$
$$\times\left(\beta\left(\frac{\alpha}{k}+\frac{p_2}{2r}+1,1-\frac{p_2}{2r}\right)+\frac{\alpha}{k}(1-\lambda)\beta\left(\frac{p_2}{2r}+1,1-\frac{p_2}{2r}\right)\right)^r\Bigg]^{\frac{1}{rq}}$$
$$+\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\times\Bigg[m(f'(a))^{rq}\left(\frac{1}{2}\right)^{\frac{p_1}{r}}\beta^r\left(1-\frac{p_1}{2r},\frac{\alpha}{k}+\frac{p_1}{2r}+1\right)$$
$$+\Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right)\left(\frac{1}{2}\right)^{\frac{p_2}{r}}\beta^r\left(\frac{p_2}{2r}+1,\frac{\alpha}{k}-\frac{p_2}{2r}+1\right)\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

REMARK 6. By taking particular values of parameters α,k,λ,p_1 and p_2 in above Theorems 2 and 3, several k-fractional integral inequalities associated with generalized-$\mathbf{m}$-$((h_1^{p_1},h_2^{p_2});(\Psi_1,\Psi_2))$-convex mappings can be obtained. In particular, for $k=1$, by our Theorems 2 and 3, we can get some new special Hermite-Hadamard type inequalities via fractional integrals of order $\alpha>0$. Also, for $\alpha=k=1$, we can get some new special Hermite-Hadamard type inequalities via classical integrals.

REMARK 7. Also, applying our Theorems 2 and 3, for $f'(x)\leqslant L$, for all $x\in I$, we can get some new k-fractional integral inequalities.

3. Applications to special means

DEFINITION 16. [2] A function $M:\mathbb{R}_+^2\longrightarrow\mathbb{R}_+$, is called a Mean function if it has the following properties:

1. Homogeneity: $M(ax,ay)=aM(x,y)$, for all $a>0$,
2. Symmetry: $M(x,y)=M(y,x)$,
3. Reflexivity: $M(x,x)=x$,
4. Monotonicity: If $x\leqslant x'$ and $y\leqslant y'$, then $M(x,y)\leqslant M(x',y')$,
5. Internality: $\min\{x,y\}\leqslant M(x,y)\leqslant\max\{x,y\}$.

Let consider some special means for arbitrary positive real numbers $\alpha \neq \beta$ as follows: The arithmetic mean $A := A(\alpha,\beta)$; The geometric mean $G := G(\alpha,\beta)$; The harmonic mean $H := H(\alpha,\beta)$; The power mean $P_r := P_r(\alpha,\beta)$; The identric mean $I := I(\alpha,\beta)$; The logarithmic mean $L := L(\alpha,\beta)$; The generalized log-mean $L_p := L_p(\alpha,\beta)$; The weighted p-power mean $M = M_p$. Now, let a and b be positive real numbers such that $a < b$. Let consider continuous functions $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$, $\theta : I \longrightarrow K$ and $\Psi_1 : K \times K \longrightarrow \mathbb{R}$, $\Psi_2 : f(K) \times f(K) \longrightarrow \mathbb{R}$, where $\overline{M} := M(\theta(a),\theta(b)) : [\theta(a),\theta(a)+\Psi_1(\theta(b),\theta(a))] \times [\theta(a),\theta(a)+\Psi_1(\theta(b),\theta(a))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means. Therefore one can obtain various inequalities using the results of Sect. 2 for these means as follows. Replace $\Psi_1(\theta(y),\mathbf{m}(t)\theta(x))$ with $\Psi_1(\theta(y),\theta(x))$ where $\mathbf{m}(t) \equiv 1$, for all $t \in [0,1]$ and setting $\Psi_1(\theta(y),\theta(x)) = M(\theta(x),\theta(y))$ for all $x, y \in I$, in (19) and (27), one can obtain the following interesting inequalities involving means:

$$\begin{aligned}
&\left|T_f^{\alpha,k}(M(\cdot,\cdot),\theta,1;\lambda,a,b)\right| \\
&\leqslant \frac{\overline{M}}{2} \times \left[\frac{\alpha}{k}(1-\lambda) + 2\sqrt[p]{\frac{k}{p\alpha+k}}\right] \\
&\quad \times \sqrt[rq]{(f'(a))^{rq} I^r(h_1(\xi)) + \Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right) I^r(h_2(\xi))},
\end{aligned} \tag{35}$$

$$\begin{aligned}
&\left|T_f^{\alpha,k}(M(\cdot,\cdot),\theta,1;\lambda,a,b)\right| \\
&\leqslant \frac{\overline{M}}{2} \times \left\{ \left(\frac{k}{\alpha+k} + \frac{\alpha}{k}(1-\lambda)\right)^{1-\frac{1}{q}} \right. \\
&\quad \times \sqrt[rq]{(f'(a))^{rq} F^r(h_1(\xi)) + \Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right) F^r(h_2(\xi))} \\
&\quad \left. + \left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}} \times \sqrt[rq]{(f'(a))^{rq} G^r(h_1(\xi)) + \Psi_2\left((f'(b))^{rq},(f'(a))^{rq}\right) G^r(h_2(\xi))} \right\}.
\end{aligned} \tag{36}$$

Letting $\overline{M} := A, G, H, P_r, I, L, L_p, M_p$ in (35) and (36), we get the inequalities involving means for a particular choices of $(f'(x))^q$ that are generalized-1--$((h_1^{p_1}, h_2^{p_2}); (\Psi_1, \Psi_2))$-convex mappings.

REMARK 8. Also, applying our Theorems 2 and 3 for appropriate choices of functions h_1 and h_2 (see Remark 4) such that $(f'(x))^q$ to be generalized-1--$((h_1^{p_1}, h_2^{p_2}); (\Psi_1, \Psi_2))$-convex mappings (see examples: $f(x) = x^\alpha$, where $\alpha > 1, \forall x > 0$; $f(x) = -\frac{1}{x}, \forall x > 0$; $f(x) = e^x, \forall x \in \mathbb{R}$; $f(x) = \ln x, \forall x > 0$; etc.), we can deduce some new inequalities using above special means. The details are left to the interested reader.

4. Conclusions

The authors discovered a new identity concerning differentiable mappings defined on $\mathbf{m}$-invex set via k-fractional integrals. By using the obtained identity as an auxiliary result, some k-fractional integral inequalities for generalized-$\mathbf{m}$-$((h_1^p, h_2^q);(\eta_1,\eta_2))$-convex mappings are presented. Also, some new special cases are given. At the end, some applications to special means for different positive real numbers are provided as well. Motivated by this interesting class we can indeed see to be vital for fellow researchers and scientists working in the same domain. We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard, Ostrowski and Simpson type integral inequalities for various kinds of convex and preinvex functions involving local fractional integrals, fractional integral operators, Caputo k-fractional derivatives, q-calculus, (p,q)-calculus, time scale calculus and conformable fractional integrals.

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