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AN ORDERING ON GREEN'S FUNCTION AND A LYAPUNOV–TYPE INEQUALITY FOR A FAMILY OF NABLA FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. In this article, we consider a family of two-point Riemann–Liouville type nabla fractional boundary value problems involving a fractional difference boundary condition. We construct the corresponding Green's function and deduce its ordering property. Then, we obtain a Lyapunov-type inequality using the properties of the Green's function, and illustrate a few of its applications.

1. Introduction

In this article, we construct the Green's function $G(b,\beta;t,s)$ of the following two-point nabla fractional boundary value problem

$$\begin{cases} \big(\nabla_a^{\alpha} u\big)(t) + h(t) = 0, \quad t \in \mathbb{N}_{a+2}^{b}, \\ u(a) = 0, \ \big(\nabla_a^{\beta} u\big)(b) = 0. \end{cases} \tag{1.1}$$

Here $1 < \alpha < 2$, $0 \leqslant \beta \leqslant 1$, a, $b \in \mathbb{R}$ with $b - a \in \mathbb{N}_2$, $h : \mathbb{N}_{a+2}^{b} \to \mathbb{R}$, ∇_a^{α} and ∇_a^{β} are the Riemann–Liouville type α^{th} and β^{th}-order nabla difference operators, respectively. Observe that the pair of boundary conditions in (1.1) reduces to conjugate [6, 12, 20], right-focal [18] and right-focal type [19] boundary conditions as $\beta \to 0^{+}$, $\beta \to 1^{-}$ and $\beta \to (\alpha - 1)$, respectively. In Section 3, we obtain an ordering property on $G(b,\beta;t,s)$ with respect to b and β.

Lately, there has been an increased interest in establishing Lyapunov-type inequalities for discrete fractional boundary value problems. For the first time, Ferreira [10] deduced a Lyapunov-type inequality for a discrete boundary value problem involving the Riemann–Liouville type α^{th}-order $(1 < \alpha \leqslant 2)$ forward difference operator. Following Ferreira's work, authors of [8, 11] established Lyapunov-type inequalities for various classes of delta fractional boundary value problems. In this line, Ikram [16]

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developed Lyapunov-type inequalities for certain nabla fractional boundary value problems of Caputo type. Recently, the author [18, 19] obtained Lyapunov-type inequalities for the nabla fractional difference equation

$$\big(\nabla_a^{\alpha} u\big)(t) + q(t)u(t) = 0, \quad t \in \mathbb{N}_{a+2}^{b},$$

associated with two-point conjugate (C), left focal (LF), right focal (RF), left-focal type (LFT) and right-focal type(RFT) boundary conditions:

(C) $u(a) = u(b) = 0;$

(LF) $\big(\nabla u\big)(a+1) = u(b) = 0;$

(RF) $u(a) = \big(\nabla u\big)(b) = 0;$

(LFT) $\big(\nabla_a^{\alpha-1} u\big)(a+1) = u(b) = 0;$

(RF) $u(a) = \big(\nabla_a^{\alpha-1} u\big)(b) = 0.$

Motivated by these developments, in this article, we obtain a Lyapunov-type inequality for the two-point nabla fractional boundary value problem

$$\begin{cases} \big(\nabla_a^{\alpha} u\big)(t) + q(t)u(t) = 0, & t \in \mathbb{N}_{a+2}^{b}, \\ u(a) = 0, \ \big(\nabla_a^{\beta} u\big)(b) = 0, \end{cases} \tag{1.2}$$

where $q : \mathbb{N}_{a+2}^{b} \to \mathbb{R}$, and demonstrate a few of its applications.

2. Preliminaries

Denote the set of all real numbers by $\mathbb{R}$. Define

$$\mathbb{N}_a := \{a, a+1, a+2, \ldots\} \text{ and } \mathbb{N}_a^b := \{a, a+1, a+2, \ldots, b\}$$

for any a, $b \in \mathbb{R}$ such that $b - a \in \mathbb{N}_1$. Assume that empty sums and products are taken to be 0 and 1, respectively.

DEFINITION 2.1. (See [7]) The backward jump operator $\rho : \mathbb{N}_a \to \mathbb{N}_a$ is defined by

$$\rho(t) = \max\{a, (t-1)\}, \quad t \in \mathbb{N}_a.$$

DEFINITION 2.2. (See [22, 23]) The Euler gamma function is defined by

$$\Gamma(z) := \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \Re(z) > 0.$$

Using the reduction formula

$$\Gamma(z+1) = z\Gamma(z), \quad \Re(z) > 0,$$

the Euler gamma function can be extended to the half-plane $\Re(z) \leqslant 0$ except for $z \neq 0, -1, -2, \ldots$

DEFINITION 2.3. (See [14]) For $t\in\mathbb{R}\setminus\{\dots,-2,-1,0\}$ and $r\in\mathbb{R}$ such that $(t+r)\in\mathbb{R}\setminus\{\dots,-2,-1,0\}$, the generalized rising function is defined by

$$t^{\overline{r}}=\frac{\Gamma(t+r)}{\Gamma(t)}.$$

Also, we use the convention that if $t\in\{\dots,-2,-1,0\}$ and $r\in\mathbb{R}$ such that $(t+r)\in\mathbb{R}\setminus\{\dots,-2,-1,0\}$, then

$$t^{\overline{r}}:=0.$$

DEFINITION 2.4. (See [7]) Let $u:\mathbb{N}_a\to\mathbb{R}$ and $N\in\mathbb{N}_1$. The first order backward (nabla) difference of u is defined by

$$\big(\nabla u\big)(t):=u(t)-u(t-1),\quad t\in\mathbb{N}_{a+1},$$

and the N^{th}-order nabla difference of u is defined recursively by

$$\big(\nabla^N u\big)(t):=\Big(\nabla\big(\nabla^{N-1}u\big)\Big)(t),\quad t\in\mathbb{N}_{a+N}.$$

DEFINITION 2.5. (See [14]) Let $u:\mathbb{N}_{a+1}\to\mathbb{R}$ and $N\in\mathbb{N}_1$. The N^{th}-order nabla sum of u based at a is given by

$$\big(\nabla_a^{-N}u\big)(t):=\frac{1}{(N-1)!}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{N-1}}u(s),\quad t\in\mathbb{N}_a,$$

where by convention $\big(\nabla_a^{-N}u\big)(a)=0$. We define $\big(\nabla_a^{-0}u\big)(t)=u(t)$ for all $t\in\mathbb{N}_{a+1}$.

DEFINITION 2.6. (See [14]) Let $u:\mathbb{N}_{a+1}\to\mathbb{R}$ and $\nu>0$. The ν^{th}-order nabla sum of u based at a is given by

$$\big(\nabla_a^{-\nu}u\big)(t):=\frac{1}{\Gamma(\nu)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\nu-1}}u(s),\quad t\in\mathbb{N}_a,$$

where by convention $\big(\nabla_a^{-\nu}u\big)(a)=0$.

DEFINITION 2.7. (See [14]) Let $u:\mathbb{N}_{a+1}\to\mathbb{R}$, $\nu>0$ and choose $N\in\mathbb{N}_1$ such that $N-1<\nu\leqslant N$. The Riemann–Liouville type ν^{th}-order nabla difference of u is given by

$$\big(\nabla_a^{\nu}u\big)(t):=\Big(\nabla^N\big(\nabla_a^{-(N-\nu)}u\big)\Big)(t),\quad t\in\mathbb{N}_{a+N}.$$

THEOREM 2.1. (See [2]) *Assume* $u:\mathbb{N}_a\to\mathbb{R}$, $\nu>0$, $\nu\notin\mathbb{N}_1$, *and choose* $N\in\mathbb{N}_1$ *such that* $N-1<\nu<N$. *Then,*

$$\big(\nabla_a^{\nu}u\big)(t)=\frac{1}{\Gamma(-\nu)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{-\nu-1}}u(s),\quad t\in\mathbb{N}_{a+1}.$$

THEOREM 2.2. (See [14]) *Let* $\nu,\ \mu > 0$ *and* $u : \mathbb{N}_a \to \mathbb{R}$. *Then,*

$$\Big(\nabla_a^{\nu}\big(\nabla_a^{-\mu}u\big)\Big)(t) = \big(\nabla_a^{\nu-\mu}u\big)(t), \quad t \in \mathbb{N}_a.$$

THEOREM 2.3. (See [14, 17]) *We observe the following properties of gamma and generalized rising functions.*

1. $\Gamma(t) > 0$ *for all* $t > 0$.
2. $t^{\overline{\alpha}}(t+\alpha)^{\overline{\beta}} = t^{\overline{\alpha+\beta}}$.
3. *If* $t \leqslant r$, *then* $t^{\overline{\alpha}} \leqslant r^{\overline{\alpha}}$.
4. *If* $\alpha < t \leqslant r$, *then* $r^{\overline{-\alpha}} \leqslant t^{\overline{-\alpha}}$.
5. $\nabla(t+\alpha)^{\overline{\beta}} = \beta(t+\alpha)^{\overline{\beta-1}}$.
6. $\nabla(\alpha-t)^{\overline{\beta}} = -\beta(\alpha-\rho(t))^{\overline{\beta-1}}$.

THEOREM 2.4. (See [14]) *Let* $\nu \in \mathbb{R}^+$ *and* $\mu \in \mathbb{R}$ *such that* $\mu,\ \mu+\nu$ *and* $\mu-\nu$ *are nonnegative integers. Then,*

$$\nabla_a^{-\nu}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a)^{\overline{\mu+\nu}}, \quad t \in \mathbb{N}_a,$$
$$\nabla_a^{\nu}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)}(t-a)^{\overline{\mu-\nu}}, \quad t \in \mathbb{N}_a.$$

THEOREM 2.5. (See [14]) *Assume* $\nu > 0$ *and* $N-1 < \nu \leqslant N$. *Then, a general solution of*

$$\big(\nabla_a^{\nu}u\big)(t) = 0, \quad t \in \mathbb{N}_{a+N},$$

is given by

$$u(t) = C_1(t-a)^{\overline{\nu-1}} + C_2(t-a)^{\overline{\nu-2}} + \ldots + C_N(t-a)^{\overline{\nu-N}}, \quad t \in \mathbb{N}_a,$$

where $C_1,\ C_2,\ \cdots,\ C_N \in \mathbb{R}$.

3. Properties of Green's function

First, we deduce the unique solution of (1.1).

THEOREM 3.1. *The discrete boundary value problem* (1.1) *has the unique solution*

$$u(t) = \sum_{s=a+2}^{b} G(b,\beta;t,s)h(s), \quad t \in \mathbb{N}_a^b, \tag{3.1}$$

where the Green's function $G(b,\beta;t,s)$ is given by

$$G(b,\beta;t,s) = \begin{cases} \frac{1}{\Gamma(\alpha)} \frac{(b-s+1)^{\overline{\alpha-\beta-1}}}{(b-a)^{\overline{\alpha-\beta-1}}}(t-a)^{\overline{\alpha-1}}, & t \in \mathbb{N}_a^{\rho(s)}, \\ \frac{1}{\Gamma(\alpha)} \Big[\frac{(b-s+1)^{\overline{\alpha-\beta-1}}}{(b-a)^{\overline{\alpha-\beta-1}}}(t-a)^{\overline{\alpha-1}} - (t-s+1)^{\overline{\alpha-1}}\Big], & t \in \mathbb{N}_s^b. \end{cases} \tag{3.2}$$

Proof. Applying $\nabla_a^{-\alpha}$ on both sides of (1.1) and using Theorem 2.5, we have

$$u(t) = -\big(\nabla_a^{-\alpha} h\big)(t) + C_1(t-a)^{\overline{\alpha-1}} + C_2(t-a)^{\overline{\alpha-2}}, \quad t \in \mathbb{N}_a, \tag{3.3}$$

for some C_1, $C_2 \in \mathbb{R}$. Using $u(a) = 0$ in (3.3), we get $C_2 = 0$. Applying ∇_a^{β} on both sides of (3.3) and using Theorems 2.2 and 2.4, we have

$$\big(\nabla_a^{\beta} u\big)(t) = -\big(\nabla_a^{\beta-\alpha} h\big)(t) + C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(t-a)^{\overline{\alpha-\beta-1}}, \quad t \in \mathbb{N}_a. \tag{3.4}$$

Using $\big(\nabla_a^{\beta} u\big)(b) = 0$ in (3.4), we get

$$C_1 = \frac{1}{(b-a)^{\overline{\alpha-\beta-1}}\Gamma(\alpha)} \sum_{s=a+1}^{b} (b-s+1)^{\overline{\alpha-\beta-1}} h(s).$$

Substituting the values of C_1 and C_2 in (3.3), we have

$$\begin{aligned} u(t) =& \frac{(t-a)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-\beta-1}}\Gamma(\alpha)} \sum_{s=a+1}^{b} (b-s+1)^{\overline{\alpha-\beta-1}} h(s) - \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-s+1)^{\overline{\alpha-1}} h(s) \\ =& \frac{1}{\Gamma(\alpha)} \sum_{s=a+2}^{t} \Big[\frac{(b-s+1)^{\overline{\alpha-\beta-1}}}{(b-a)^{\overline{\alpha-\beta-1}}}(t-a)^{\overline{\alpha-1}} - (t-s+1)^{\overline{\alpha-1}}\Big] h(s) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=t+1}^{b} \Big[\frac{(b-s+1)^{\overline{\alpha-\beta-1}}}{(b-a)^{\overline{\alpha-\beta-1}}}(t-a)^{\overline{\alpha-1}}\Big] h(s) \\ =& \sum_{s=a+2}^{b} G(b,\beta;t,s)(t,s)h(s). \end{aligned}$$

The proof is complete. □

REMARK 1. Observe that

1. $G(b,\beta;t,a+1) = 0$ for $t \in \mathbb{N}_a^b$.
2. $G(b,\beta;a,s) = 0$ for $s \in \mathbb{N}_{a+2}^b$.

Brackins [6], Gholami et al. [12] and the author [18, 19, 20] have derived the Green's functions $G(b,0;t,s)$, $G(b,1;t,s)$ and $G(b,\alpha-1;t,s)$ of the two-point nabla fractional boundary value problem associated with conjugate, right-focal and right-focal type boundary conditions, respectively, and also obtained a few properties.

THEOREM 3.2. (See [6, 12, 18, 19, 20]) *$G(b,0;t,s)$, $G(b,1;t,s)$ and $G(b,\alpha-1;t,s)$ are nonnegative for $(t,s)\in\mathbb{N}_a^b\times\mathbb{N}_{a+2}^b$.*

Next, we obtain a few properties of $G(b,\beta;t,s)$.

LEMMA 3.3. *If $0\leqslant\beta_1<\beta_2\leqslant 1$, then $G(b,\beta_1;t,s)<G(b,\beta_2;t,s)$ for $(t,s)\in\mathbb{N}_{a+1}^b\times\mathbb{N}_{a+2}^b$.*

Proof. Using (2) of Theorem 2.3, we rewrite $G(b,\beta_1;t,s)$ in terms of $G(b,\beta_2;t,s)$ as follows:

$$G(b,\beta_1;t,s)$$
$$=\begin{cases}\frac{1}{\Gamma(\alpha)}\frac{(b-a+\alpha-\beta_1-1)^{\overline{\beta_1-\beta_2}}}{(b-s+\alpha-\beta_1)^{\overline{\beta_1-\beta_2}}}\frac{(b-s+1)^{\overline{\alpha-\beta_2-1}}}{(b-a)^{\overline{\alpha-\beta_2-1}}}(t-a)^{\overline{\alpha-1}}, & t\in\mathbb{N}_a^{\rho(s)},\\ \frac{1}{\Gamma(\alpha)}\Big[\frac{(b-a+\alpha-\beta_1-1)^{\overline{\beta_1-\beta_2}}}{(b-s+\alpha-\beta_1)^{\overline{\beta_1-\beta_2}}}\frac{(b-s+1)^{\overline{\alpha-\beta_2-1}}}{(b-a)^{\overline{\alpha-\beta_2-1}}}(t-a)^{\overline{\alpha-1}}-(t-s+1)^{\overline{\alpha-1}}\Big], & t\in\mathbb{N}_s^b.\end{cases}$$

Since $\beta_2-\beta_1<(b-s+\alpha-\beta_1)<(b-a+\alpha-\beta_1-1)$, from (4) of Theorem 2.3, we have

$$(b-a+\alpha-\beta_1-1)^{\overline{\beta_1-\beta_2}}<(b-s+\alpha-\beta_1)^{\overline{\beta_1-\beta_2}}, \tag{3.5}$$

implying that

$$G(b,\beta_1;t,s)<G(b,\beta_2;t,s),\quad (t,s)\in\mathbb{N}_{a+1}^b\times\mathbb{N}_{a+2}^b.$$

The proof is complete. □

THEOREM 3.4. *$G(b,\beta;t,s)\geqslant 0$ for $(t,s)\in\mathbb{N}_a^b\times\mathbb{N}_{a+2}^b$.*

Proof. The proof follows from Remark 1, Theorem 3.2 and Lemma 3.3. □

LEMMA 3.5. *Assume $b_1<b_2$ and $(t,s)\in\mathbb{N}_{a+1}^b\times\mathbb{N}_{a+2}^b$.*

1. *If $0\leqslant\beta<(\alpha-1)$, then $G(b_1,\beta;t,s)<G(b_2,\beta;t,s)$.*
2. *If $(\alpha-1)<\beta\leqslant 1$, then $G(b_1,\beta;t,s)>G(b_2,\beta;t,s)$.*
3. *If $\beta=(\alpha-1)$, then $G(b,\beta;t,s)$ is independent of b.*

Proof. Consider

$$\begin{aligned}\nabla_b\big[G(b,\beta;t,s)\big]&=\frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\nabla_b\Big[\frac{(b-s+1)^{\overline{\alpha-\beta-1}}}{(b-a)^{\overline{\alpha-\beta-1}}}\Big]\\&=\frac{(b-a)^{\overline{\alpha-\beta-2}}(b-s+1)^{\overline{\alpha-\beta-2}}(t-a)^{\overline{\alpha-1}}(s-a-1)(\alpha-\beta-1)}{(b-a)^{\overline{\alpha-\beta-1}}(b-a-1)^{\overline{\alpha-\beta-1}}\Gamma(\alpha)}\\&=\frac{(b-s+1)^{\overline{\alpha-\beta-2}}(t-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-\beta}}\Gamma(\alpha)}(s-a-1)(\alpha-\beta-1).\end{aligned}$$

Clearly, $(s-a-1)>0$, $\Gamma(\alpha)>0$ and it follows from (1) of Theorem 2.3 that

$$(t-a)^{\overline{\alpha-1}} = \frac{\Gamma(t-a+\alpha-1)}{\Gamma(t-a)} > 0,$$
$$(b-s+1)^{\overline{\alpha-\beta-2}} = \frac{\Gamma(b-s+\alpha-\beta-1)}{\Gamma(b-s+1)} > 0,$$

and

$$(b-a-1)^{\overline{\alpha-\beta}} = \frac{\Gamma(b-a+\alpha-\beta-1)}{\Gamma(b-a-1)} > 0.$$

Thus, if $0 \leqslant \beta < (\alpha-1)$, then $\nabla_b\big[G(b,\beta;t,s)\big] > 0$ implying that (1) follows. If $(\alpha-1) < \beta \leqslant 1$, then $\nabla_b\big[G(b,\beta;t,s)\big] < 0$ implying that (2) follows. If $\beta = (\alpha-1)$, then $\nabla_b\big[G(b,\beta;t,s)\big] = 0$ implying that $G(b,\beta;t,s)$ is independent of b. The proof is complete. □

DEFINITION 3.1. Denote by

$$H(b,\beta;s) = \frac{(b-s+1)^{\overline{\alpha-\beta-1}}}{(b-a)^{\overline{\alpha-\beta-1}}}, \quad s \in \mathbb{N}_{a+2}^{b}.$$

REMARK 2. We have

$$H(b,\beta;s) = \frac{\Gamma(b-s+\alpha-\beta)\Gamma(b-a)}{\Gamma(b-s+1)\Gamma(b-a+\alpha-\beta-1)}, \quad s \in \mathbb{N}_{a+2}^{b}.$$

(i) It follows from (1) of Theorem 2.3 that $H(b,\beta;s) > 0$ for $s \in \mathbb{N}_{a+2}^{b}$.

(ii) Since $(b-s+1) < (b-a)$, from (3) of Theorem 2.3, we have

$$(b-s+1)^{\overline{\alpha-1}} < (b-a)^{\overline{\alpha-1}},$$

implying that $H(b,0;s) < 1$.

(iii) Since $(2-\alpha) < (b-s+1) < (b-a)$, from (4) of Theorem 2.3, we have

$$(b-a)^{\overline{\alpha-2}} < (b-s+1)^{\overline{\alpha-2}},$$

implying that $H(b,1;s) > 1$.

LEMMA 3.6. *If* $0 \leqslant \beta_1 < \beta_2 \leqslant 1$, *then* $H(b,\beta_1;s) < H(b,\beta_2;s)$ *for* $s \in \mathbb{N}_{a+2}^{b}$.

Proof. Using (2) of Theorem 2.3, we rewrite $H(b,\beta_1;s)$ in terms of $H(b,\beta_2;s)$ as follows:

$$\begin{aligned} H(b,\beta_1;s) = \frac{(b-s+1)^{\overline{\alpha-\beta_1-1}}}{(b-a)^{\overline{\alpha-\beta_1-1}}} &= \frac{(b-a+\alpha-\beta_1-1)^{\overline{\beta_1-\beta_2}}}{(b-s+\alpha-\beta_1)^{\overline{\beta_1-\beta_2}}} \frac{(b-s+1)^{\overline{\alpha-\beta_2-1}}}{(b-a)^{\overline{\alpha-\beta_2-1}}} \\ &= \frac{(b-a+\alpha-\beta_1-1)^{\overline{\beta_1-\beta_2}}}{(b-s+\alpha-\beta_1)^{\overline{\beta_1-\beta_2}}} H(b,\beta_2;s). \end{aligned}$$

It follows from (3.5) that

$$H(b,\beta_1;s) < H(b,\beta_2;s), \quad s \in \mathbb{N}_{a+2}^{b}.$$

The proof is complete. $\square$

LEMMA 3.7. *Assume* $s \in \mathbb{N}_{a+2}^{b}$.

1. *If* $0 \leqslant \beta < (\alpha-1)$, *then* $H(b,\beta;s) < 1$.
2. *If* $(\alpha-1) < \beta \leqslant 1$, *then* $H(b,\beta;s) > 1$.
3. *If* $\beta = (\alpha-1)$, *then* $H(b,\beta;s) = 1$.

Proof.

1. Since $(b-s+1) < (b-a)$, from (3) of Theorem 2.3, we have

$$(b-s+1)^{\overline{\alpha-\beta-1}} < (b-a)^{\overline{\alpha-\beta-1}},$$

implying that $H(b,\beta;s) < 1$.

2. Since $-(\alpha-\beta-1) < (b-s+1) < (b-a)$, from (4) of Theorem 2.3, we have

$$(b-a)^{\overline{\alpha-\beta-1}} < (b-s+1)^{\overline{\alpha-\beta-1}},$$

implying that $H(b,\beta;s) > 1$.

3. The proof of (3) is trivial. $\square$

LEMMA 3.8. *Assume* $b_1 < b_2$.

1. *If* $0 \leqslant \beta < (\alpha-1)$, *then* $H(b_1,\beta;s) < H(b_2,\beta;s)$ *for* $s \in \mathbb{N}_{a+2}^{b}$.
2. *If* $(\alpha-1) < \beta \leqslant 1$, *then* $H(b_1,\beta;s) > H(b_2,\beta;s)$ *for* $s \in \mathbb{N}_{a+2}^{b}$.

Proof. Consider

$$\begin{aligned}
\nabla_b\big[H(b,\beta;s)\big] &= \nabla_b\Big[\frac{(b-s+1)^{\overline{\alpha-\beta-1}}}{(b-a)^{\overline{\alpha-\beta-1}}}\Big] \\
&= \frac{(b-a)^{\overline{\alpha-\beta-2}}(b-s+1)^{\overline{\alpha-\beta-2}}(s-a-1)(\alpha-\beta-1)}{(b-a)^{\overline{\alpha-\beta-1}}(b-a-1)^{\overline{\alpha-\beta-1}}} \\
&= \frac{(b-s+1)^{\overline{\alpha-\beta-2}}}{(b-a-1)^{\overline{\alpha-\beta}}}(s-a-1)(\alpha-\beta-1).
\end{aligned}$$

Clearly, $(s-a-1) > 0$ and it follows from (1) of Theorem 2.3 that

$$(b-s+1)^{\overline{\alpha-\beta-2}} = \frac{\Gamma(b-s+\alpha-\beta-1)}{\Gamma(b-s+1)} > 0,$$

and

$$(b-a-1)^{\overline{\alpha-\beta}} = \frac{\Gamma(b-a+\alpha-\beta-1)}{\Gamma(b-a-1)} > 0.$$

Thus, if $0 \leqslant \beta < (\alpha-1)$, then $\nabla_b\big[H(b,\beta;s)\big] > 0$ implying that (1) follows. If $(\alpha-1) < \beta \leqslant 1$, then $\nabla_b\big[H(b,\beta;s)\big] < 0$ implying that (2) follows. The proof is complete. $\square$

THEOREM 3.9. *The maximum of the Green's function $G(b,\beta;t,s)$ defined in (3.2) is given by*

$$\max_{(t,s)\in\mathbb{N}_{a+1}^b\times\mathbb{N}_{a+2}^b} G(b,\beta;t,s) = \begin{cases} \Omega, & 0 \leqslant \beta \leqslant (\alpha-1), \\ \max\{\Omega, \Lambda-1\}, & (\alpha-1) < \beta \leqslant 1, \end{cases}$$

where

$$\Omega = G\left(b,\beta;\left\lfloor\frac{(a+b+3)(\alpha-\beta-1)+b\beta}{(2\alpha-2-\beta)}\right\rfloor - 1, \left\lfloor\frac{(a+b+3)(\alpha-\beta-1)+b\beta}{(2\alpha-2-\beta)}\right\rfloor\right),$$

and

$$\Lambda = G\left(b,\beta;\left\lfloor\frac{(a+b+3)(\alpha-\beta-1)+b\beta+1}{(2\alpha-2-\beta)}\right\rfloor, \left\lfloor\frac{(a+b+3)(\alpha-\beta-1)+b\beta+1}{(2\alpha-2-\beta)}\right\rfloor\right).$$

Proof. Assume $(t,s) \in \mathbb{N}_{a+1}^b \times \mathbb{N}_{a+2}^b$. First, we show that for any fixed $s \in \mathbb{N}_{a+2}^b$, $G(b,\beta;t,s)$ is an increasing function of t between $a+1$ and $s-1$. Consider the first order nabla difference of $G(b,\beta;t,s)$ with respect to t.

$$\begin{aligned}\nabla_t\big[G(b,\beta;t,s)\big] &= \frac{H(b,\beta;s)}{\Gamma(\alpha)}\nabla_t(t-a)^{\overline{\alpha-1}} = \frac{H(b,\beta;s)(t-a)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} \\ &= \frac{H(b,\beta;s)\Gamma(t-a+\alpha-2)}{\Gamma(\alpha-1)\Gamma(t-a)}. \end{aligned} \tag{3.6}$$

It follows from Remark 2 and (1) of Theorem 2.3 that $\nabla_t\big[G(b,\beta;t,s)\big] > 0$ implying that $G(b,\beta;t,s)$ is an increasing function of t between $a+1$ and $s-1$. Next, we show that for any fixed $s \in \mathbb{N}_{a+2}^b$, $G(b,\beta;t,s)$ is a decreasing function of t between s and b. Consider the first order nabla difference of $G(b,\beta;t,s)$ with respect to t.

$$\begin{aligned}\nabla_t\big[G(b,\beta;t,s)\big] &= \frac{1}{\Gamma(\alpha)}\Big[H(b,\beta;s)\nabla_t(t-a)^{\overline{\alpha-1}} - \nabla_t(t-s+1)^{\overline{\alpha-1}}\Big] \\ &= \frac{1}{\Gamma(\alpha-1)}\Big[H(b,\beta;s)(t-a)^{\overline{\alpha-2}} - (t-s+1)^{\overline{\alpha-2}}\Big] \\ &= \frac{H(b,\beta;s)(t-a)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}\left[1 - \frac{H(t,1;s)}{H(b,\beta;s)}\right]. \end{aligned} \tag{3.7}$$

Clearly, $\Gamma(\alpha-1) > 0$ and it follows from (3.6) that

$$\frac{H(b,\beta;s)(t-a)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} > 0.$$

We consider two different cases based on α and β.

(i) Suppose $0 \leqslant \beta \leqslant (\alpha-1)$. Since $t \in \mathbb{N}_s^b$ and $s \in \mathbb{N}_{a+2}^b$, from Remark 2 and Lemma 3.7, we obtain

$$H(t,1;s) > 1 \text{ and } H(b,\beta;s) < 1,$$

implying that $\nabla_t \left[G(b,\beta;t,s)\right] < 0$.

(ii) Suppose $(\alpha-1) < \beta \leqslant 1$. Since $t \in \mathbb{N}_s^b$ and $s \in \mathbb{N}_{a+2}^b$, from Lemmas 3.6 and 3.8, we have

$$H(t,1;s) > H(t,\beta;s) > H(b,\beta;s),$$

implying that $\nabla_t \left[G(b,\beta;t,s)\right] < 0$.

Thus, $G(b,\beta;t,s)$ is a decreasing function of t between s and b. Therefore, we have demonstrated that for any fixed $s \in \mathbb{N}_{a+2}^b$, $G(b,\beta;t,s)$ increases from $G(b,\beta;a+1,s)$ to $G(b,\beta;s-1,s)$ and then decreases from $G(b,\beta;s,s)$ to $G(b,\beta;b,s)$. Now, we examine $G(b,\beta;t,s)$ to determine whether the maximum for a fixed t will occur at $(s-1,s)$ or (s,s). We have

$$G(b,\beta;s-1,s) = \frac{H(b,\beta;s)(s-a-1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}$$

and

$$G(b,\beta;s,s) = \frac{H(b,\beta;s)(s-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} - 1.$$

We consider two different cases based on α and β.

(i) Suppose $0 \leqslant \beta \leqslant (\alpha-1)$. Consider

$$\begin{aligned} G(b,\beta;s-1,s) - G(b,\beta;s,s) &= \frac{H(b,\beta;s)}{\Gamma(\alpha)}\left[(s-a-1)^{\overline{\alpha-1}} - (s-a)^{\overline{\alpha-1}}\right] + 1 \\ &= -\frac{H(b,\beta;s)}{\Gamma(\alpha)}\nabla_s\left[(s-a)^{\overline{\alpha-1}}\right] + 1 \\ &= -\frac{H(b,\beta;s)}{\Gamma(\alpha-1)}(s-a)^{\overline{\alpha-2}} + 1. \end{aligned} \tag{3.8}$$

Using Lemma 3.7 in (3.8), we obtain

$$G(b,\beta;s-1,s) - G(b,\beta;s,s) \geqslant -\frac{1}{\Gamma(\alpha-1)}(s-a)^{\overline{\alpha-2}} + 1.$$

Since $(2-\alpha) < (s-a) < 1$, from (4) of Theorem 2.3, we have

$$(s-a)^{\overline{\alpha-2}} < 1^{\overline{\alpha-2}},$$

implying that $G(b,\beta;s,s) \leqslant G(b,\beta;s-1,s)$.

Now we wish to maximize $G(b,\beta;s-1,s)$ for $s \in \mathbb{N}_{a+2}^{b}$. Consider the first order nabla difference of $G(b,\beta;s-1,s)$ with respect to s.

$$\begin{aligned}\nabla_s\left[G(b,\beta;s-1,s)\right] =& \frac{1}{\Gamma(\alpha)(b-a)^{\overline{\alpha-\beta-1}}}\nabla_s\left[(b-s+1)^{\overline{\alpha-\beta-1}}(s-a-1)^{\overline{\alpha-1}}\right]\\ =& \frac{(b-s+2)^{\overline{\alpha-\beta-2}}(s-a-1)^{\overline{\alpha-2}}}{\Gamma(\alpha)(b-a)^{\overline{\alpha-\beta-1}}}\\ &\left[(\alpha-1)(b-s+\alpha-\beta)-(\alpha-\beta-1)(s-a+\alpha-3)\right].\end{aligned}$$

In this expression, $\Gamma(\alpha) > 0$,

$$(b-s+2)^{\overline{\alpha-\beta-2}} = \frac{\Gamma(b-s+\alpha-\beta)}{\Gamma(b-s+2)} > 0,$$

$$(s-a-1)^{\overline{\alpha-2}} = \frac{\Gamma(s-a+\alpha-3)}{\Gamma(s-a-1)} > 0,$$

and

$$(b-a)^{\overline{\alpha-\beta-1}} = \frac{\Gamma(b-a+\alpha-\beta-1)}{\Gamma(b-a)} > 0.$$

The equation $(\alpha-1)(b-s+\alpha-\beta)-(\alpha-\beta-1)(s-a+\alpha-3) = 0$ has the solution

$$s = \frac{(a+b+3)(\alpha-\beta-1)+b\beta}{(2\alpha-2-\beta)},$$

so we consider

$$s = \left\lfloor \frac{(a+b+3)(\alpha-\beta-1)+b\beta}{(2\alpha-2-\beta)} \right\rfloor.$$

If

$$s \leqslant \left\lfloor \frac{(a+b+3)(\alpha-\beta-1)+b\beta}{(2\alpha-2-\beta)} \right\rfloor,$$

the expression $(\alpha-1)(b-s+\alpha-\beta)-(\alpha-\beta-1)(s-a+\alpha-3)$ is positive, and thus the expression $(b-s+1)^{\overline{\alpha-\beta-1}}(s-a-1)^{\overline{\alpha-1}}$ is increasing. If

$$s \geqslant \left\lfloor \frac{(a+b+3)(\alpha-\beta-1)+b\beta}{(2\alpha-2-\beta)} \right\rfloor,$$

the expression $(\alpha-1)(b-s+\alpha-\beta)-(\alpha-\beta-1)(s-a+\alpha-3)$ is negative, and thus the expression $(b-s+1)^{\overline{\alpha-\beta-1}}(s-a-1)^{\overline{\alpha-1}}$ is decreasing. Hence the maximum of the expression $(b-s+1)^{\overline{\alpha-\beta-1}}(s-a-1)^{\overline{\alpha-1}}$ occurs at

$$s = \left\lfloor \frac{(a+b+3)(\alpha-\beta-1)+b\beta}{(2\alpha-2-\beta)} \right\rfloor.$$

Thus, we have

$$\max_{(t,s)\in\mathbb{N}_{a+1}^{b}\times\mathbb{N}_{a+2}^{b}} G(b,\beta;t,s) = \max_{s\in\mathbb{N}_{a+2}^{b}} G(b,\beta;s-1,s) = \Omega. \tag{3.9}$$

(ii) Suppose $(\alpha-1)<\beta\leqslant 1$. First, we maximize $G(b,\beta;s,s)$ for $s\in\mathbb{N}_{a+2}^{b}$. Consider the first order nabla difference of $G(b,\beta;s,s)$ with respect to s.

$$\begin{aligned}\nabla_s\big[G(b,\beta;s,s)\big] &= \frac{1}{\Gamma(\alpha)(b-a)^{\overline{\alpha-\beta-1}}}\nabla_s\big[(b-s+1)^{\overline{\alpha-\beta-1}}(s-a)^{\overline{\alpha-1}}\big]\\ &= \frac{(b-s+2)^{\overline{\alpha-\beta-2}}(s-a)^{\overline{\alpha-2}}}{\Gamma(\alpha)(b-a)^{\overline{\alpha-\beta-1}}}\\ &\quad \big[(\alpha-1)(b-s+\alpha-\beta)-(\alpha-\beta-1)(s-a+\alpha-2)\big].\end{aligned}$$

In this expression, $\Gamma(\alpha)>0$,

$$(b-s+2)^{\overline{\alpha-\beta-2}} = \frac{\Gamma(b-s+\alpha-\beta)}{\Gamma(b-s+2)} > 0,$$

$$(s-a)^{\overline{\alpha-2}} = \frac{\Gamma(s-a+\alpha-2)}{\Gamma(s-a)} > 0,$$

and

$$(b-a)^{\overline{\alpha-\beta-1}} = \frac{\Gamma(b-a+\alpha-\beta-1)}{\Gamma(b-a)} > 0.$$

The equation $(\alpha-1)(b-s+\alpha-\beta)-(\alpha-\beta-1)(s-a+\alpha-2)=0$ has the solution

$$s = \frac{(a+b+3)(\alpha-\beta-1)+b\beta+1}{(2\alpha-2-\beta)},$$

so we consider

$$s = \left\lfloor \frac{(a+b+3)(\alpha-\beta-1)+b\beta+1}{(2\alpha-2-\beta)} \right\rfloor.$$

If

$$s \leqslant \left\lfloor \frac{(a+b+3)(\alpha-\beta-1)+b\beta+1}{(2\alpha-2-\beta)} \right\rfloor,$$

the expression $(\alpha-1)(b-s+\alpha-\beta)-(\alpha-\beta-1)(s-a+\alpha-2)$ is positive, and thus the expression $(b-s+1)^{\overline{\alpha-\beta-1}}(s-a)^{\overline{\alpha-1}}$ is increasing. If

$$s \geqslant \left\lfloor \frac{(a+b+3)(\alpha-\beta-1)+b\beta+1}{(2\alpha-2-\beta)} \right\rfloor,$$

the expression $(\alpha-1)(b-s+\alpha-\beta)-(\alpha-\beta-1)(s-a+\alpha-2)$ is negative, and thus the expression $(b-s+1)^{\overline{\alpha-\beta-1}}(s-a)^{\overline{\alpha-1}}$ is decreasing. Hence the maximum of the expression $(b-s+1)^{\overline{\alpha-\beta-1}}(s-a)^{\overline{\alpha-1}}$ occurs at

$$s = \left\lfloor \frac{(a+b+3)(\alpha-\beta-1)+b\beta+1}{(2\alpha-2-\beta)} \right\rfloor.$$

Thus, from (3.9), we have

$$\max_{(t,s)\in\mathbb{N}_{a+1}^b\times\mathbb{N}_{a+2}^b} G(b,\beta;t,s) = \max\left\{\max_{s\in\mathbb{N}_{a+2}^b} G(b,\beta;s-1,s),\ \max_{s\in\mathbb{N}_{a+2}^b} G(b,\beta;s,s)\right\}$$
$$= \max\{\Omega,\Lambda-1\}.$$

The proof is complete. □

THEOREM 3.10. *The following inequality holds for $G(b,\beta;t,s)$:*

$$\max_{t\in\mathbb{N}_{a+1}^b}\sum_{s=a+2}^{b} G(b,\beta;t,s) = \frac{(b-a-1)^{\overline{\alpha}}}{(\alpha-\beta)\Gamma(\alpha)}.$$

Proof. Consider

$$\begin{aligned}
&\sum_{s=a+2}^{b} G(b,\beta;t,s)\\
&= \sum_{s=a+2}^{t} G(b,\beta;t,s) + \sum_{s=t+1}^{b} G(b,\beta;t,s)\\
&=\frac{1}{\Gamma(\alpha)}\sum_{s=a+2}^{t}\left[\frac{(b-s+1)^{\overline{\alpha-\beta-1}}}{(b-a)^{\overline{\alpha-\beta-1}}}(t-a)^{\overline{\alpha-1}}-(t-s+1)^{\overline{\alpha-1}}\right]\\
&\quad+\frac{1}{\Gamma(\alpha)}\sum_{s=t+1}^{b}\frac{(b-s+1)^{\overline{\alpha-\beta-1}}}{(b-a)^{\overline{\alpha-\beta-1}}}(t-a)^{\overline{\alpha-1}}\\
&=\frac{\Gamma(\alpha-\beta)(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)(b-a)^{\overline{\alpha-\beta-1}}}\sum_{s=a+2}^{b}\frac{(b-s+1)^{\overline{\alpha-\beta-1}}}{\Gamma(\alpha-\beta)}-\sum_{s=a+2}^{t}\frac{(t-s+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\\
&=\frac{(t-a)^{\overline{\alpha-1}}}{(\alpha-\beta)\Gamma(\alpha)(b-a)^{\overline{\alpha-\beta-1}}}(b-a-1)^{\overline{\alpha-\beta}}-\frac{(t-a-1)^{\overline{\alpha}}}{\Gamma(\alpha+1)}\\
&=\frac{(b-a-1)(t-a)^{\overline{\alpha-1}}}{(\alpha-\beta)\Gamma(\alpha)}-\frac{(t-a-1)^{\overline{\alpha}}}{\Gamma(\alpha+1)}.
\end{aligned}$$

We now find the maximum of this expression with respect to $t\in\mathbb{N}_{a+1}^b$. Since

$$\frac{(t-a-1)^{\overline{\alpha}}}{\Gamma(\alpha+1)} = \frac{\Gamma(t-a+\alpha-1)}{\Gamma(t-a-1)\Gamma(\alpha+1)} \geqslant 0,\quad t\in\mathbb{N}_{a+1}^b,$$

we have

$$\max_{t\in\mathbb{N}_{a+1}^b}\sum_{s=a+2}^{b} G(b,\beta;t,s) = \max_{t\in\mathbb{N}_{a+1}^b}\frac{(b-a-1)(t-a)^{\overline{\alpha-1}}}{(\alpha-\beta)\Gamma(\alpha)} = \frac{(b-a-1)^{\overline{\alpha}}}{(\alpha-\beta)\Gamma(\alpha)}.$$

The proof is complete. □

We are now able to formulate a Lyapunov-type inequality for the discrete boundary value problem (1.2).

THEOREM 3.11. *If* (1.2) *has a nontrivial solution, then*

$$\sum_{s=a+2}^{b} |q(s)| \geqslant \begin{cases} \frac{1}{\Omega}, & 0 \leqslant \beta \leqslant (\alpha-1), \\ \frac{1}{\max\{\Omega, \Lambda-1\}}, & (\alpha-1) < \beta \leqslant 1. \end{cases}$$

Proof. Let $\mathfrak{B}$ be the Banach space of functions $u : \mathbb{N}_a^b \to \mathbb{R}$ endowed with norm

$$\|u\| = \max_{t \in \mathbb{N}_a^b} |u(t)|.$$

It follows from Theorem 3.1 that a solution to (1.2) satisfies the equation

$$u(t) = \sum_{s=a+2}^{b} G(b, \beta; t, s) q(s) u(s).$$

Hence,

$$\begin{aligned} \|u\| &= \max_{t \in \mathbb{N}_a^b} \Big| \sum_{s=a+2}^{b} G(b, \beta; t, s) q(s) u(s) \Big| = \max_{t \in \mathbb{N}_{a+1}^b} \Big| \sum_{s=a+2}^{b} G(b, \beta; t, s) q(s) u(s) \Big| \\ &\leqslant \max_{t \in \mathbb{N}_{a+1}^b} \Big[\sum_{s=a+2}^{b} G(b, \beta; t, s) |q(s)| |u(s)| \Big] \leqslant \|u\| \Big[\max_{t \in \mathbb{N}_{a+1}^b} \sum_{s=a+2}^{b} G(b, \beta; t, s) |q(s)| \Big] \\ &\leqslant \|u\| \Big[\max_{(t,s) \in \mathbb{N}_{a+1}^b \times \mathbb{N}_{a+2}^b} G(b, \beta; t, s) \Big] \sum_{s=a+2}^{b} |q(s)|, \end{aligned}$$

or, equivalently,

$$1 \leqslant \Big[\max_{(t,s) \in \mathbb{N}_{a+1}^b \times \mathbb{N}_{a+2}^b} G(b, \beta; t, s) \Big] \sum_{s=a+2}^{b} |q(s)|.$$

An application of Theorem 3.9 yields the result. □

Now, we discuss two applications of Theorem 3.11. First, we obtain a criterion for the nonexistence of nontrivial solutions of (1.2).

THEOREM 3.12. *Assume* $1 < \alpha < 2$ *and*

$$\sum_{s=a+2}^{b} |q(s)| < \begin{cases} \Omega, & 0 \leqslant \beta \leqslant (\alpha-1), \\ \max\{\Omega, \Lambda-1\}, & (\alpha-1) < \beta \leqslant 1. \end{cases} \tag{3.10}$$

Then, the discrete fractional boundary value problem (1.2) *has no nontrivial solution on* $\mathbb{N}_a^b$.

Next, we estimate a lower bound for eigenvalues of the eigenvalue problem corresponding to (1.2).

THEOREM 3.13. *Assume* $1<\alpha<2$ *and* u *is a nontrivial solution of the eigenvalue problem*

$$\begin{cases} \left(\nabla_a^{\alpha} u\right)(t)+\lambda u(t)=0, & t\in\mathbb{N}_{a+2}^{b}, \\ u(a)=0,\ \left(\nabla_a^{\beta} u\right)(b)=0, \end{cases} \tag{3.11}$$

where $u(t)\neq 0$ *for each* $t\in\mathbb{N}_{a+2}^{b-1}$. *Then,*

$$|\lambda|\geqslant\begin{cases} \frac{1}{\Omega}, & 0\leqslant\beta\leqslant(\alpha-1), \\ \frac{1}{\max\{\Omega,\Lambda-1\}}, & (\alpha-1)<\beta\leqslant 1. \end{cases} \tag{3.12}$$

Conclusion

In this article we established a Lyapunov-type inequality for (1.2) using the properties of the corresponding Green's function. This inequality is a generalization of those Lyapunov-type inequalities obtained in [18, 19]. Two applications are provided to illustrate the applicability of established results.

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REFERENCES

[1] T. ABDELJAWAD AND F. M. ATICI, *On the definitions of nabla fractional operators*, Abstr. Appl. Anal., Art. ID 406757, 2012.
[2] K. AHRENDT, L. CASTLE, M. HOLM AND K. YOCHMAN, *Laplace transforms for the nabla-difference operator and a fractional variation of parameters formula*, Commun. Appl. Anal. **16**, 3 (2012), 317–347.
[3] G. A. ANASTASSIOU, *Nabla discrete fractional calculus and nabla inequalities*, Math. Comput. Modelling **51**, 5-6 (2010), 562–571.
[4] F. M. ATICI AND P. W. ELOE, *Discrete fractional calculus with the nabla operator*, Electron. J. Qual. Theory Differ. Equ., Special Edition I, 3 (2009), 12 pp.
[5] F. M. ATICI AND P. W. ELOE, *Two-point boundary value problems for finite fractional difference equations*, J. Difference Equ. Appl. **17**, 4 (2011), 445–456.
[6] A. BRACKINS, *Boundary Value Problems of Nabla Fractional Difference Equations*, Ph.D. Thesis - The University of Nebraska-Lincoln, 2014.
[7] M. BOHNER AND A. PETERSON, *Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2001.
[8] A. CHIDOUH AND D. F. M. TORRES, *Existence of positive solutions to a discrete fractional boundary value problem and corresponding Lyapunov-type inequalities*, Opuscula Math. **38**, 1 (2018), 31–40.
[9] P. W. ELOE, J. W. LYONS AND J. T. NEUGEBAUER, *An ordering on Green's functions for a family of two-point boundary value problems for fractional differential equations*, Commun. Appl. Anal. **19**, (2015), 453–462.
[10] R. A. C. FERREIRA, *Some discrete fractional Lyapunov-type inequalities*, Fract. Differ. Calc. **5**, 1 (2015), 87–92.
[11] K. GHANBARI AND Y. GHOLAMI, *New classes of Lyapunov type inequalities of fractional* Δ*-difference Sturm-Liouville problems with applications*, Bull. Iranian Math. Soc. **43**, 2 (2017), 385–408.
[12] Y. GHOLAMI AND K. GHANBARI, *Coupled systems of fractional* ∇*-difference boundary value problems*, Differ. Equ. Appl. **8**, 4 (2016), 459–470.
[13] C. GOODRICH, *On a fractional boundary value problem with fractional boundary conditions*, Appl. Math. Lett. **25**, 8 (2012), 1101–1105.
[14] C. GOODRICH AND A. C. PETERSON, *Discrete Fractional Calculus*, Springer, Cham, 2015.

[15] C. GOODRICH, *Solutions to a discrete right-focal fractional boundary value problem*, Int. J. Difference Equ. **5**, 2 (2010), 195–216.
[16] A. IKRAM, *Green's Functions and Lyapunov Inequalities for Nabla Caputo Boundary Value Problems*, Ph.D. Thesis - The University of Nebraska-Lincoln, 2018.
[17] JAGAN MOHAN JONNALAGADDA, *Analysis of a system of nonlinear fractional nabla difference equations*, Int. J. Dyn. Syst. Differ. Equ. **5**, 2 (2015), 149–174.
[18] JAGAN MOHAN JONNALAGADDA, *Discrete fractional Lyapunov-type inequalities in nabla sense*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., Manuscript submitted for publication.
[19] JAGAN MOHAN JONNALAGADDA, *Lyapunov-type inequalities for discrete Riemann-Liouville fractional boundary value problems*, Int. J. Difference Equ. **13**, 2 (2018), 85–103.
[20] JAGAN MOHAN JONNALAGADDA, *On two-point Riemann-Liouville type nabla fractional boundary value problems*, Adv. Dyn. Syst. Appl. **13**, 2 (2018), 141–166.
[21] W. G. KELLEY AND A. C. PETERSON, *Difference Equations*, Academic Press, San Diego, 2001.
[22] A. A. KILBAS, H. M. SRIVASTAVA AND J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Amsterdam, 2006.
[23] I. PODLUBNY, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

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