Fractional Differential Calculus
Volume 9, Number 1 (2019), 163–186 doi:10.7153/fdc-2019-09-12

# SOME INEQUALITIES FOR THE GENERALIZED $k$–$g$–FRACTIONAL INTEGRALS OF CONVEX FUNCTIONS

SILVESTRU SEVER DRAGOMIR

(*Communicated by M. Kirane*)

*Abstract.* Let $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the $k$-$g$-*left-sided fractional integral* of $f$ by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ x\in(a,b]$$

and the $k$-$g$-*right-sided fractional integral* of $f$ by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ x\in[a,b),$$

where the kernel $k$ is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval.

In this paper we establish some trapezoid and Ostrowski type inequalities for the $k$-$g$-fractional integrals of convex functions. Applications for Hermite-Hadamard type inequalities for generalized $g$-means and examples for Riemann-Liouville and exponential fractional integrals are also given.

## 1. Introduction

Assume that the kernel $k$ is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. We define the function $K:[0,\infty)\to\mathbb{C}$ by

$$K(t):=\begin{cases}\int_0^t k(s)\,ds \text{ if } 0<t,\\[2ex] 0 \text{ if } t=0.\end{cases}$$

As a simple example, if $k(t)=t^{\alpha-1}$ then for $\alpha\in(0,1)$ the function $k$ is defined on $(0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$. If $\alpha\geqslant 1$, then $k$ is defined on $[0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$.

*Mathematics subject classification* (2010): 26D15, 26A51, 26D07, 26A33.

*Keywords and phrases*: Generalized Riemann-Liouville fractional integrals, Hadamard fractional integrals, functions of bounded variation, Ostrowski type inequalities, trapezoid inequalities.


Paper FDC-09-12

Let $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the $k$-$g$-*left-sided fractional integral* of $f$ by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))g'(t)f(t)\,dt,\ x\in(a,b] \tag{1}$$

and the $k$-$g$-*right-sided fractional integral* of $f$ by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))g'(t)f(t)\,dt,\ x\in[a,b). \tag{2}$$

If we take $k(t)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, where $\Gamma$ is the *Gamma function*, then

$$\begin{aligned}S_{k,g,a+}f(x)&=\frac{1}{\Gamma(\alpha)}\int_a^x[g(x)-g(t)]^{\alpha-1}g'(t)f(t)\,dt\\&=:I^{\alpha}_{a+,g}f(x),\ a<x\leqslant b\end{aligned} \tag{3}$$

and

$$S_{k,g,b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b[g(t)-g(x)]^{\alpha-1}g'(t)f(t)\,dt=:I^{\alpha}_{b-,g}f(x),\ a\leqslant x<b, \tag{4}$$

which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function $f$ with respect to another function $g$ on $[a,b]$ as defined in [23, p. 100].

For $g(t)=t$ in (4) we have the classical *Riemann-Liouville fractional integrals* while for the logarithmic function $g(t)=\ln t$ we have the *Hadamard fractional integrals* [23, p. 111]

$$H^{\alpha}_{a+}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0\leqslant a<x\leqslant b \tag{5}$$

and

$$H^{\alpha}_{b-}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b\left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0\leqslant a<x<b. \tag{6}$$

One can consider the function $g(t)=-t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R^{\alpha}_{a+}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_a^x\frac{f(t)\,dt}{(x-t)^{1-\alpha}t^{\alpha+1}},\ 0\leqslant a<x\leqslant b \tag{7}$$

and

$$R^{\alpha}_{b-}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_x^b\frac{f(t)\,dt}{(t-x)^{1-\alpha}t^{\alpha+1}},\ 0\leqslant a<x<b. \tag{8}$$

Also, for $g(t)=\exp(\beta t)$, $\beta>0$, we can consider the *"$\beta$-Exponential fractional integrals"*

$$E^{\alpha}_{a+,\beta}f(x):=\frac{\beta}{\Gamma(\alpha)}\int_a^x[\exp(\beta x)-\exp(\beta t)]^{\alpha-1}\exp(\beta t)f(t)\,dt, \tag{9}$$

for $a < x \leqslant b$ and

$$E^{\alpha}_{b-,\beta} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_x^b [\exp(\beta t) - \exp(\beta x)]^{\alpha-1} \exp(\beta t) f(t)\, dt, \tag{10}$$

for $a \leqslant x < b$.

If we take $g(t) = t$ in (1) and (2), then we can consider the following $k$*-fractional integrals*

$$S_{k,a+} f(x) = \int_a^x k(x-t) f(t)\, dt,\ x \in (a,b] \tag{11}$$

and

$$S_{k,b-} f(x) = \int_x^b k(t-x) f(t)\, dt,\ x \in [a,b). \tag{12}$$

In [26], Raina studied a class of functions defined formally by

$$\mathscr{F}^{\sigma}_{\rho,\lambda}(x) := \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k,\ \ |x| < R, \text{ with } R > 0, \tag{13}$$

for $\rho,\ \lambda > 0$ where the coefficients $\sigma(k)$ generate a bounded sequence of positive real numbers. With the help of (13), Raina defined the following left-sided fractional integral operator

$$\mathscr{J}^{\sigma}_{\rho,\lambda,a+;w} f(x) := \int_a^x (x-t)^{\lambda-1} \mathscr{F}^{\sigma}_{\rho,\lambda}\left(w(x-t)^{\rho}\right) f(t)\, dt,\ x > a, \tag{14}$$

where $\rho,\ \lambda > 0$, $w \in \mathbb{R}$ and $f$ is such that the integral on the right side exists.

In [1], the right-sided fractional operator was also introduced as

$$\mathscr{J}^{\sigma}_{\rho,\lambda,b-;w} f(x) := \int_x^b (t-x)^{\lambda-1} \mathscr{F}^{\sigma}_{\rho,\lambda}\left(w(t-x)^{\rho}\right) f(t)\, dt,\ x < b, \tag{15}$$

where $\rho,\ \lambda > 0$, $w \in \mathbb{R}$ and $f$ is such that the integral on the right side exists. Several Ostrowski type inequalities were also established.

We observe that for $k(t) = t^{\lambda-1} \mathscr{F}^{\sigma}_{\rho,\lambda}(wt^{\rho})$ we re-obtain the definitions of (14) and (15) from (11) and (12).

In [24], Kirane and Torebek introduced the following *exponential fractional integrals*

$$\mathscr{T}^{\alpha}_{a+} f(x) := \frac{1}{\alpha} \int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\} f(t)\, dt,\ x > a \tag{16}$$

and

$$\mathscr{T}^{\alpha}_{b-} f(x) := \frac{1}{\alpha} \int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\} f(t)\, dt,\ x < b, \tag{17}$$

where $\alpha \in (0,1)$.

We observe that for $k(t) = \frac{1}{\alpha} \exp\left(-\frac{1-\alpha}{\alpha} t\right)$, $t \in \mathbb{R}$ we re-obtain the definitions of (16) and (17) from (11) and (12).

Let $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. We can define the more general exponential fractional integrals

$$\mathscr{T}_{g,a+}^{\alpha} f(x) := \frac{1}{\alpha} \int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(g(x)-g(t))\right\} g'(t) f(t)\,dt,\ x>a \tag{18}$$

and

$$\mathscr{T}_{g,b-}^{\alpha} f(x) := \frac{1}{\alpha} \int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(g(t)-g(x))\right\} g'(t) f(t)\,dt,\ x<b, \tag{19}$$

where $\alpha \in (0,1)$.

Let $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. Assume that $\alpha > 0$. We can also define the *logarithmic fractional integrals*

$$\mathscr{L}_{g,a+}^{\alpha} f(x) := \int_a^x (g(x)-g(t))^{\alpha-1} \ln(g(x)-g(t))\, g'(t) f(t)\,dt, \tag{20}$$

for $0 < a < x \leqslant b$ and

$$\mathscr{L}_{g,b-}^{\alpha} f(x) := \int_x^b (g(t)-g(x))^{\alpha-1} \ln(g(t)-g(x))\, g'(t) f(t)\,dt, \tag{21}$$

for $0 < a \leqslant x < b$, where $\alpha > 0$. These are obtained from (11) and (12) for the kernel $k(t) = t^{\alpha-1}\ln t,\ t > 0$.

For $\alpha = 1$ we get

$$\mathscr{L}_{g,a+} f(x) := \int_a^x \ln(g(x)-g(t))\, g'(t) f(t)\,dt,\ 0<a<x\leqslant b \tag{22}$$

and

$$\mathscr{L}_{g,b-} f(x) := \int_x^b \ln(g(t)-g(x))\, g'(t) f(t)\,dt,\ 0<a\leqslant x<b. \tag{23}$$

For $g(t) = t$, we have the simple forms

$$\mathscr{L}_{a+}^{\alpha} f(x) := \int_a^x (x-t)^{\alpha-1} \ln(x-t) f(t)\,dt,\ 0<a<x\leqslant b, \tag{24}$$

$$\mathscr{L}_{b-}^{\alpha} f(x) := \int_x^b (t-x)^{\alpha-1} \ln(t-x) f(t)\,dt,\ 0<a\leqslant x<b, \tag{25}$$

$$\mathscr{L}_{a+} f(x) := \int_a^x \ln(x-t) f(t)\,dt,\ 0<a<x\leqslant b \tag{26}$$

and

$$\mathscr{L}_{b-} f(x) := \int_x^b \ln(t-x) f(t)\,dt,\ 0<a\leqslant x<b. \tag{27}$$

Recall the classical Riemann-Liouville fractional integrals defined for $\alpha > 0$ by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt$$

for $a < x \leqslant b$ and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt,$$

for $a \leqslant x < b$, where $\Gamma$ is the *Gamma function*. For $\alpha = 0$, they are defined as

$$J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x) \text{ for } x \in (a,b).$$

In the recent paper [17] we obtained the following results for convex functions and the classical Riemann-Liouville fractional integrals:

THEOREM 1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function and $x \in (a,b)$, then we have the inequalities*

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+2)} \left[ f'_{+}(x)(b-x)^{\alpha+1} - f'_{-}(x)(x-a)^{\alpha+1} \right] \\ \leqslant &\frac{1}{\Gamma(\alpha+1)} \left[ (x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b) \right] - J_{a+}^{\alpha} f(x) - J_{b-}^{\alpha} f(x) \\ \leqslant &\frac{1}{\Gamma(\alpha+2)} \left[ f'_{-}(b)(b-x)^{\alpha+1} - f'_{+}(a)(x-a)^{\alpha+1} \right] \end{aligned} \tag{28}$$

*and*

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+2)} \left[ f'_{+}(x)(b-x)^{\alpha+1} - f'_{-}(x)(x-a)^{\alpha+1} \right] \\ \leqslant &J_{x-}^{\alpha} f(a) + J_{x+}^{\alpha} f(b) - \frac{1}{\Gamma(\alpha+1)} \left[ (x-a)^{\alpha} + (b-x)^{\alpha} \right] f(x) \\ \leqslant &\frac{1}{\Gamma(\alpha+2)} \left[ f'_{-}(b)(b-x)^{\alpha+1} - f'_{+}(a)(x-a)^{\alpha+1} \right], \end{aligned} \tag{29}$$

*where $f'_{\pm}(\cdot)$ are the lateral derivatives of $f$.*

In particular, we have:

COROLLARY 1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function, then we have the inequalities*

$$\begin{aligned} 0 &\leqslant \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)} \left[ f'_{+}\left(\frac{a+b}{2}\right) - f'_{-}\left(\frac{a+b}{2}\right) \right] (b-a)^{\alpha+1} \\ &\leqslant \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} \frac{f(a)+f(b)}{2} (b-a)^{\alpha} - J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right) - J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right) \\ &\leqslant \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)} \left[ f'_{-}(b) - f'_{+}(a) \right] (b-a)^{\alpha+1}, \end{aligned} \tag{30}$$

$$0 \leqslant \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right](b-a)^{\alpha+1} \tag{31}$$
$$\leqslant J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)(b-a)^{\alpha}$$
$$\leqslant \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[f'_{-}(b)-f'_{+}(a)\right](b-a)^{\alpha+1}$$

*and*

$$0 \leqslant \frac{1}{\Gamma(\alpha+1)}\frac{f(b)+f(a)}{2}(b-a)^{\alpha}-\frac{J^{\alpha}_{b-}f(a)+J^{\alpha}_{a+}f(b)}{2} \tag{32}$$
$$\leqslant \frac{2^{\alpha}-1}{2^{\alpha+1}\Gamma(\alpha+2)}\left(f'_{-}(b)-f'_{+}(a)\right)(b-a)^{\alpha+1}.$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [2]-[18], [21]-[34] and the references therein.

In this paper we establish some trapezoid and Ostrowski type inequalities for the $k$-$g$-fractional integrals of convex functions. Applications for Hermite-Hadamard type inequalities for generalized $g$-means and examples for Riemann-Liouville and exponential fractional integrals are also given.

## 2. Some identities

For $k$ and $g$ as at the beginning of Introduction, we consider the mixed operator

$$S_{k,g,a+,b-}f(x) \tag{33}$$
$$:=\frac{1}{2}\left[S_{k,g,a+}f(x)+S_{k,g,b-}f(x)\right]$$
$$=\frac{1}{2}\left[\int_a^x k(g(x)-g(t))g'(t)f(t)\,dt+\int_x^b k(g(t)-g(x))g'(t)f(t)\,dt\right],$$

for the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$ and $x\in(a,b)$.

Observe that

$$S_{k,g,x+}f(b)=\int_x^b k(g(b)-g(t))g'(t)f(t)\,dt,\; x\in[a,b) \tag{34}$$

and

$$S_{k,g,x-}f(a)=\int_a^x k(g(t)-g(a))g'(t)f(t)\,dt,\; x\in(a,b]. \tag{35}$$

We can define also the mixed operator

$$\breve{S}_{k,g,a+,b-}f(x) \tag{36}$$
$$:=\frac{1}{2}\left[S_{k,g,x+}f(b)+S_{k,g,x-}f(a)\right]$$
$$=\frac{1}{2}\left[\int_x^b k(g(b)-g(t))g'(t)f(t)\,dt+\int_a^x k(g(t)-g(a))g'(t)f(t)\,dt\right],$$

for any $x \in (a,b)$.

The following two parameters representation for the operators $S_{k,g,a+,b-}$ and $\check{S}_{k,g,a+,b-}$ hold [20]:

LEMMA 1. *Assume that the kernel $k$ is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval and $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. If $f:[a,b]\to\mathbb{C}$ is absolutely continuous on $[a,b]$, then we have for $x\in(a,b)$ that*

$$S_{k,g,a+,b-}f(x) = \frac{1}{2}\left[K(g(x)-g(a))f(a)+K(g(b)-g(x))f(b)\right]$$
$$+\frac{1}{2}\lambda\int_a^x K(g(x)-g(t))\,dt-\frac{1}{2}\gamma\int_x^b K(g(t)-g(x))\,dt$$
$$+\frac{1}{2}\int_a^x K(g(x)-g(t))\left[f'(t)-\lambda\right]dt+\frac{1}{2}\int_x^b K(g(t)-g(x))\left[\gamma-f'(t)\right]dt \tag{37}$$

*and*

$$\check{S}_{k,g,a+,b-}f(x) = \frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)$$
$$+\frac{1}{2}\gamma\int_x^b K(g(b)-g(t))\,dt-\frac{1}{2}\lambda\int_a^x K(g(t)-g(a))\,dt$$
$$+\frac{1}{2}\int_x^b K(g(b)-g(t))\left[f'(t)-\gamma\right]dt+\frac{1}{2}\int_a^x K(g(t)-g(a))\left[\lambda-f'(t)\right]dt, \tag{38}$$

*for any $\lambda,\ \gamma\in\mathbb{C}$.*

*Proof.* Using the integration by parts formula, we have

$$\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt \tag{39}$$
$$=-\int_a^x \left[K(g(x)-g(t))\right]' f(t)\,dt$$
$$=-\left[K(g(x)-g(t))\,f(t)\big|_a^x-\int_a^x K(g(x)-g(t))\,f'(t)\,dt\right]$$
$$=K(g(x)-g(a))\,f(a)+\int_a^x K(g(x)-g(t))\,f'(t)\,dt$$

and

$$\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt \tag{40}$$
$$=\int_x^b \left[K(g(t)-g(x))\right]' f(t)\,dt$$
$$=\left[K(g(t)-g(x))\right]f(t)\big|_x^b-\int_x^b \left[K(g(t)-g(x))\right]f'(t)\,dt$$
$$=\left[K(g(b)-g(x))\right]f(b)-\int_x^b \left[K(g(t)-g(x))\right]f'(t)\,dt,$$

for any $x \in (a,b)$.

From (39) and (40) we get

$$\int_a^x k(g(x)-g(t))g'(t)f(t)\,dt \tag{41}$$
$$=K(g(x)-g(a))f(a)+\lambda\int_a^x K(g(x)-g(t))\,dt+\int_a^x K(g(x)-g(t))\left[f'(t)-\lambda\right]dt$$

and

$$\int_x^b k(g(t)-g(x))g'(t)f(t)\,dt \tag{42}$$
$$=[K(g(b)-g(x))]f(b)-\gamma\int_x^b K(g(t)-g(x))\,dt-\int_x^b K(g(t)-g(x))\left[f'(t)-\gamma\right]dt,$$

for any $x \in (a,b)$.

If we add the equalities (41) and (42) and divide by 2 then we get the desired result (37).

Using the integration by parts formula, we have

$$\begin{aligned}\int_x^b k(g(b)-g(t))g'(t)f(t)\,dt &\\ =-\int_x^b [K(g(b)-g(t))]'f(t)\,dt &\\ =-\left[K(g(b)-g(t))f(t)|_x^b-\int_x^b K(g(b)-g(t))f'(t)\,dt\right] &\\ =K(g(b)-g(x))f(x)+\int_x^b K(g(b)-g(t))f'(t)\,dt &\end{aligned} \tag{43}$$

and

$$\int_a^x k(g(t)-g(a))g'(t)f(t)\,dt \tag{44}$$
$$=\int_a^x [K(g(t)-g(a))]'f(t)\,dt=K(g(t)-g(a))f(t)|_a^x-\int_a^x K(g(t)-g(a))f'(t)\,dt$$
$$=K(g(x)-g(a))f(x)-\int_a^x K(g(t)-g(a))f'(t)\,dt,$$

for any $x \in (a,b)$.

From (43) and (44) we have

$$\int_x^b k(g(b)-g(t))g'(t)f(t)\,dt \tag{45}$$
$$=K(g(b)-g(x))f(x)+\gamma\int_x^b K(g(b)-g(t))\,dt+\int_x^b K(g(b)-g(t))\left[f'(t)-\gamma\right]dt$$

and

$$\begin{aligned}&\int_a^x k(g(t)-g(a))\,g'(t)\,f(t)\,dt \qquad (46)\\ =&K(g(x)-g(a))\,f(x)-\lambda\int_a^x K(g(t)-g(a))\,dt-\int_a^x K(g(t)-g(a))\left[f'(t)-\lambda\right]dt,\end{aligned}$$

for any $x\in(a,b)$.

If we add the equalities (45) and (46) and divide by $2$ then we get the desired result (38). $\square$

If $g$ is a function which maps an interval $I$ of the real line to the real numbers, and is both continuous and injective then we can define the *$g$-mean of two numbers* $a$, $b\in I$ as

$$M_g(a,b):=g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent $p$*. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right),$$

the *LogMeanExp function*.

Using the $g$-mean of two numbers we can introduce

$$\begin{aligned}P_{k,g,a+,b-}f:=&S_{k,g,a+,b-}f\left(M_g(a,b)\right) \qquad (47)\\ =&\frac{1}{2}\int_a^{M_g(a,b)} k\left(\frac{g(a)+g(b)}{2}-g(t)\right)g'(t)\,f(t)\,dt\\ &+\frac{1}{2}\int_{M_g(a,b)}^b k\left(g(t)-\frac{g(a)+g(b)}{2}\right)g'(t)\,f(t)\,dt\end{aligned}$$

and

$$\begin{aligned}\check{P}_{k,g,a+,b-}f:=&\check{S}_{k,g,a+,b-}f\left(M_g(a,b)\right) \qquad (48)\\ =&\frac{1}{2}\int_{M_g(a,b)}^b k(g(b)-g(t))\,g'(t)\,f(t)\,dt\\ &+\frac{1}{2}\int_a^{M_g(a,b)} k(g(t)-g(a))\,g'(t)\,f(t)\,dt.\end{aligned}$$

COROLLARY 2. *With the assumptions of Lemma 1 we have*

$$
\begin{aligned}
P_{k,g,a+,b-}f =& K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(a)+f(b)}{2}+\frac{1}{2}\lambda\int_a^{M_g(a,b)} K\left(\frac{g(a)+g(b)}{2}-g(t)\right)dt \\
& -\frac{1}{2}\gamma\int_{M_g(a,b)}^b K\left(g(t)-\frac{g(a)+g(b)}{2}\right)dt \\
& +\frac{1}{2}\int_a^{M_g(a,b)} K\left(\frac{g(a)+g(b)}{2}-g(t)\right)\left[f'(t)-\lambda\right]dt \\
& +\frac{1}{2}\int_{M_g(a,b)}^b K\left(g(t)-\frac{g(a)+g(b)}{2}\right)\left[\gamma-f'(t)\right]dt \qquad (49)
\end{aligned}
$$

*and*

$$
\begin{aligned}
\breve{P}_{k,g,a+,b-}f =& K\left(\frac{g(b)-g(a)}{2}\right)f(M_g(a,b)) \\
& +\frac{1}{2}\left[\gamma\int_{M_g(a,b)}^b K(g(b)-g(t))\,dt-\lambda\int_a^{M_g(a,b)} K(g(t)-g(a))\,dt\right] \\
& +\frac{1}{2}\int_{M_g(a,b)}^b K(g(b)-g(t))\left[f'(t)-\gamma\right]dt \\
& +\frac{1}{2}\int_a^{M_g(a,b)} K(g(t)-g(a))\left[\lambda-f'(t)\right]dt, \qquad (50)
\end{aligned}
$$

*for any* $\lambda,\ \gamma\in\mathbb{C}$.

For $x=\frac{a+b}{2}$ we can consider

$$
\begin{aligned}
M_{k,g,a+,b-}f :=& S_{k,g,a+,b-}f\left(\frac{a+b}{2}\right) \qquad (51) \\
=& \frac{1}{2}\int_a^{\frac{a+b}{2}} k\left(g\left(\frac{a+b}{2}\right)-g(t)\right)g'(t)f(t)\,dt \\
& +\frac{1}{2}\int_{\frac{a+b}{2}}^b k\left(g(t)-g\left(\frac{a+b}{2}\right)\right)g'(t)f(t)\,dt
\end{aligned}
$$

and

$$
\begin{aligned}
\breve{M}_{k,g,a+,b-}f :=& \breve{S}_{k,g,a+,b-}f\left(\frac{a+b}{2}\right) \qquad (52) \\
=& \frac{1}{2}\int_{\frac{a+b}{2}}^b k(g(b)-g(t))g'(t)f(t)\,dt+\frac{1}{2}\int_a^{\frac{a+b}{2}} k(g(t)-g(a))g'(t)f(t)\,dt.
\end{aligned}
$$

We have the mid-point representation as well:

COROLLARY 3. *With the assumptions of Lemma 1 we have*

$$\begin{aligned}
&M_{k,g,a+,b-}f \\
=&\frac{1}{2}\left[K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)f(a)+K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)f(b)\right] \\
&+\frac{1}{2}\left[\lambda\int_a^{\frac{a+b}{2}}K\left(g\left(\frac{a+b}{2}\right)-g(t)\right)dt-\gamma\int_{\frac{a+b}{2}}^{b}K\left(g(t)-g\left(\frac{a+b}{2}\right)\right)dt\right] \\
&+\frac{1}{2}\int_a^{\frac{a+b}{2}}K\left(g\left(\frac{a+b}{2}\right)-g(t)\right)\left[f'(t)-\lambda\right]dt \\
&+\frac{1}{2}\int_{\frac{a+b}{2}}^{b}K\left(g(t)-g\left(\frac{a+b}{2}\right)\right)\left[\gamma-f'(t)\right]dt
\end{aligned}\tag{53}$$

*and*

$$\begin{aligned}
&\breve{M}_{k,g,a+,b-}f \\
=&\frac{1}{2}\left[K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)+K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right]f\left(\frac{a+b}{2}\right) \\
&+\frac{1}{2}\left[\gamma\int_{\frac{a+b}{2}}^{b}K(g(b)-g(t))\,dt-\lambda\int_a^{\frac{a+b}{2}}K(g(t)-g(a))\,dt\right] \\
&+\frac{1}{2}\int_{\frac{a+b}{2}}^{b}K(g(b)-g(t))\left[f'(t)-\gamma\right]dt+\frac{1}{2}\int_a^{\frac{a+b}{2}}K(g(t)-g(a))\left[\lambda-f'(t)\right]dt,
\end{aligned}\tag{54}$$

*for any* $\lambda,\ \gamma\in\mathbb{C}$.

## 3. Trapezoid type inequality for convex functions

We have the following trapezoid type inequality for convex functions:

THEOREM 2. *Assume that the kernel $k$ is defined either on $(0,\infty)$ or on $[0,\infty)$ with nonnegative values and integrable on any finite subinterval and $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. If $f:[a,b]\to\mathbb{R}$ is a continuous convex function on $[a,b]$, then we have*

$$\begin{aligned}
&\frac{1}{2}\left[f'_+(x)\int_x^b K(g(t)-g(x))\,dt-f'_-(x)\int_a^x K(g(x)-g(t))\,dt\right] \\
\leqslant&\frac{1}{2}\left[K(g(x)-g(a))f(a)+K(g(b)-g(x))f(b)\right]-S_{k,g,a+,b-}f(x) \\
\leqslant&\frac{1}{2}\left[f'_-(b)\int_x^b K(g(t)-g(x))\,dt-f'_+(a)\int_a^x K(g(x)-g(t))\,dt\right],
\end{aligned}\tag{55}$$

*for any* $x\in(a,b)$.

*Proof.* Since $f:[a,b]\to\mathbb{R}$ is a continuous convex function on $[a,b]$, then the lateral derivatives $f'_{\pm}$ exist on $(a,b)$ and they are equal except at most a countably subset of $(a,b)$. Also $f'_{+}(a)$ and $f'_{-}(b)$ exist and we have $f'_{+}(a)\leqslant f'_{-}(t)\leqslant f'_{+}(t)\leqslant f'_{-}(b)$ for any $t\in(a,b)$.

Observe that by the positivity of the kernel $k$ we have $K(g(x)-g(t))\geqslant 0$ for $t\in(a,x)$ and $K(g(t)-g(x))\geqslant 0$ for $t\in(x,b)$.

If we use the equality (37) for $\lambda=f'_{+}(a)$ and $\gamma=f'_{-}(b)$, then we have for $x\in(a,b)$ that

$$\begin{aligned}
S_{k,g,a+,b-}f(x)=&\frac{1}{2}\left[K(g(x)-g(a))f(a)+K(g(b)-g(x))f(b)\right]\\
&+\frac{1}{2}f'_{+}(a)\int_a^x K(g(x)-g(t))\,dt-\frac{1}{2}f'_{-}(b)\int_x^b K(g(t)-g(x))\,dt\\
&+\frac{1}{2}\int_a^x K(g(x)-g(t))\left[f'(t)-f'_{+}(a)\right]dt\\
&+\frac{1}{2}\int_x^b K(g(t)-g(x))\left[f'_{-}(b)-f'(t)\right]dt\\
\geqslant&\frac{1}{2}\left[K(g(x)-g(a))f(a)+K(g(b)-g(x))f(b)\right]\\
&+\frac{1}{2}f'_{+}(a)\int_a^x K(g(x)-g(t))\,dt-\frac{1}{2}f'_{-}(b)\int_x^b K(g(t)-g(x))\,dt,
\end{aligned}$$

which proves the second part of (55).

If we use the equality (37) for $\lambda=f'_{-}(x)$ and $\gamma=f'_{+}(x)$, then we have for $x\in(a,b)$ that

$$\begin{aligned}
S_{k,g,a+,b-}f(x)=&\frac{1}{2}\left[K(g(x)-g(a))f(a)+K(g(b)-g(x))f(b)\right]\\
&+\frac{1}{2}f'_{-}(x)\int_a^x K(g(x)-g(t))\,dt-\frac{1}{2}f'_{+}(x)\int_x^b K(g(t)-g(x))\,dt\\
&+\frac{1}{2}\int_a^x K(g(x)-g(t))\left[f'(t)-f'_{-}(x)\right]dt\\
&+\frac{1}{2}\int_x^b K(g(t)-g(x))\left[f'_{+}(x)-f'(t)\right]dt\\
\leqslant&\frac{1}{2}\left[K(g(x)-g(a))f(a)+K(g(b)-g(x))f(b)\right]\\
&+\frac{1}{2}f'_{-}(x)\int_a^x K(g(x)-g(t))\,dt-\frac{1}{2}f'_{+}(x)\int_x^b K(g(t)-g(x))\,dt,
\end{aligned}$$

which proves the first part of (55). □

REMARK 1. If the functions is differentiable convex on $(a,b)$, then the first inequality in (55) becomes

$$\frac{1}{2}\left[\int_x^b K(g(t)-g(x))\,dt-\int_a^x K(g(x)-g(t))\,dt\right]f'(x)$$
$$\leqslant\frac{1}{2}\left[K(g(x)-g(a))f(a)+K(g(b)-g(x))f(b)\right]-S_{k,g,a+,b-}f(x), \tag{56}$$

for any $x\in(a,b)$.

COROLLARY 4. *With the assumptions of Theorem 2 we have the Hermite--Hadamard type inequality for the $g$-mean $M_g(a,b)$*

$$\frac{1}{2}\left[f'_+(M_g(a,b))\int_{M_g(a,b)}^b K\left(g(t)-\frac{g(a)+g(b)}{2}\right)dt\right.$$
$$\left.-f'_-(M_g(a,b))\int_a^{M_g(a,b)} K\left(\frac{g(a)+g(b)}{2}-g(t)\right)dt\right]$$
$$\leqslant K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(a)+f(b)}{2}-P_{k,g,a+,b-}f$$
$$\leqslant\frac{1}{2}\left[f'_-(b)\int_{M_g(a,b)}^b K\left(g(t)-\frac{g(a)+g(b)}{2}\right)dt\right.$$
$$\left.-f'_+(a)\int_a^{M_g(a,b)} K\left(\frac{g(a)+g(b)}{2}-g(t)\right)dt\right]. \tag{57}$$

*In particular, if $f$ is differentiable in $M_g(a,b)$, then we have the simpler inequality*

$$\frac{1}{2}f'_-(M_g(a,b))$$
$$\times\left[\int_{M_g(a,b)}^b K\left(g(t)-\frac{g(a)+g(b)}{2}\right)dt-\int_a^{M_g(a,b)} K\left(\frac{g(a)+g(b)}{2}-g(t)\right)dt\right]$$
$$\leqslant K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(a)+f(b)}{2}-P_{k,g,a+,b-}f. \tag{58}$$

We also have:

COROLLARY 5. *With the assumptions of Theorem 2 we have*

$$\frac{1}{2}\left[f'_+\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^b K\left(g(t)-g\left(\frac{a+b}{2}\right)\right)dt\right.$$
$$\left.-f'_-\left(\frac{a+b}{2}\right)\int_a^{\frac{a+b}{2}} K\left(g\left(\frac{a+b}{2}\right)-g(t)\right)dt\right]$$
$$\leqslant\frac{1}{2}\left[K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)f(a)+K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)f(b)\right]-M_{k,g,a+,b-}f$$
$$\leqslant\frac{1}{2}\left[f'_-(b)\int_{\frac{a+b}{2}}^b K\left(g(t)-g\left(\frac{a+b}{2}\right)\right)dt-f'_+(a)\int_a^{\frac{a+b}{2}} K\left(g\left(\frac{a+b}{2}\right)-g(t)\right)dt\right]. \tag{59}$$

*In particular, if $f$ is differentiable in $\frac{a+b}{2}$, then we have the simpler inequality*

$$\begin{aligned} &\frac{1}{2} f'\left(\frac{a+b}{2}\right) \\ &\times\left[\int_{\frac{a+b}{2}}^{b} K\left(g(t)-g\left(\frac{a+b}{2}\right)\right) dt-\int_{a}^{\frac{a+b}{2}} K\left(g\left(\frac{a+b}{2}\right)-g(t)\right) dt\right] \\ \leqslant & \frac{1}{2}\left[K\left(g\left(\frac{a+b}{2}\right)-g(a)\right) f(a)+K\left(g(b)-g\left(\frac{a+b}{2}\right)\right) f(b)\right]-M_{k,g,a+,b-} f. \end{aligned} \tag{60}$$

## 4. Ostrowski type inequalities for convex functions

We also have:

THEOREM 3. *Assume that the kernel $k$ is defined either on $(0,\infty)$ or on $[0,\infty)$ with nonnegative values and integrable on any finite subinterval and $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. If $f:[a,b]\to\mathbb{R}$ is a continuous convex function on $[a,b]$, then we have*

$$\begin{aligned} &\frac{1}{2}\left[f_{+}'(x)\int_{x}^{b} K(g(b)-g(t))\,dt-f_{-}'(x)\int_{a}^{x} K(g(t)-g(a))\,dt\right] \\ \leqslant & \breve{S}_{k,g,a+,b-} f(x)-\frac{1}{2}[K(g(b)-g(x))+K(g(x)-g(a))] f(x) \\ \leqslant & \frac{1}{2}\left[f_{-}'(b)\int_{x}^{b} K(g(b)-g(t))\,dt-f_{+}'(a)\int_{a}^{x} K(g(t)-g(a))\,dt\right], \end{aligned} \tag{61}$$

*for $x\in(a,b)$.*

*Proof.* Observe that by the positivity of the kernel $k$ we have $K(g(b)-g(t))\geqslant 0$ for $t\in(x,b)$ and $K(g(t)-g(a))\geqslant 0$ for $t\in(a,x)$.

Using the identity (38), we have for $\gamma=f_{+}'(x)$ and $\lambda=f_{-}'(x)$ that

$$\begin{aligned} \breve{S}_{k,g,a+,b-} f(x)= & \frac{1}{2}[K(g(b)-g(x))+K(g(x)-g(a))] f(x) \\ & +\frac{1}{2} f_{+}'(x)\int_{x}^{b} K(g(b)-g(t))\,dt-\frac{1}{2} f_{-}'(x)\int_{a}^{x} K(g(t)-g(a))\,dt \\ & +\frac{1}{2}\int_{x}^{b} K(g(b)-g(t))\left[f'(t)-f_{+}'(x)\right] dt \\ & +\frac{1}{2}\int_{a}^{x} K(g(t)-g(a))\left[f_{-}'(x)-f'(t)\right] dt \\ \geqslant & \frac{1}{2}[K(g(b)-g(x))+K(g(x)-g(a))] f(x) \\ & +\frac{1}{2} f_{+}'(x)\int_{x}^{b} K(g(b)-g(t))\,dt-\frac{1}{2} f_{-}'(x)\int_{a}^{x} K(g(t)-g(a))\,dt, \end{aligned}$$

which proves the first inequality in (61).

Using the identity (38), we have for $\gamma = f'_-(b)$ and $\lambda = f'_+(a)$ that

$$\begin{aligned}
\check{S}_{k,g,a+,b-}f(x) =& \frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\\
&+\frac{1}{2}f'_-(b)\int_x^b K(g(b)-g(t))\,dt-\frac{1}{2}f'_+(a)\int_a^x K(g(t)-g(a))\,dt\\
&+\frac{1}{2}\int_x^b K(g(b)-g(t))\left[f'(t)-f'_-(b)\right]dt\\
&+\frac{1}{2}\int_a^x K(g(t)-g(a))\left[f'_+(a)-f'(t)\right]dt\\
\leqslant& \frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\\
&+\frac{1}{2}f'_-(b)\int_x^b K(g(b)-g(t))\,dt-\frac{1}{2}f'_+(a)\int_a^x K(g(t)-g(a))\,dt,
\end{aligned}$$

which proves the second inequality in (61). □

REMARK 2. If the function is differentiable convex on $(a,b)$, then the first inequality in (61) becomes

$$\begin{aligned}
&\frac{1}{2}\left[\int_x^b K(g(b)-g(t))\,dt-\int_a^x K(g(t)-g(a))\,dt\right]f'(x)\\
\leqslant& \check{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x), \qquad (62)
\end{aligned}$$

for any $x\in(a,b)$.

COROLLARY 6. *With the assumptions of Theorem 3 we have the Hermite--Hadamard type inequality for the $g$-mean $M_g(a,b)$*

$$\begin{aligned}
&\frac{1}{2}\left[f'_+(M_g(a,b))\int_{M_g(a,b)}^b K(g(b)-g(t))\,dt-f'_-(M_g(a,b))\int_a^{M_g(a,b)} K(g(t)-g(a))\,dt\right]\\
\leqslant& \check{P}_{k,g,a+,b-}f-K\left(\frac{g(b)-g(a)}{2}\right)f(M_g(a,b))\\
\leqslant& \frac{1}{2}\left[f'_-(b)\int_{M_g(a,b)}^b K(g(b)-g(t))\,dt-f'_+(a)\int_a^{M_g(a,b)} K(g(t)-g(a))\,dt\right]. \qquad (63)
\end{aligned}$$

*In particular, if $f$ is differentiable in $M_g(a,b)$, then we have the simpler inequality*

$$\begin{aligned}
&\frac{1}{2}f'(M_g(a,b))\left[\int_{M_g(a,b)}^b K(g(b)-g(t))\,dt-\int_a^{M_g(a,b)} K(g(t)-g(a))\,dt\right]\\
\leqslant& \check{P}_{k,g,a+,b-}f-K\left(\frac{g(b)-g(a)}{2}\right)f(M_g(a,b)). \qquad (64)
\end{aligned}$$

We also have:

COROLLARY 7. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}
&\frac{1}{2}\left[f_+'\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^{b}K\left(g\left(b\right)-g\left(t\right)\right)dt-f_-'\left(\frac{a+b}{2}\right)\int_{a}^{\frac{a+b}{2}}K\left(g\left(t\right)-g\left(a\right)\right)dt\right]\\
\leqslant&\breve{M}_{k,g,a+,b-}f-\frac{1}{2}\left[K\left(g\left(b\right)-g\left(\frac{a+b}{2}\right)\right)+K\left(g\left(\frac{a+b}{2}\right)-g\left(a\right)\right)\right]f\left(\frac{a+b}{2}\right)\\
\leqslant&\frac{1}{2}\left[f_-'\left(b\right)\int_{\frac{a+b}{2}}^{b}K\left(g\left(b\right)-g\left(t\right)\right)dt-f_+'\left(a\right)\int_{a}^{\frac{a+b}{2}}K\left(g\left(t\right)-g\left(a\right)\right)dt\right]. \qquad (65)
\end{aligned}$$

*In particular, if $f$ is differentiable in $\frac{a+b}{2}$, then we have the simpler inequality*

$$\begin{aligned}
&\frac{1}{2}f'\left(\frac{a+b}{2}\right)\left[\int_{\frac{a+b}{2}}^{b}K\left(g\left(b\right)-g\left(t\right)\right)dt-\int_{a}^{\frac{a+b}{2}}K\left(g\left(t\right)-g\left(a\right)\right)dt\right]\\
\leqslant&\breve{M}_{k,g,a+,b-}f-\frac{1}{2}\left[K\left(g\left(b\right)-g\left(\frac{a+b}{2}\right)\right)+K\left(g\left(\frac{a+b}{2}\right)-g\left(a\right)\right)\right]f\left(\frac{a+b}{2}\right). \qquad (66)
\end{aligned}$$

## 5. Applications for generalized Riemann-Liouville fractional integrals

If we take $k\left(t\right)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, where $\Gamma$ is the *Gamma function*, then

$$S_{k,g,a+}f\left(x\right)=I_{a+,g}^{\alpha}f(x):=\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{x}\left[g\left(x\right)-g\left(t\right)\right]^{\alpha-1}g'\left(t\right)f\left(t\right)dt,$$

for $a<x\leqslant b$ and

$$S_{k,g,b-}f\left(x\right)=I_{b-,g}^{\alpha}f(x):=\frac{1}{\Gamma\left(\alpha\right)}\int_{x}^{b}\left[g\left(t\right)-g\left(x\right)\right]^{\alpha-1}g'\left(t\right)f\left(t\right)dt,$$

for $a\leqslant x<b$, which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function $f$ with respect to another function $g$ on $[a,b]$ as defined in [23, p. 100].

We consider the mixed operators

$$I_{g,a+,b-}^{\alpha}f\left(x\right):=\frac{1}{2}\left[I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)\right] \qquad (67)$$

and

$$\breve{I}_{g,a+,b-}^{\alpha}f\left(x\right):=\frac{1}{2}\left[I_{x+,g}^{\alpha}f\left(b\right)+I_{x-,g}^{\alpha}f(a)\right], \qquad (68)$$

for $x\in\left(a,b\right)$.

We observe that for $\alpha > 0$ we have

$$K(t) = \frac{1}{\Gamma(\alpha)} \int_0^t s^{\alpha-1} ds = \frac{t^\alpha}{\alpha \Gamma(\alpha)} = \frac{t^\alpha}{\Gamma(\alpha+1)},\ t \geqslant 0.$$

Let $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. If $f:[a,b] \to \mathbb{R}$ is a continuous convex function on $[a,b]$, then by Theorem 2 we have the trapezoid type inequalities

$$\begin{aligned} &\frac{1}{2\Gamma(\alpha+1)}\left[f'_+(x)\int_x^b (g(t)-g(x))^\alpha dt - f'_-(x)\int_a^x (g(x)-g(t))^\alpha dt\right] \\ &\leqslant \frac{1}{2\Gamma(\alpha+1)}\left[(g(x)-g(a))^\alpha f(a) + (g(b)-g(x))^\alpha f(b)\right] - I^\alpha_{g,a+,b-} f(x) \\ &\leqslant \frac{1}{2\Gamma(\alpha+1)}\left[f'_-(b)\int_x^b (g(t)-g(x))^\alpha dt - f'_+(a)\int_a^x (g(x)-g(t))^\alpha dt\right], \end{aligned} \tag{69}$$

for $x \in (a,b)$.

In particular, if $f$ is differentiable convex on $(a,b)$, then by the first inequality in (69) we have

$$\begin{aligned} &\frac{1}{2\Gamma(\alpha+1)}\left[\int_x^b (g(t)-g(x))^\alpha dt - \int_a^x (g(x)-g(t))^\alpha dt\right] f'(x) \\ &\leqslant \frac{1}{2\Gamma(\alpha+1)}\left[(g(x)-g(a))^\alpha f(a) + (g(b)-g(x))^\alpha f(b)\right] - I^\alpha_{g,a+,b-} f(x), \end{aligned} \tag{70}$$

for $x \in (a,b)$.

If we take in (69) and (70) $x = M_g(a,b)$, then we get

$$\begin{aligned} &\frac{1}{2\Gamma(\alpha+1)} f'(M_g(a,b)) \\ &\times \left[\int_{M_g(a,b)}^b \left(g(t) - \frac{g(a)+g(b)}{2}\right)^\alpha dt - \int_a^{M_g(a,b)} \left(\frac{g(a)+g(b)}{2} - g(t)\right)^\alpha dt\right] \\ &\leqslant \frac{(g(b)-g(a))^\alpha}{2^\alpha \Gamma(\alpha+1)} \frac{f(a)+f(b)}{2} - I^\alpha_{g,a+,b-} f(M_g(a,b)) \\ &\leqslant \frac{1}{2\Gamma(\alpha+1)}\left[f'_-(b)\int_{M_g(a,b)}^b \left(g(t) - \frac{g(a)+g(b)}{2}\right)^\alpha dt\right. \\ &\left. - f'_+(a)\int_a^{M_g(a,b)} \left(\frac{g(a)+g(b)}{2} - g(t)\right)^\alpha dt\right]. \end{aligned} \tag{71}$$

If we take in (69) and (70) $x=\frac{a+b}{2}$, then we also get

$$
\begin{aligned}
&\frac{1}{2\Gamma(\alpha+1)} f'\left(\frac{a+b}{2}\right)\\
&\times\left[\int_{\frac{a+b}{2}}^{b}\left(g(t)-g\left(\frac{a+b}{2}\right)\right)^{\alpha} dt-\int_{a}^{\frac{a+b}{2}}\left(g\left(\frac{a+b}{2}\right)-g(t)\right)^{\alpha} dt\right]\\
\leqslant&\frac{1}{2\Gamma(\alpha+1)}\times\left[\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\alpha} f(a)+\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\alpha} f(b)\right]\\
&-I_{g,a+,b-}^{\alpha} f\left(\frac{a+b}{2}\right)\\
\leqslant&\frac{1}{2\Gamma(\alpha+1)}\left[f'_{-}(b)\int_{\frac{a+b}{2}}^{b}\left(g(t)-g\left(\frac{a+b}{2}\right)\right)^{\alpha} dt\right.\\
&\left.-f'_{+}(a)\int_{a}^{\frac{a+b}{2}}\left(g\left(\frac{a+b}{2}\right)-g(t)\right)^{\alpha} dt\right]. \qquad (72)
\end{aligned}
$$

Let $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. If $f:[a,b]\to\mathbb{R}$ is a continuous convex function on $[a,b]$, then on making use of Theorem 3 we can state the following Ostrowski type inequality

$$
\begin{aligned}
&\frac{1}{2\Gamma(\alpha+1)}\left[f'_{+}(x)\int_{x}^{b}(g(b)-g(t))^{\alpha} dt-f'_{-}(x)\int_{a}^{x}(g(t)-g(a))^{\alpha} dt\right]\\
\leqslant&\breve{I}_{g,a+,b-}^{\alpha} f(x)-\frac{1}{2\Gamma(\alpha+1)}\left[(g(b)-g(x))^{\alpha}+(g(x)-g(a))^{\alpha}\right] f(x)\\
\leqslant&\frac{1}{2\Gamma(\alpha+1)}\left[f'_{-}(b)\int_{x}^{b}(g(b)-g(t))^{\alpha} dt-f'_{+}(a)\int_{a}^{x}(g(t)-g(a))^{\alpha} dt\right], \qquad (73)
\end{aligned}
$$

for $x\in(a,b)$.

In particular, if $f$ is differentiable convex on $(a,b)$, then by the first inequality in (73) we have

$$
\begin{aligned}
&\frac{1}{2\Gamma(\alpha+1)}\left[\int_{x}^{b}(g(b)-g(t))^{\alpha} dt-\int_{a}^{x}(g(t)-g(a))^{\alpha} dt\right] f'(x)\\
\leqslant&\breve{I}_{g,a+,b-}^{\alpha} f(x)-\frac{1}{2\Gamma(\alpha+1)}\left[(g(b)-g(x))^{\alpha}+(g(x)-g(a))^{\alpha}\right] f(x), \qquad (74)
\end{aligned}
$$

for $x\in(a,b)$.

If we take in (73) and (74) $x = M_g(a,b)$, then we get

$$\begin{aligned}&\frac{1}{2\Gamma(\alpha+1)} f'(M_g(a,b)) \times \left[\int_{M_g(a,b)}^{b} (g(b)-g(t))^{\alpha} dt - \int_{a}^{M_g(a,b)} (g(t)-g(a))^{\alpha} dt\right] \\ \leqslant & \breve{I}^{\alpha}_{g,a+,b-} f(M_g(a,b)) - \frac{(g(b)-g(a))^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)} f(M_g(a,b)) \\ \leqslant & \frac{1}{2\Gamma(\alpha+1)} \left[f'_{-}(b) \int_{M_g(a,b)}^{b} (g(b)-g(t))^{\alpha} dt - f'_{+}(a) \int_{a}^{M_g(a,b)} (g(t)-g(a))^{\alpha} dt\right]. \end{aligned} \tag{75}$$

If we take in (73) and (74) $x = \frac{a+b}{2}$, then we also get

$$\begin{aligned}&\frac{1}{2\Gamma(\alpha+1)} f'\left(\frac{a+b}{2}\right) \times \left[\int_{\frac{a+b}{2}}^{b} (g(b)-g(t))^{\alpha} dt - \int_{a}^{\frac{a+b}{2}} (g(t)-g(a))^{\alpha} dt\right] \\ \leqslant & \breve{I}^{\alpha}_{g,a+,b-} f\left(\frac{a+b}{2}\right) \\ & - \frac{1}{2\Gamma(\alpha+1)} \left[\left(g(b) - g\left(\frac{a+b}{2}\right)\right)^{\alpha} + \left(g\left(\frac{a+b}{2}\right) - g(a)\right)^{\alpha}\right] f\left(\frac{a+b}{2}\right) \\ \leqslant & \frac{1}{2\Gamma(\alpha+1)} \left[f'_{-}(b) \int_{\frac{a+b}{2}}^{b} (g(b)-g(t))^{\alpha} dt - f'_{+}(a) \int_{a}^{\frac{a+b}{2}} (g(t)-g(a))^{\alpha} dt\right]. \end{aligned} \tag{76}$$

If we take in these inequalities $g(t) = t$, we recapture the results for the classical Riemann-Liouville fractional integrals outlined in Introduction.

## 6. Example for an exponential kernel

For $\alpha \in \mathbb{R}$ we consider the kernel $k(t) := \exp(\alpha t)$, $t \in \mathbb{R}$. We have

$$|k(s)| = \exp(\alpha s) \text{ for } s \in \mathbb{R}$$

and

$$K(t) = \frac{\exp(\alpha t) - 1}{\alpha}, \text{ if } t \in \mathbb{R},$$

for $\alpha \neq 0$.

Let $f : [a,b] \to \mathbb{C}$ be an integrable function on $[a,b]$ and $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. Define

$$\begin{aligned}\mathscr{H}^{\alpha}_{g,a+,b-} f(x) =& \frac{1}{2} \int_{x}^{b} \exp[\alpha(g(t)-g(x))] g'(t) f(t) dt \\ &+ \frac{1}{2} \int_{a}^{x} \exp[\alpha(g(x)-g(t))] g'(t) f(t) dt,\end{aligned} \tag{77}$$

for $x \in (a,b)$.

If $g = \ln h$ where $h : [a,b] \to (0,\infty)$ is a strictly increasing function on $(a,b)$, having a continuous derivative $h'$ on $(a,b)$, then we can consider the following operator as well

$$\kappa^{\alpha}_{h,a+,b-} f(x) := \mathscr{H}^{\alpha}_{\ln h,a+,b-} f(x) = \frac{1}{2}\left[\int_x^b \left(\frac{h(t)}{h(x)}\right)^{\alpha} \frac{h'(t)}{h(t)} f(t)\,dt + \int_a^x \left(\frac{h(x)}{h(t)}\right)^{\alpha} \frac{h'(t)}{h(t)} f(t)\,dt\right], \tag{78}$$

for $x \in (a,b)$.

Let $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. If $f : [a,b] \to \mathbb{R}$ is differentiable convex function on $(a,b)$, then by Theorem 2 we have the trapezoid type inequalities

$$\begin{aligned}
&\frac{1}{2} f'(x) \times \left(\int_x^b \frac{\exp(\alpha(g(t)-g(x)))-1}{\alpha} dt - \int_a^x \frac{\exp(\alpha(g(x)-g(t)))-1}{\alpha} dt\right) \\
&\leqslant \frac{1}{2}\left[\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha} f(a) + \frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha} f(b)\right] - \mathscr{H}^{\alpha}_{g,a+,b-} f(x) \\
&\leqslant \frac{1}{2}\left[f'_-(b) \int_x^b \frac{\exp(\alpha(g(t)-g(x)))-1}{\alpha} dt - f'_+(a) \int_a^x \frac{\exp(\alpha(g(x)-g(t)))-1}{\alpha} dt\right],
\end{aligned} \tag{79}$$

for any $x \in (a,b)$.

If $g = \ln h$ where $h : [a,b] \to (0,\infty)$ is a strictly increasing function on $(a,b)$, having a continuous derivative $h'$ on $(a,b)$, then by (79) we get

$$\begin{aligned}
&\frac{1}{2} f'(x) \left(\int_x^b \frac{\left(\frac{h(t)}{h(x)}\right)^{\alpha} - 1}{\alpha} dt - \int_a^x \frac{\left(\frac{h(x)}{h(t)}\right)^{\alpha} - 1}{\alpha} dt\right) \\
&\leqslant \frac{1}{2}\left[\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha} - 1}{\alpha} f(a) + \frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha} - 1}{\alpha} f(b)\right] - \kappa^{\alpha}_{h,a+,b-} f(x) \\
&\leqslant \frac{1}{2}\left[f'_-(b) \int_x^b \frac{\left(\frac{h(t)}{h(x)}\right)^{\alpha} - 1}{\alpha} dt - f'_+(a) \int_a^x \frac{\left(\frac{h(x)}{h(t)}\right)^{\alpha} - 1}{\alpha} dt\right],
\end{aligned} \tag{80}$$

for any $x \in (a,b)$.

Let $f : [a,b] \to \mathbb{C}$ be an integrable function on $[a,b]$ and $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. Also define

$$\breve{\mathscr{H}}^{\alpha}_{g,a+,b-} f(x) := \frac{1}{2}\int_x^b \exp[\alpha(g(b)-g(t))] g'(t) f(t)\,dt + \frac{1}{2}\int_a^x \exp[\alpha(g(t)-g(a))] g'(t) f(t)\,dt, \tag{81}$$

for any $x \in (a,b)$.

If $g = \ln h$ where $h : [a,b] \to (0,\infty)$ is a strictly increasing function on $(a,b)$, having a continuous derivative $h'$ on $(a,b)$, then we can consider the following operator as well

$$\breve{\kappa}_{h,a+,b-}^{\alpha} f(x) := \breve{\mathscr{H}}_{\ln h,a+,b-}^{\alpha} f(x) \tag{82}$$
$$= \frac{1}{2}\left[\int_x^b \left(\frac{h(b)}{h(t)}\right)^{\alpha} \frac{h'(t)}{h(t)} f(t)\,dt + \int_a^x \left(\frac{h(t)}{h(a)}\right)^{\alpha} \frac{h'(t)}{h(t)} f(t)\,dt\right],$$

for any $x \in (a,b)$.

If $f : [a,b] \to \mathbb{R}$ is differentiable convex function on $(a,b)$, then by Theorem 3 we have the Ostrowski type inequalities

$$\frac{1}{2} f'(x)\left[\int_x^b \exp[\alpha(g(b)-g(t))]\,dt - \int_a^x \exp[\alpha(g(t)-g(a))]\,dt\right]$$
$$\leqslant \breve{\mathscr{H}}_{g,a+,b-}^{\alpha} f(x) - \frac{1}{2}\left[\frac{\exp(\alpha(g(b)-g(x))) + \exp(\alpha(g(x)-g(a))) - 2}{\alpha}\right] f(x)$$
$$\leqslant \frac{1}{2}\left[f'_-(b)\int_x^b \exp[\alpha(g(b)-g(t))]\,dt - f'_+(a)\int_a^x \exp[\alpha(g(t)-g(a))]\,dt\right], \tag{83}$$

for any $x \in (a,b)$.

If $g = \ln h$ where $h : [a,b] \to (0,\infty)$ is a strictly increasing function on $(a,b)$, having a continuous derivative $h'$ on $(a,b)$, then by (83) we get

$$\frac{1}{2} f'(x)\left[\int_x^b \left(\frac{h(b)}{h(t)}\right)^{\alpha} dt - \int_a^x \left(\frac{h(t)}{h(a)}\right)^{\alpha} dt\right]$$
$$\leqslant \breve{\kappa}_{h,a+,b-}^{\alpha} f(x) - \frac{1}{2}\left[\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha} + \left(\frac{h(x)}{h(a)}\right)^{\alpha} - 2}{\alpha}\right] f(x)$$
$$\leqslant \frac{1}{2}\left[f'_-(b)\int_x^b \left(\frac{h(b)}{h(t)}\right)^{\alpha} dt - f'_+(a)\int_a^x \left(\frac{h(t)}{h(a)}\right)^{\alpha} dt\right], \tag{84}$$

for any $x \in (a,b)$.

Finally, if we take $x_h := h^{-1}\left(\sqrt{h(a)h(b)}\right) = h^{-1}(G(h(a),h(b))) \in (a,b)$, where

$G$ is the geometric mean, in (80) and (85), then we get

$$
\begin{aligned}
&\frac{1}{2}f'(x_h)\left(\int_{x_h}^{b}\frac{\left(\frac{h(t)}{G(h(a),h(b))}\right)^{\alpha}-1}{\alpha}dt-\int_{a}^{x_h}\frac{\left(\frac{G(h(a),h(b))}{h(t)}\right)^{\alpha}-1}{\alpha}dt\right)\\
&\leqslant\frac{\left(\frac{h(b)}{h(a)}\right)^{\alpha/2}-1}{\alpha}\frac{f(a)+f(b)}{2}-\kappa_{h,a+,b-}^{\alpha}f(x_h)\\
&\leqslant\frac{1}{2}\left[f'_{-}(b)\int_{x_h}^{b}\frac{\left(\frac{h(t)}{G(h(a),h(b))}\right)^{\alpha}-1}{\alpha}dt-f'_{+}(a)\int_{a}^{x_h}\frac{\left(\frac{G(h(a),h(b))}{h(t)}\right)^{\alpha}-1}{\alpha}dt\right] \qquad (85)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2}f'(x_h)\left[\int_{x_h}^{b}\left(\frac{h(b)}{h(t)}\right)^{\alpha}dt-\int_{a}^{x_h}\left(\frac{h(t)}{h(a)}\right)^{\alpha}dt\right]\\
&\leqslant\breve{\kappa}_{h,a+,b-}^{\alpha}f(x_h)-\frac{\left(\frac{h(b)}{h(a)}\right)^{\alpha/2}-1}{\alpha}f(x_h)\\
&\leqslant\frac{1}{2}\left[f'_{-}(b)\int_{x_h}^{b}\left(\frac{h(b)}{h(t)}\right)^{\alpha}dt-f'_{+}(a)\int_{a}^{x_h}\left(\frac{h(t)}{h(a)}\right)^{\alpha}dt\right]. \qquad (86)
\end{aligned}
$$

## REFERENCES


[1] R. P. AGARWAL, M.-J. LUO AND R. K. RAINA, *On Ostrowski type inequalities*, Fasc. Math. **56** (2016), 5–27.

[2] A. AGLIĆ ALJINOVIĆ, *Montgomery identity and Ostrowski type inequalities for Riemann-Liouville fractional integral*, J. Math.**2014**, Art. ID 503195, 6 pp.

[3] T. M. APOSTOL, *Mathematical Analysis*, Second Edition, Addison-Wesley Publishing Company, 1975.

[4] A. O. AKDEMIR, *Inequalities of Ostrowski's type for $m$-and $(\alpha,m)$-logarithmically convex functions via Riemann-Liouville fractional integrals*, J. Comput. Anal. Appl. **16** (2014), no. 2, 375–383.

[5] G. A. ANASTASSIOU, *Fractional representation formulae under initial conditions and fractional Ostrowski type inequalities*, Demonstr. Math. **48** (2015), no. 3, 357–378.

[6] G. A. ANASTASSIOU, *The reduction method in fractional calculus and fractional Ostrowski type inequalities*, Indian J. Math. **56** (2014), no. 3, 333–357.

[7] H. BUDAK, M. Z. SARIKAYA, E. SET, *Generalized Ostrowski type inequalities for functions whose local fractional derivatives are generalized $s$-convex in the second sense*, J. Appl. Math. Comput. Mech. **15**(2016), no. 4, 11–21.

[8] P. CERONE AND S. S. DRAGOMIR, *Midpoint-type rules from an inequalities point of view*, Handbook of analytic-computational methods in applied mathematics, 135–200, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[9] S. S. DRAGOMIR, *The Ostrowski's integral inequality for Lipschitzian mappings and applications*, Comput. Math. Appl. **38** (1999), no. 11–12, 33–37.

[10] S. S. DRAGOMIR, *The Ostrowski integral inequality for mappings of bounded variation*, Bull. Austral. Math. Soc. **60** (1999), No. 3, 495–508.

[11] S. S. DRAGOMIR, *On the midpoint quadrature formula for mappings with bounded variation and applications*, Kragujevac J. Math. **22** (2000), 13–19.

[12] S. S. DRAGOMIR, *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, Math. Ineq. Appl. **4** (2001), No. 1, 59–66. Preprint: RGMIA Res. Rep. Coll. **2** (1999), Art. 7, [Online: http://rgmia.org/papers/v2n1/v2n1-7.pdf].

[13] S. S. DRAGOMIR, *Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation*, Arch. Math. (Basel) **91** (2008), no. 5, 450–460.

[14] S. S. DRAGOMIR, *Refinements of the Ostrowski inequality in terms of the cumulative variation and applications*, Analysis (Berlin) **34** (2014), No. 2, 223–240. Preprint: RGMIA Res. Rep. Coll. **16** (2013), Art. 29 [Online:http://rgmia.org/papers/v16/v16a29.pdf].

[15] S. S. DRAGOMIR, *Ostrowski type inequalities for Lebesgue integral: a survey of recent results*, Australian J. Math. Anal. Appl. Volume **14**, Issue 1, Article 1, pp. 1–287, 2017. [Online http://ajmaa.org/cgi-bin/paper.pl?string=v14n1/V14I1P1.tex].

[16] S. S. DRAGOMIR, *Ostrowski type inequalities for Riemann-Liouville fractional integrals of bounded variation, Hölder and Lipschitzian functions*, Preprint RGMIA Res. Rep. Coll. **20** (2017), Art. 48. [Online http://rgmia.org/papers/v20/v20a48.pdf].

[17] S. S. DRAGOMIR, *Ostrowski and trapezoid type inequalities for Riemann-Liouville fractional integrals of absolutely continuous functions with bounded derivatives*, RGMIA Res. Rep. Coll. **20** (2017), Art. 53. [Online http://rgmia.org/papers/v20/v20a53.pdf].

[18] S. S. DRAGOMIR, *Ostrowski type inequalities for generalized Riemann-Liouville fractional integrals of functions with bounded variation*, RGMIA Res. Rep. Coll. **20** (2017), Art. 58. [Online http://rgmia.org/papers/v20/v20a58.pdf].

[19] S. S. DRAGOMIR, *Further Ostrowski and trapezoid type inequalities for the generalized Riemann-Liouville fractional integrals of functions with bounded variation*, RGMIA Res. Rep. Coll. **20** (2017), Art. 84. [Online http://rgmia.org/papers/v20/v20a84.pdf].

[20] S. S. DRAGOMIR, *Some inequalities for the generalized $k$-$g$-fractional integrals of functions under complex boundedness conditions*, RGMIA Res. Rep. Coll. **20** (2017), Art. 119. [Online http://rgmia.org/papers/v20/v20a119.pdf].

[21] A. GUEZANE-LAKOUD AND F. AISSAOUI, *New fractional inequalities of Ostrowski type*, Transylv. J. Math. Mech. **5** (2013), no. 2, 103–106.

[22] A. KASHURI AND R. LIKO, *Ostrowski type fractional integral inequalities for generalized $(s,m,\varphi)$-preinvex functions*, Aust. J. Math. Anal. Appl. **13** (2016), no. 1, Art. 16, 11 pp.

[23] A. KILBAS, H. M. SRIVASTAVA AND J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006. xvi+523 pp. ISBN: 978-0-444-51832-3; 0-444-51832-0.

[24] M. KIRANE, B. T. TOREBEK, *Hermite-Hadamard, Hermite-Hadamard-Fejer, Dragomir-Agarwal and Pachpatte type Inequalities for convex functions via fractional integrals*, Preprint arXiv:1701.00092.

[25] M. A. NOOR, K. I. NOOR AND S. IFTIKHAR, *Fractional Ostrowski inequalities for harmonic $h$-preinvex functions*, Facta Univ. Ser. Math. Inform. **31** (2016), no. 2, 417–445.

[26] R. K. RAINA, *On generalized Wright's hypergeometric functions and fractional calculus operators*, East Asian Math. J., **21**(2)(2005), 191–203.

[27] M. Z. SARIKAYA AND H. FILIZ, *Note on the Ostrowski type inequalities for fractional integrals*, Vietnam J. Math. **42** (2014), no. 2, 187–190.

[28] M. Z. SARIKAYA AND H. BUDAK, *Generalized Ostrowski type inequalities for local fractional integrals*, Proc. Amer. Math. Soc. **145** (2017), no. 4, 1527–1538.

[29] E. SET, *New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals*, Comput. Math. Appl. 63 (2012), no. 7, 1147–1154.

[30] M. TUNÇ, *On new inequalities for $h$-convex functions via Riemann-Liouville fractional integration*, Filomat **27**:4 (2013), 559–565.

[31] M. TUNÇ, *Ostrowski type inequalities for $m$- and $(\alpha,m)$-geometrically convex functions via Riemann-Louville fractional integrals*, Afr. Mat. **27** (2016), no. 5–6, 841–850.

[32] H. YILDIRIM AND Z. KIRTAY, *Ostrowski inequality for generalized fractional integral and related inequalities*, Malaya J. Mat., **2**(3)(2014), 322–329.

[33] C. YILDIZ, E, ÖZDEMIR AND Z. S. MUHAMET, *New generalizations of Ostrowski-like type inequalities for fractional integrals*, Kyungpook Math. J. **56** (2016), no. 1, 161–172.

[34] H. YUE, *Ostrowski inequality for fractional integrals and related fractional inequalities*, Transylv. J. Math. Mech. **5** (2013), no. 1, 85–89.

(Received March 27, 2019)

*Silvestru Sever Dragomir*
*Mathematics, College of Engineering & Science*
*Victoria University*
*PO Box 14428, Melbourne City, MC 8001, Australia*
*e-mail:* sever.dragomir@vu.edu.au
*DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences*
*School of Computer Science & Applied Mathematics,*
*University of the Witwatersrand*
*Private Bag 3, Johannesburg 2050, South Africa*