

FRACTIONAL RESOLVENT OPERATOR WITH $\alpha \in (0, 1)$ AND APPLICATIONS

JOSÉ PAULO CARVALHO DOS SANTOS

(Communicated by V. Keyantuo)

Abstract. In this paper we study an analytic resolvent family for abstract fractional integro-differential system using the perturbation theory of sectorial operators. We apply this resolvent family on the existence of mild solutions for abstract semilinear Cauchy problem

$$D_t^\alpha u(t) = Au(t) + \int_0^t B(t-s)u(s)ds + f(t, u(t)), t \in (0, \tau),$$

$$u(0) = u_0 \in X,$$

where $D_t^\alpha u$ represents the Caputo derivative of u for $\alpha \in (0, 1)$, $A, (B(t))_{t \geq 0}$ are closed linear operators defined on a common domain which is dense in a Banach space X and f satisfies appropriated conditions. In the end, we apply the ours abstract results in the existence of mild solution of two partial integro-differential systems.

1. Introduction

In this paper we study the existence of a resolvent family for the abstract fractional integro-differential system

$$D_t^\alpha u(t) = Au(t) + \int_0^t B(t-s)u(s)ds, \quad t \geq 0, \quad (1)$$

$$u(0) = u_0, \quad (2)$$

where $A, (B(t))_{t \geq 0}$ are closed linear operators defined on a common domain which is dense in a Banach space $(X, \|\cdot\|)$, and $D_t^\alpha h(t)$ represents the Caputo derivative of h for $\alpha \in (0, 1)$ defined by

$$D_t^\alpha h(t) := \int_0^t g_{1-\alpha}(t-s)h'(s)ds,$$

where $g_{1-\alpha}$ is the Gelfand-Shilov function $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}$, $t > 0$, with $\beta = 1 - \alpha$.

In the past decades, considerable attention has been attracted to the theory of resolvent operator for integro-differential equations. We refer to the book by Gripenberg

Mathematics subject classification (2010): 26A33, 34K30, 35R11, 45N05.

Keywords and phrases: Integro-differential equations, resolvent of operators, fractional differential equation, mild solution.

et. al. [16] for the case where the underlying space X has finite dimension. For abstract integro-differential equations on infinite dimensional spaces, we cite the book by J. Prüss [27] and the papers of Da Prato et al. [8, 7], Grimmer et al. [13, 14, 15], Lunardi [23, 24], Sforza [29] and Dos Santos et al. [9, 10, 11].

With a resolvent family of fractional integro-differential equations it is possible study an existence of solutions for semilinear fractional integro-differential Cauchy problem

$$D_t^\alpha u(t) = Au(t) + \int_0^t B(t-s)u(s)ds + f(t, u(t)), t \in (0, \tau), \tag{3}$$

$$u(0) = u_0 \in X, \tag{4}$$

where f satisfies the appropriate conditions. Regarding the fractional differential equations in spaces of infinite dimension, this problem has been extensive studied, we can mention the pioner thesis of Bajlekova [5] and the works of [17, 18, 19, 21, 30, 31] and references therein. For abstract fractional integro-differential equations in infinite dimension, we suggest the articles Agarwal et al. [1] in the case of $\alpha \in (1, 2)$, the book of Kostić [20], Ponce [26] and Herzallah et al. [12] when $B(t) = a(t)A, t \geq 0$. To the best of the authors' knowledge, the existence of an analytic resolvent operator by perturbation theory for the abstract integro-differential fractional equation (1)-(2) and the existence of mild solutions of (3)-(4), with $\alpha \in (0, 1)$, is a subject that has not been treated in the literature. This is the principal motivation of this paper.

This work has four sections. In Section 2, by perturbations of the sectorial operators and assuming some conditions on family operator $(B(t))_{t \geq 0}$, we prove the existence and qualitative properties of a resolvent operator and the auxiliary resolvent operators for the fractional integro-differential system (1)-(2). In Section 3, the existence of mild solution for the nonhomogeneous equation associated to (3)-(4) is discussed. In the last Section some applications in a partial integro-differential equation of Jeffrey's type, which arise in the theory of heat equation with memory and a partial integro-differential fractional coupled system, are considered.

By $D_t^\alpha h(t)$ we denoted the Caputo derivative of h for $\alpha > 0$, defined by

$$D_t^\alpha h(t) := \int_0^t g_{n-\alpha}(t-s) \frac{d^n}{ds^n} h(s) ds,$$

where n is the smallest integer greater than or equal to α and $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}, t > 0, \beta \geq 0$. These functions satisfy the semigroup property

$$g_\alpha * g_\beta = g_{\alpha+\beta}.$$

If we denote

$$J_t^\alpha f(t) = (g_\alpha * f)(t) = \int_0^t g_\alpha(t-s) f(s) ds, \tag{5}$$

we have

$$D_t^\alpha J_t^\alpha f(t) = f(t), \tag{6}$$

$$J_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}. \tag{7}$$

Applying the properties of the Laplace transform and taking into account that $\widehat{g_\alpha}(\Lambda) = \Lambda^{-\alpha}$, we obtain

$$\widehat{D_t^\alpha} f(\Lambda) = \Lambda^\alpha f(\Lambda) - \sum_{k=0}^{n-1} f^{(k)}(0) \Lambda^{\alpha-1-k}, \tag{8}$$

(see [5, 28] for details.)

Throughout this paper, let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. We denote by $\mathcal{L}(Z, W)$ the space of bounded linear operators from Z into W endowed with norm of operators, and we write simply $\mathcal{L}(Z)$ when $Z = W$. By $\mathbf{R}(Q)$ we denote the range of a map Q and for a closed linear operator $P : D(P) \subseteq Z \rightarrow W$, the notation $[D(P)]$ represents the domain of P endowed with the graph norm, $\|z\|_1 = \|z\|_Z + \|Pz\|_W$, $z \in D(P)$. The notation, $B(x, R)$ and $B[x, R]$ represent the open ball and the closed ball, respectively, with center at x and radius $R > 0$ in X . Let $I \subset \mathbb{R}$, by $C(I, X)$ we denote the space of continuous functions defined on I into X , and $C^1(I, X)$ stands for the space of continuous functions from I to X having continuous derivative. We define the space $C^\alpha(I, X)$, by

$$C^\alpha(I, X) := \{x \in C(I, X) : D_t^\alpha x \in C(I, X)\}.$$

We denote by $L^p(I, X)$ the set of all measurable functions $u(\cdot)$ on I into X such that $\|u(t)\|^p$ is integrable, and its norm is given by $\|u\|_{L^p(I, X)} = (\int_I \|u(t)\|^p)^{\frac{1}{p}}$; similarly, by $L^p_{loc}(\mathbb{R}_+, X)$ we denote the space of the functions belonging $L^p(I, X)$, for any compact set $I \subset \mathbb{R}_+$. When $X = \mathbb{R}^n$, for some n , we denote for simplicity by $C(I), C^1(I), C^\alpha(I), L^p(I)$ and $L^p_{loc}(\mathbb{R}_+)$, respectively. The notation $\rho(P)$ stands for the resolvent set of P and $R(\Lambda, P) = (\Lambda I - P)^{-1}$ is the resolvent operator of P . Furthermore, for appropriate functions $K : [0, \infty) \rightarrow Z$ and $S : [0, \infty) \rightarrow \mathcal{L}(Z, W)$, the notation \widehat{K} denotes the Laplace transform of K , and $S * K$ the convolution between S and K , which is defined by $S * K(t) = \int_0^t S(t-s)K(s)ds$.

2. Fractional resolvent operator

To begin, we introduce the following concept of resolvent operator for the abstract fractional integro-differential problem (1)-(2).

DEFINITION 1. A one parameter family of bounded linear operators $(\mathcal{R}_\alpha(t))_{t \geq 0}$ on X is called a α -resolvent operator of (1)-(2) if the following conditions are verified.

- (a) The function $\mathcal{R}_\alpha(\cdot) : [0, \infty) \rightarrow \mathcal{L}(X)$ is strongly continuous and $\mathcal{R}_\alpha(0)x = x$ for all $x \in X$ and $\alpha \in (0, 1)$.
- (b) For $x \in D(A)$, $\mathcal{R}_\alpha(\cdot)x \in C([0, \infty), [D(A)]) \cap C^\alpha((0, \infty), X)$, and

$$D_t^\alpha \mathcal{R}_\alpha(t)x = A\mathcal{R}_\alpha(t)x + \int_0^t B(t-s)\mathcal{R}_\alpha(s)xds \tag{9}$$

$$= \mathcal{R}_\alpha(t)Ax + \int_0^t \mathcal{R}_\alpha(t-s)B(s)xds, \tag{10}$$

for every $t \geq 0$.

In this work we always assume that the following conditions are verified.

(H1) The operator $A : D(A) \subseteq X \rightarrow X$ is a closed linear operator with $[D(A)]$ dense in X , for some $\phi \in (\frac{\pi}{2}, \pi)$ there is positive constants $C_0 = C_0(\phi)$ such that $\Lambda \in \rho(A)$ for each

$$\Sigma_{0,\phi} = \{\Lambda \in \mathbb{C} : |\arg(\Lambda)| < \phi\} \subset \rho(A),$$

and $\|R(\Lambda, A)\| \leq \frac{C_0}{|\Lambda|}$ for all $\Lambda \in \Sigma_{0,\phi}$.

(H2) For all $t \geq 0$, $B(t) : D(B(t)) \subseteq X \rightarrow X$ is a closed linear operator, $D(A) \subseteq D(B(t))$ and $B(\cdot)x$ is strongly measurable on $(0, \infty)$ for each $x \in D(A)$. There exists $b(\cdot) \in L^1_{loc}(\mathbb{R}^+)$ such that $\hat{b}(\Lambda)$ exists for $Re(\Lambda) > 0$ and $\|B(t)x\| \leq b(t)\|x\|_1$ for all $t > 0$ and $x \in D(A)$. Moreover, the operator valued function $\hat{B} : \Sigma_{0,\pi/2} \rightarrow \mathcal{L}([D(A)], X)$ has an analytical extension (still denoted by \hat{B}) to $\Sigma_{0,\phi}$ such that $\|\hat{B}(\Lambda)x\| \leq \|\hat{B}(\Lambda)\| \|x\|_1$ for all $x \in D(A)$, and $\|\hat{B}(\Lambda)\| = O(\frac{1}{|\Lambda|})$, as $|\Lambda| \rightarrow \infty$.

(H3) There exists a subspace $D \subseteq D(A)$ dense in $[D(A)]$ and positive constants C_i , $i = 1, 2$, such that $A(D) \subseteq D(A)$, $\hat{B}(\Lambda)(D) \subseteq D(A)$, $\|A\hat{B}(\Lambda)x\| \leq C_1\|x\|$ for every $x \in D$ and all $\Lambda \in \Sigma_{0,\phi}$.

REMARK 1. We note that conditions of type **(H2)** and **(H3)** have been previously considered in the literature; see [9, 10, 11, 14] for details.

In the sequel, for $r > 0$ and $\theta \in (\frac{\pi}{2}, \phi)$,

$$\Sigma_{r,\theta} = \{\Lambda \in \mathbb{C} : |\Lambda| \geq r, \text{ and } |\arg(\Lambda)| < \theta\}.$$

In addition, $\rho(F_\alpha)$ and $\rho(G_\alpha)$ are the sets

$$\rho(F_\alpha) = \{\Lambda \in \mathbb{C} : F_\alpha(\Lambda) := (\Lambda^\alpha I - A - \hat{B}(\Lambda))^{-1} \in \mathcal{L}(X)\} \text{ and}$$

$$\rho(G_\alpha) = \{\Lambda \in \mathbb{C} : G_\alpha(\Lambda) := \Lambda^{\alpha-1}(\Lambda^\alpha I - A - \hat{B}(\Lambda))^{-1} \in \mathcal{L}(X)\}.$$

We next study some preliminary properties needed to establish existence of a α -resolvent operator for the problem (1)-(2). The proof of the next Lemma is immediate, but we put it to better understanding.

LEMMA 1. Suppose that condition **(H1)** holds, then $\Lambda^\alpha \in \rho(A)$ for each $\Lambda \in \Sigma_{0,\phi}$ and there exists $M_0 = M_0(\phi)$ such that

$$\|R(\Lambda^\alpha, A)\| \leq \frac{M_0}{|\Lambda|^\alpha}, \tag{11}$$

for all $\Lambda \in \Sigma_{0,\phi}$.

Proof. Let $z = \Lambda^\alpha$, we have $\Lambda^\alpha = e^{\alpha \log(\Lambda)} = |\Lambda|^\alpha e^{i\alpha \arg(\Lambda)}$. We infer $|z| = |\Lambda|^\alpha$ and

$$|\arg(z)| = |\alpha \arg(\Lambda)| \leq |\arg(\Lambda)| < \phi.$$

This implies $z = \Lambda^\alpha \in \rho(A)$ and (11) is verified. \square

Following the same arguments used in the proof of Lemma 2.2 in [1], we have the next result. We will include a proof to that the work stay more complete.

LEMMA 2. *There exists $r_1 > 0$ such that $\Sigma_{r_1, \phi} \subseteq \rho(F_\alpha)$ and the function $F_\alpha : \Sigma_{r_1, \phi} \rightarrow \mathcal{L}(X)$ is analytic. Moreover,*

$$F_\alpha(\Lambda) = R(\Lambda^\alpha, A)[I - \widehat{B}(\Lambda)R(\Lambda^\alpha, A)]^{-1}, \quad (12)$$

and there exists constants M_i , for $i = 1, 2, 3$, such that

$$\|F_\alpha(\Lambda)\| \leq \frac{M_1}{|\Lambda|^\alpha}, \quad (13)$$

$$\|AF_\alpha(\Lambda)x\| \leq \frac{M_2}{|\Lambda|^\alpha} \|x\|_1, \quad x \in D(A), \quad (14)$$

$$\|AF_\alpha(\Lambda)\| \leq M_3, \quad (15)$$

for every $\Lambda \in \Sigma_{r_1, \phi}$.

Proof. We have

$$\begin{aligned} \|\widehat{B}(\Lambda)R(\Lambda^\alpha, A)\| &\leq \|\widehat{B}(\Lambda)\| \|R(\Lambda^\alpha, A)\| \\ &\leq \|\widehat{B}(\Lambda)\| (\|R(\Lambda^\alpha, A)\| + \|\Lambda^\alpha R(\Lambda^\alpha, A)\| + 1) \\ &\leq \left(\frac{M_0 \|\widehat{B}(\Lambda)\|}{|\Lambda|^\alpha} + M_0 \|\widehat{B}(\Lambda)\| + \|\widehat{B}(\Lambda)\| \right). \end{aligned}$$

From (H2) fixed $\varepsilon < 1$, there exists a positive number $r_1 > 1$ such that $\|\widehat{B}(\Lambda)R(\Lambda^\alpha, A)\| \leq \varepsilon$ for $\Lambda \in \Sigma_{r_1, \phi}$, consequently, the operator $I - \widehat{B}(\Lambda)R(\Lambda^\alpha, A)$ has a continuous inverse with

$$\|(I - \widehat{B}(\Lambda)R(\Lambda^\alpha, A))^{-1}\| \leq \frac{1}{1 - \varepsilon}.$$

Moreover, for $x \in X$, we have

$$(\Lambda^\alpha I - \widehat{B}(\Lambda) - A)R(\Lambda^\alpha, A)(I - \widehat{B}(\Lambda)R(\Lambda^\alpha, A))^{-1}x = x,$$

and for $x \in D(A)$

$$R(\Lambda^\alpha, A)(I - \widehat{B}(\Lambda)R(\Lambda^\alpha, A))^{-1}(\Lambda^\alpha I - \widehat{B}(\Lambda) - A)x = x,$$

which shows (12), that $\Sigma_{r_1, \phi} \subseteq \rho(F_\alpha)$ and estimate (13) is valid. Now, from (12) we obtain $\mathbf{R}(F_\alpha(\Lambda)) \subseteq D(A)$, and

$$AF_\alpha(\Lambda) = (\Lambda^\alpha R(\Lambda^\alpha, A) - I)(I - \widehat{B}(\Lambda)R(\Lambda^\alpha, A))^{-1}.$$

Consequently, the functions $AG : \Sigma_{r_1, \phi} \rightarrow \mathcal{L}(X)$ is analytic, and

$$\begin{aligned} \|AF_\alpha(\Lambda)\| &\leq \frac{1}{1-\varepsilon} \|\Lambda^\alpha R(\Lambda^\alpha, A) - I\| \\ &\leq \frac{1}{1-\varepsilon} \left(\frac{M_0 |\Lambda|^\alpha}{|\Lambda|^\alpha} + 1 \right) \\ &\leq \frac{M_0 + 1}{1-\varepsilon}. \end{aligned}$$

In addition, for $x \in D(A)$, we can write

$$\begin{aligned} \|AF_\alpha(\Lambda)x\| &\leq \|AR(\Lambda^\alpha, A)(I - \widehat{B}(\Lambda)R(\Lambda^\alpha, A))^{-1}x - AR(\Lambda^\alpha, A)x\| \\ &\quad + \|R(\Lambda^\alpha, A)Ax\| \\ &= \|AR(\Lambda^\alpha, A)(I - \widehat{B}(\Lambda)R(\Lambda^\alpha, A))^{-1}(I - (I - \widehat{B}(\Lambda)R(\Lambda^\alpha, A)))x\| \\ &\quad + \|R(\Lambda^\alpha, A)Ax\| \\ &\leq \|AF_\alpha(\Lambda)\widehat{B}(\Lambda)R(\Lambda^\alpha, A)x\| + \|R(\Lambda^\alpha, A)Ax\| \\ &\leq \frac{M_0 + 1}{1-\varepsilon} \|\widehat{B}(\Lambda)\| \|R(\Lambda^\alpha, A)x\|_1 + \|R(\Lambda^\alpha, A)Ax\| \\ &\leq \frac{M_0 + 1}{1-\varepsilon} \|\widehat{B}(\Lambda)\| (\|R(\Lambda^\alpha, A)x\| + 2\|R(\Lambda^\alpha, A)Ax\|) \\ &\leq \frac{M_2}{|\Lambda|^\alpha} \|x\|_1, \end{aligned}$$

for $|\Lambda|$ sufficiently large. This proves (14) and completes the proof. \square

Using the previous result we have the next Lemma.

LEMMA 3. *There exists $r_1 > 0$ such that $\Sigma_{r_1, \vartheta} \subseteq \rho(G_\alpha)$ and the function $G_\alpha : \Sigma_{r_1, \vartheta} \rightarrow \mathcal{L}(X)$ is analytic. Moreover,*

$$G_\alpha(\Lambda) = \Lambda^{\alpha-1}F_\alpha(\Lambda) = \Lambda^{\alpha-1}R(\Lambda^\alpha, A)[I - \widehat{B}(\Lambda)R(\Lambda^\alpha, A)]^{-1}, \tag{16}$$

and there exists constants M_i for $i = 4, 5, 6$ such that

$$\|G_\alpha(\Lambda)\| \leq \frac{M_4}{|\Lambda|}, \tag{17}$$

$$\|AG_\alpha(\Lambda)x\| \leq \frac{M_5}{|\Lambda|} \|x\|_1, \quad x \in D(A), \tag{18}$$

$$\|AG_\alpha(\Lambda)\| \leq \frac{M_6}{|\Lambda|^{1-\alpha}}, \tag{19}$$

for every $\Lambda \in \Sigma_{r_1, \vartheta}$.

Proof. Since $G_\alpha(\Lambda) = \Lambda^{\alpha-1}F_\alpha(\Lambda)$ it is easy to see that (16) is satisfied with $\Sigma_{r_1, \vartheta} \subseteq \rho(G_\alpha)$ and estimate (17) is valid. Now, from (16) we obtain $\mathbf{R}(G_\alpha(\Lambda)) \subseteq D(A)$, and

$$AG_\alpha(\Lambda) = \Lambda^{\alpha-1}(\Lambda^\alpha R(\Lambda^\alpha, A) - I)(I - \widehat{B}(\Lambda)R(\Lambda^\alpha, A))^{-1}.$$

Consequently, the functions $AG_\alpha : \Sigma_{r_1, \vartheta} \rightarrow \mathcal{L}(X)$ is analytic, and

$$\|AG_\alpha(\Lambda)\| = \|\Lambda^{\alpha-1}AF_\alpha(\Lambda)\| \leq \frac{M_3}{|\Lambda|^{1-\alpha}}.$$

For $x \in D(A)$, we can write

$$\|AG_\alpha(\Lambda)x\| = \|\Lambda^{\alpha-1}AG_\alpha(\Lambda)x\| \leq \frac{|\Lambda|^{\alpha-1}M_2}{|\Lambda|^\alpha} \|x\|_1 \leq \frac{M_2}{|\Lambda|} \|x\|_1,$$

this implies (18). \square

In the rest of this paper we assume the conditions **(Hi)**, $i = 1, 2, 3$, holds, r, θ are numbers such that $r > r_1$ and $\theta \in (\pi/2, \phi)$. Moreover, we denote by C a generic constant that represent any of the constants involved in the statements of Lemma 3 as well as any other constant that arises in the estimate that follows. By $\Gamma_{r, \theta}, \Gamma_{r, \theta}^i$, $i = 1, 2, 3$, we define the paths

$$\Gamma_{r, \theta}^1 = \{te^{i\theta} : t \geq r\}, \quad \Gamma_{r, \theta}^2 = \{re^{i\xi} : -\theta \leq \xi \leq \theta\} \text{ and } \Gamma_{r, \theta}^3 = \{te^{-i\theta} : t \geq r\},$$

and $\Gamma_{r, \theta} = \bigcup_{i=1}^3 \Gamma_{r, \theta}^i$ oriented counterclockwise.

We start with generalization of the analytic resolvent operator associated a integro-differential equations [14] for the fractional integro-differential problem (1)-(2) with $\alpha \in (0, 1)$.

DEFINITION 2. We define the operator family $(\mathcal{R}_\alpha(t))_{t \geq 0}$ by

$$\mathcal{R}_\alpha(t) = \frac{1}{2\pi i} \int_{\Gamma_{r, \theta}} e^{\Lambda t} G_\alpha(\Lambda) d\Lambda, t \geq 0, \tag{20}$$

and the auxiliary resolvent operator family $(\mathcal{S}_\alpha(t))_{t \geq 0}$ by

$$\mathcal{S}_\alpha(t) = \frac{t^{1-\alpha}}{2\pi i} \int_{\Gamma_{r, \theta}} e^{\Lambda t} F_\alpha(\Lambda) d\Lambda, t \geq 0. \tag{21}$$

REMARK 2. When $B(t) = 0$, for all $t \geq 0$, the operators family $(\mathcal{R}_\alpha(t))_{t \geq 0}$ and $(\mathcal{S}_\alpha(t))_{t \geq 0}$ coincide with operators family $(E_\alpha(t^\alpha A))_{t \geq 0}$ and $(E_{\alpha, \alpha}(t^\alpha A))_{t \geq 0}$ respectively, for more details by $(E_\alpha(t^\alpha A))_{t \geq 0}$ and $(E_{\alpha, \alpha}(t^\alpha A))_{t \geq 0}$ see [2, 5, 6] and the references therein.

We next will establish some properties of $(\mathcal{R}_\alpha(t))_{t \geq 0}$ and $(\mathcal{S}_\alpha(t))_{t \geq 0}$ family.

THEOREM 1. The operator function $\mathcal{R}_\alpha(\cdot)$ is:

- (i) exponentially bounded in $\mathcal{L}(X)$;
- (ii) exponentially bounded in $\mathcal{L}([D(A)])$;

(iii) strongly continuous on $[0, \infty)$ and uniformly continuous on $(0, \infty)$;

(iv) strongly continuous on $[0, \infty)$ in $\mathcal{L}([D(A)])$.

Proof. Proof of (i). If $t > 1$, from (20) and estimate (17) we get

$$\begin{aligned} \|\mathcal{R}_\alpha(t)\| &= \left\| \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{\Lambda t} G_\alpha(\Lambda) d\Lambda \right\| \\ &\leq \frac{C}{\pi} \int_r^\infty e^{ts \cos \theta} \frac{ds}{s} + \frac{C}{2\pi} \int_{-\theta}^\theta e^{tr \cos \xi} d\xi \\ &\leq \left(\frac{C}{\pi r |\cos \theta|} + \frac{C\theta}{\pi} \right) e^{rt}. \end{aligned}$$

If $t \in (0, 1)$, using that $G_\alpha(\cdot)$ is analytic on $\Sigma_{r,\theta}$, we get

$$\begin{aligned} \|\mathcal{R}_\alpha(t)\| &= \left\| \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{\Lambda t} G_\alpha(\Lambda) d\Lambda \right\| \\ &\leq \frac{C}{\pi} \int_{\frac{t}{r}}^\infty e^{ts \cos \theta} \frac{ds}{s} + \frac{C}{2\pi} \int_{-\theta}^\theta e^{r \cos \xi} d\xi \\ &\leq \left(\frac{C}{\pi} \int_r^\infty e^{u \cos \theta} \frac{du}{u} + \frac{C}{2\pi} \int_{-\theta}^\theta e^{r \cos \xi} d\xi \right) \\ &\leq \left(\frac{C}{\pi r |\cos \theta|} + \frac{C\theta}{\pi} \right) e^r. \end{aligned}$$

This shows (i).

Proof of (ii). From (18) that the integral in

$$R(t) = \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{\Lambda t} A G_\alpha(\Lambda) d\Lambda, \quad t > 0,$$

is absolutely convergent in $\mathcal{L}([D(A)], X)$ and defines a linear operator

$$R(t) \in \mathcal{L}([D(A)], X).$$

Using that A is closed, we can affirm that $R(t) = A\mathcal{R}_\alpha(t)$.

From Lemma 3, $G_\alpha : \Sigma_{r,\theta} \rightarrow \mathcal{L}([D(A)])$ is analytic and $\|G_\alpha(\Lambda)\|_1 \leq C|\Lambda|^{-1}$. If $t > 1$ and $x \in D(A)$, we get

$$\begin{aligned} \|A\mathcal{R}_\alpha(t)x\| &= \left\| \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{\Lambda t} A G_\alpha(\Lambda) x d\Lambda \right\| \\ &\leq \left(\frac{C}{\pi} \int_r^\infty e^{ts \cos \theta} \frac{ds}{s} + \frac{C}{2\pi} \int_{-\theta}^\theta e^{tr \cos \xi} d\xi \right) \|x\|_1 \\ &\leq \left(\frac{C}{\pi r |\cos \theta|} + \frac{C\theta}{\pi} \right) e^{rt} \|x\|_1. \end{aligned}$$

For $t \in (0, 1)$ and $x \in D(A)$ we get

$$\begin{aligned} \|A\mathcal{R}_\alpha(t)x\| &= \left\| \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{\Lambda t} A G_\alpha(\Lambda) x d\Lambda \right\| \\ &\leq \frac{C}{\pi} \int_{\frac{t}{r}}^\infty e^{s \cos \theta} \frac{ds}{s} \|x\|_1 \\ &\quad + \frac{C}{2\pi} \int_{-\theta}^\theta e^{r \cos \xi} d\xi \|x\|_1 \\ &\leq \left(\frac{C}{\pi r |\cos \theta|} + \frac{C\theta}{\pi} \right) e^r \|x\|_1. \end{aligned}$$

From before we obtain $\mathcal{R}(\cdot)$ is exponentially bounded in $\mathcal{L}([D(A)])$.

Proof of (iii.) It is clear from (20) that $\mathcal{R}_\alpha(\cdot)x$ is uniformly and strongly continuous at $t > 0$ for every $x \in X$. We next establish the strongly continuity at $t = 0$. Using that

$$\frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} \Lambda^{-1} e^{\Lambda t} d\Lambda = \lim_{N \rightarrow \infty} \frac{1}{2\pi i} \int_{\{\Gamma_{r,\theta}: r \leq s \leq N\} \cup C_{N,\theta}} \Lambda^{-1} e^{\Lambda t} d\Lambda = 1,$$

where $C_{N,\theta}$ represent the curve $Ne^{i\xi}$ for $\theta \leq \xi \leq 2\pi - \theta$. For $x \in D(A)$ and $0 < t \leq 1$ we get

$$\begin{aligned} \mathcal{R}_\alpha(t)x - x &= \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} \left(e^{\Lambda t} G_\alpha(\Lambda)x - \Lambda^{-1} e^{\Lambda t} x \right) d\Lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{\Lambda t} \Lambda^{-1} F_\alpha(\Lambda) (A + \widehat{B}(\Lambda)) x d\Lambda. \end{aligned}$$

Furthermore, it follows from (13), and assumption **(H2)** that

$$\|e^{\Lambda t} \Lambda^{-1} F_\alpha(\Lambda) (A + \widehat{B}(\Lambda)) x\| \leq e^r C \left(\frac{1}{|\Lambda|^{\alpha+1}} \right) = H(\Lambda),$$

where $H(\cdot)$ is integrable for $\Lambda \in \Gamma_{r,\theta}$. From the Lebesgue dominated convergence theorem we infer that

$$\lim_{t \rightarrow 0^+} (\mathcal{R}_\alpha(t)x - x) = \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} \Lambda^{-1} F_\alpha(\Lambda) (A + \widehat{B}(\Lambda)) x d\Lambda. \tag{22}$$

Let now $C_{L,\theta}$ be the curve $Le^{i\xi}$ for $-\theta \leq \xi \leq \theta$. Turning to apply the Cauchy's Theorem combining with the estimate

$$\left\| \int_{C_{L,\theta}} \Lambda^{-1} F_\alpha(\Lambda) (A + \widehat{B}(\Lambda)) x d\Lambda \right\| \leq \frac{C\theta}{L^\alpha}$$

we obtain

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} \Lambda^{-1} F_\alpha(\Lambda) (A + \widehat{B}(\Lambda)) x d\Lambda \\ &= \lim_{L \rightarrow \infty} \frac{1}{2\pi i} \int_{\{\Gamma_{r,\theta}: r \leq s \leq L\} \cup C_{L,\theta}} \Lambda^{-1} F_\alpha(\Lambda) (A + \widehat{B}(\Lambda)) x d\Lambda = 0, \end{aligned}$$

we can affirm that $\lim_{t \rightarrow 0^+} \|\mathcal{R}_\alpha(t)x - x\| = 0$ for all $x \in D(A)$, which completes the proof of the strongly continuity on $\mathcal{L}(X)$ since $D(A)$ is dense in X and $\mathcal{R}_\alpha(\cdot)$ is bounded on $[0, 1]$ by (i).

Proof of (iv). For $x \in D$, proceeding as in the proof of (iii), we have

$$A\mathcal{R}_\alpha(t)x - Ax = \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{\Lambda t} \Lambda^{-1} A F_\alpha(\Lambda) (A + \widehat{B}(\Lambda)) x d\Lambda.$$

Using now that $(A + \widehat{B}(\Lambda))x \in D(A)$, the inequality (14) and the assumption **(H3)** and proceeding as in the proof of (iii) we can conclude that $A\mathcal{R}_\alpha(t)x - Ax \rightarrow 0$ as $t \rightarrow 0$. The above remarks shows that $\|\mathcal{R}_\alpha(t)x - x\|_1 \rightarrow 0$ as $t \rightarrow 0$ for all $x \in D(A)$, since D is dense in $[D(A)]$ and $\mathcal{R}_\alpha(\cdot)$ is exponentially bounded in $\mathcal{L}([D(A)])$. \square

THEOREM 2. *The operator function $t \rightarrow t^{\alpha-1} \mathcal{S}_\alpha(t)$ is exponentially bounded in $\mathcal{L}(X)$ and uniformly (strong) continuous on $(0, \infty)$.*

Proof. For $t \geq 1$, from (13) we have

$$\begin{aligned} \|t^{\alpha-1} \mathcal{S}_\alpha(t)\| &= \left\| \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{\Lambda t} F_\alpha(\Lambda) d\Lambda \right\| \\ &\leq \left(\frac{C}{\pi} \int_r^\infty e^{st \cos \theta} \frac{ds}{s^\alpha} + \frac{C}{2\pi} \int_{-\theta}^\theta e^{tr \cos \xi} r^{1-\alpha} d\xi \right) \\ &\leq \left(\frac{C}{\pi r^\alpha |\cos \theta|} + \frac{C\theta r^{1-\alpha}}{\pi} \right) e^{rt}. \end{aligned}$$

Since $F_\alpha(\cdot)$ is analytic on $\Sigma_{r,\theta}$, for $t \in (0, 1)$ we get

$$\begin{aligned} \|t^{\alpha-1} \mathcal{S}_\alpha(t)\| &= \left\| \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{\Lambda t} F_\alpha(\Lambda) d\Lambda \right\| \\ &\leq \frac{C}{\pi} \int_t^\infty e^{s \cos \theta} \frac{ds}{s^\alpha} + \frac{C}{2\pi} \int_{-\theta}^\theta e^{r \cos \xi} d\xi \\ &\leq \left(\frac{C}{\pi r^\alpha |\cos \theta|} + \frac{C\theta}{\pi} r^{1-\alpha} \right) e^r. \end{aligned}$$

This completes the proof of exponential boundedness.

For the uniform continuity, let $t > 0$ and $x \in X$, we have for $R > r$ and $s > 0$,

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_{r,\theta} \cap \{\Lambda \in \mathbb{C} : |\Lambda| \geq R\}} e^{\Lambda t} F_\alpha(\Lambda) d\Lambda \right\| \leq \frac{C}{\pi} \int_R^\infty e^{s\sigma \cos \theta} \frac{d\sigma}{\sigma^\alpha} \leq \frac{C e^{sR \cos(\theta)}}{\pi s R^\alpha |\cos(\theta)|}.$$

Therefore, for all $\varepsilon > 0$, we can choose $R_t > r$ such that for all $s \in [\frac{t}{2}, \frac{3t}{2}]$ we have

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_{r,\theta} \cap \{\Lambda : |\Lambda| \geq R_t\}} e^{\Lambda t} F_\alpha(\Lambda) d\Lambda \right\| \leq \frac{\varepsilon}{2}. \tag{23}$$

On the other hand, $e^{\Lambda s} F_\alpha(\Lambda) \rightarrow e^{\Lambda t} F_\alpha(\Lambda)$ as $s \rightarrow t$, uniformly on $\Gamma_{r,\theta} \cap \{\Lambda \in \mathbb{C} : |\Lambda| \leq R_t\}$, this implies, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left\| \int_{\Gamma_{r,\theta} \cap \{\Lambda \in \mathbb{C} : |\Lambda| \leq R\}} e^{\Lambda s} F_\alpha(\Lambda) d\Lambda - \int_{\Gamma_{r,\theta} \cap \{\Lambda \in \mathbb{C} : |\Lambda| \leq R\}} e^{\Lambda t} F_\alpha(\Lambda) d\Lambda \right\| < \frac{\varepsilon}{2}. \quad (24)$$

By (23) and (24) we obtain for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $|t - s| < \delta$ we have

$$\| t^{\alpha-1} \mathcal{S}_\alpha(t) - s^{\alpha-1} \mathcal{S}_\alpha(s) \| < \varepsilon.$$

This completes the prove. \square

Using the Proposition 1.3.4 in [4] we give the next result.

COROLLARY 1. *Let $f \in L^1_{loc}(\mathbb{R}_+, X)$, then the convolution $t^{1-\alpha} \mathcal{S}_\alpha(t) * f(t) = \int_0^t (t-s)^{\alpha-1} \mathcal{S}_\alpha(t-s) f(s) ds$ exists (as a Bochner integral) and defines a continuous function from \mathbb{R}_+ into X .*

LEMMA 4. *For every $\Lambda \in \mathbb{C}$ with $Re(\Lambda) > \max\{0, r\}$, $\widehat{\mathcal{R}}_\alpha(\Lambda) = G_\alpha(\Lambda)$ and $(t^{\alpha-1} \widehat{\mathcal{S}}_\alpha)(\Lambda) = F_\alpha(\Lambda)$.*

Proof. Using that $G_\alpha(\cdot)$ is analytic on $\Sigma_{r,\theta}$, and that the integrals involved in the calculus are absolutely convergent, we have

$$\begin{aligned} \widehat{\mathcal{R}}_\alpha(\Lambda) &= \int_0^\infty e^{-\Lambda t} \mathcal{R}_\alpha(t) dt = \int_0^\infty \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{-(\Lambda-\gamma)t} G_\alpha(\gamma) d\gamma dt \\ &= \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} (\Lambda - \gamma)^{-1} G_\alpha(\gamma) d\gamma. \end{aligned}$$

By

$$\begin{aligned} \left\| \int_{C_{L,\theta}} (\Lambda - \gamma)^{-1} G_\alpha(\gamma) d\gamma \right\| &\leq \int_{-\theta}^\theta \frac{C}{|\Lambda - \gamma| |\gamma|} L d\xi \leq \int_{-\theta}^\theta \frac{C}{(L - |\Lambda|) L} L d\xi \\ &= \frac{2\theta C}{(L - |\Lambda|)}, \end{aligned}$$

we have $\int_{C_{L,\theta}} (\Lambda - \gamma)^{-1} G_\alpha(\gamma) d\gamma$ converges to 0 as $L \rightarrow \infty$. Therefore

$$\begin{aligned} \widehat{\mathcal{R}}_\alpha(\Lambda) &= \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} (\Lambda - \gamma)^{-1} G_\alpha(\gamma) d\gamma \\ &= \lim_{L \rightarrow \infty} \left(\frac{1}{2\pi i} \int_{\{\Gamma_{r,\theta} : r \leq s \leq L\} \cup C_{L,\theta}} (\Lambda - \gamma)^{-1} G_\alpha(\gamma) d\gamma \right) = G_\alpha(\Lambda). \end{aligned}$$

From $F_\alpha(\cdot)$ is analytic on $\Sigma_{r,\theta}$ using the same argument as before we have

$$\begin{aligned} t^{\alpha-1} \widehat{\mathcal{S}}_\alpha(\Lambda) &= \int_0^\infty e^{-\Lambda t} t^{1-\alpha} \mathcal{S}_\alpha(t) dt = \int_0^\infty \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{-(\Lambda-\gamma)t} F_\alpha(\gamma) d\gamma dt \\ &= \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} (\Lambda - \gamma)^{-1} F_\alpha(\gamma) d\gamma. \end{aligned}$$

Since

$$\begin{aligned} \left\| \int_{C_{L,\theta}} (\Lambda - \gamma)^{-1} F_\alpha(\gamma) d\gamma \right\| &\leq \int_{-\theta}^\theta \frac{C}{|\Lambda - \gamma| |\gamma|^\alpha} L d\xi \leq \int_{-\theta}^\theta \frac{C}{(L - |\Lambda|) L^\alpha} L d\xi \\ &= \frac{2\theta CL}{(L - |\Lambda|) L^\alpha}, \end{aligned}$$

we have $\int_{C_{L,\theta}} (\Lambda - \gamma)^{-1} F_\alpha(\gamma) d\gamma$ converges to 0 as $L \rightarrow \infty$. We infer

$$\begin{aligned} t^{\alpha-1} \widehat{\mathcal{S}}_\alpha(\Lambda) &= \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} (\Lambda - \gamma)^{-1} F_\alpha(\gamma) d\gamma \\ &= \lim_{L \rightarrow \infty} \left(\frac{1}{2\pi i} \int_{\{\Gamma_{r,\theta} : r \leq s \leq L\} \cup C_{L,\theta}} (\Lambda - \gamma)^{-1} F_\alpha(\gamma) d\gamma \right) = F_\alpha(\Lambda). \end{aligned}$$

□

THEOREM 3. *The function $\mathcal{R}_\alpha(\cdot)$ is a α -resolvent operator for the system (1)-(2).*

Proof. Let $x \in D(A)$. From Lemma 4, for $Re(\Lambda) > \max\{0, r\}$,

$$\widehat{\mathcal{R}}_\alpha(\Lambda) [\Lambda^{1-\alpha} (\Lambda^\alpha I - A - \widehat{B}(\Lambda))] x = x,$$

which implies

$$\Lambda \widehat{\mathcal{R}}_\alpha(\Lambda) x - x = \Lambda^{1-\alpha} \widehat{\mathcal{R}}_\alpha(\Lambda) Ax + \Lambda^{1-\alpha} \widehat{\mathcal{R}}_\alpha(\Lambda) \widehat{B}(\Lambda) x,$$

we get

$$\Lambda^\alpha \widehat{\mathcal{R}}_\alpha(\Lambda) x - \Lambda^{\alpha-1} x = \widehat{\mathcal{R}}_\alpha(\Lambda) Ax + \widehat{\mathcal{R}}_\alpha(\Lambda) \widehat{B}(\Lambda) x,$$

and applying (8) and [4, Proposition 1.6.4] we obtain

$$\widehat{D}_t^\alpha \widehat{\mathcal{R}}_\alpha(\Lambda) x = \widehat{\mathcal{R}}_\alpha(\Lambda) Ax + (\widehat{\mathcal{R}}_\alpha * B)(\Lambda) x.$$

By the uniqueness of the Laplace transform we get

$$D_t^\alpha \mathcal{R}_\alpha(t) x = \mathcal{R}_\alpha(t) Ax + \int_0^t \mathcal{R}_\alpha(t-s) B(s) x ds.$$

Arguing as above but using the equality $[\Lambda^{1-\alpha} (\Lambda^\alpha I - A - \widehat{B}(\Lambda))] \widehat{\mathcal{R}}_\alpha(\Lambda) x = x$, we obtain that (9) holds. The proof is now completed. □

We shall prove a result the existence of an analytic extension of resolvent operator.

THEOREM 4. *The function $\mathcal{R}_\alpha : (0, \infty) \rightarrow \mathcal{L}(X)$ has an analytic extension to $\Sigma_{\delta,0}$, $\delta = \min\{\phi - \frac{\pi}{2}, \pi - \phi\}$ and*

$$\mathcal{R}'_\alpha(z) = \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} \Lambda e^{\Lambda z} G_\alpha(\Lambda) d\Lambda, \quad z \in \Sigma_{\delta,0}. \tag{25}$$

Proof. For $\Lambda \in \Gamma_{r,\theta}$ and $z \in \Sigma_{\delta,0}$, we can write $\Lambda z = s |z| e^{i(\arg(z)+\xi)}$ where $\frac{\pi}{2} < \arg(z) + \xi < \pi$, $-\theta \leq \xi \leq \theta$ and $s \geq r$. If $|z| > 1$, from (20) and (13) we get

$$\begin{aligned} \|\mathcal{R}_\alpha(z)\| &= \left\| \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{\Lambda z} G_\alpha(\Lambda) d\Lambda \right\| \\ &\leq \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{Re(\Lambda z)} \frac{C}{|\Lambda|} |d\Lambda| \\ &\leq \frac{C}{\pi} \int_r^\infty e^{s|z|\cos(\arg(z)+\theta)} \frac{ds}{s} + \frac{C}{2\pi} \int_{-\theta}^\theta e^{r|z|\cos(\arg(z)+\xi)} d\xi \\ &\leq \left(\frac{C}{\pi r |\cos(\arg(z) + \theta)|} + \frac{C\theta}{\pi} \right) e^{r|z|}. \end{aligned}$$

On the other hand, using that $G_\alpha(\cdot)$ is analytic on $\Sigma_{r,\theta}$, for $0 < |z| < 1$ we obtain

$$\begin{aligned} \|\mathcal{R}_\alpha(z)\| &= \left\| \frac{1}{2\pi i} \int_{\Gamma_{\frac{r}{|z|},\theta}} e^{\Lambda z} G_\alpha(\Lambda) d\Lambda \right\| \\ &\leq \frac{C}{\pi} \int_{\frac{r}{|z|}}^\infty e^{s|z|\cos(\arg(z)+\theta)} \frac{ds}{s} + \frac{C}{2\pi} \int_{-\theta}^\theta e^{r\cos(\arg(z)+\xi)} d\xi \\ &\leq \left(\frac{C}{\pi} \int_r^\infty e^{u\cos(\arg(z)+\theta)} \frac{du}{u} + \frac{C}{2\pi} \int_{-\theta}^\theta e^{r\cos(\arg(z)+\xi)} d\xi \right) \\ &\leq \left(\frac{C}{\pi r |\cos(\arg(z) + \theta)|} + \frac{C\theta}{\pi} \right) e^r. \end{aligned}$$

This property allows us to define the extension $\mathcal{R}_\alpha(z)$ by this integral.

Similarly, the integral on the right hand side of (25) is also absolutely convergent in $\mathcal{L}(X)$ and strongly continuous on X for $|\arg z| < \delta$, we observe for $\Lambda \in \Gamma_{r,\theta}$

$$\left\| \frac{e^{\Lambda(z+h)} - e^{\Lambda z}}{h} G_\alpha(\Lambda) - \Lambda e^{\Lambda z} G_\alpha(\Lambda) \right\| \leq \left| \frac{e^{\Lambda(z+h)} - e^{\Lambda z}}{h} - \Lambda e^{\Lambda z} \right| \frac{C}{r} \rightarrow 0, |h| \rightarrow 0,$$

and

$$\left\| \frac{e^{\Lambda(z+h)} - e^{\Lambda z}}{h} G_\alpha(\Lambda) - \Lambda e^{\Lambda z} G_\alpha(\Lambda) \right\| \leq e^{Re(\Lambda z)} \frac{C}{|\Lambda|} = K(\Lambda),$$

where $K(\cdot)$ is integrable for $\Lambda \in \Gamma_{r,\theta}$. From the Lebesgue dominated convergence theorem which implies that $\mathcal{R}'_\alpha(z)$ verifies (25). \square

In the next result we show that existence of resolvent operator implies in the existence of solutions for problem (1)-(2).

THEOREM 5. *Let $x_0 \in [D(A)]$ and define $u(t) = \mathcal{R}_\alpha(t)x_0$. Then*

$$u \in C([0, \infty), [D(A)]) \cap C^\alpha((0, \infty), X),$$

and is a solutions of (1)-(2).

Proof. By Theorem 1 (iii) and Theorem 4 it is easy to see that $u(t) = \mathcal{R}_\alpha(t)x_0$ is a function in $C([0, \infty), [D(A)]) \cap C^\alpha((0, \infty), X)$. By Theorem 3 we have $u(t) = \mathcal{R}_\alpha(t)x_0$ satisfies the problem (1)-(2). \square

3. Non-homogeneous system

In this section we study the existence of mild solution for the semilinear integro-differential fractional problem

$$D_t^\alpha u(t) = Au(t) + \int_0^t B(t-s)u(s)ds + f(t, u(t)), \quad t \in (0, a), \tag{26}$$

$$u(0) = u_0, \tag{27}$$

where $\alpha \in (0, 1)$ and f is a appropriate function. In the sequel, $\mathcal{R}_\alpha(\cdot)$ and $\mathcal{S}_\alpha(\cdot)$ is the α -resolvent operators and auxiliary resolvent operator studied in previous section defined by (20) and (21) respectively.

Now we wil construct a notion of mild solution of the problem (26)-(27). Let $u : [0, \infty) \rightarrow X$ is a continuous functions satisfying (26)-(27). Then applying J_t^α at both sides of the equation (3) we have

$$\begin{aligned} u(t) &= u(0) + J_t^\alpha Au(t) + J_t^\alpha (B(t) * u(t)) + J_t^\alpha f(t, u(t)) \\ &= u(0) + g_\alpha * Au(t) + g_\alpha * (B(t) * u(t)) + g_\alpha * f(t, u(t)). \end{aligned} \tag{28}$$

Now assuming that this function is of exponential type and is locally integrable, we apply that Laplace transform os both sides we obtain

$$\widehat{u}(\Lambda) = \frac{u_0}{\Lambda} + \frac{A\widehat{u}(\Lambda)}{\Lambda^\alpha} + \frac{\widehat{B}(\Lambda)\widehat{u}(\Lambda)}{\Lambda^\alpha} + \frac{\widehat{f(u)}(\Lambda)}{\Lambda^\alpha},$$

where $\widehat{f(u)}(\Lambda)$ is a Laplace transform of $f(t, u(t))$. We infer

$$\begin{aligned} \widehat{u}(\Lambda) &= \Lambda^{\alpha-1}(\Lambda^\alpha I - A - \widehat{B}(\Lambda))^{-1}u_0 + (\Lambda^\alpha I - A - \widehat{B}(\Lambda))^{-1}\widehat{f(u)}(\Lambda) \\ &= G_\alpha(\Lambda)u_0 + F_\alpha(\Lambda)\widehat{f(u)}(\Lambda) \\ &= \widehat{\mathcal{R}_\alpha(t)}u_0 + \widehat{t^{\alpha-1}\mathcal{S}_\alpha(t)f(u)}(\Lambda) \\ &= \widehat{\mathcal{R}_\alpha(t)}u_0 + \widehat{t^{\alpha-1}\mathcal{S}_\alpha(t) * f(t, u(t))}. \end{aligned}$$

Finally applying the inverse of Laplace transform we end with the formula

$$u(t) = \mathcal{R}_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s)f(s, u(s))ds,$$

this equation inspires the next definitions.

DEFINITION 3. Let $\tau > 0$, a function $u : (0, \tau) \rightarrow X$ is called mild solution of (26)-(27) in $(0, \tau)$ if $u \in C((0, \tau), X)$ and

$$u(t) = \mathcal{R}_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s)f(s, u(s))ds, \tag{29}$$

holds for all $t \in (0, \tau)$.

The next result is about the existence of the mild solution for the problem (26)-(27).

THEOREM 6. *Let $f : (0, \infty) \times X \rightarrow X$ be a continuous function and locally Lipschitz in the second variable and uniformly with respect the first variable, that is, for each $x \in X$, there exists an open ball $B(x, R)$ and constant $L = L(B(x, R)) \geq 0$ such that*

$$\| f(t, y) - f(t, v) \| \leq L \| y - v \|,$$

for all $y, v \in B(x, R)$ and $t \in (0, \infty)$. Then, there exists $\tau_0 > 0$ such that (3)-(4) has a unique mild solutions in $(0, \tau_0)$.

Proof. Given $u_0 \in X$, let $B(u_0, r)$ and $L = L(B(u_0, r))$ be the Lipschitz constant of f . Given $b \in (0, r)$ fixed, by Theorem 1 and Theorem 2 we can choose $\tau_0 > 0$ such that

$$\| \mathcal{R}_\alpha(t)u_0 - u_0 \| \leq \frac{b}{2} \text{ and } \frac{N}{\alpha}(Lb + M)\tau_0^\alpha \leq \frac{b}{2}, \text{ for all } t \in (0, \tau_0),$$

where $M = \sup_{t \in (0, \tau_0)} \| f(s, u_0) \|$ and $N = \sup_{t \in (0, \tau_0)} \| \mathcal{S}_\alpha(t) \|$.

We define

$$S(\tau_0) = \{u \in C((0, \tau_0), X) : u(0) = u_0 \text{ and } \| u(t) - u_0 \| \leq b \text{ for all } t \in (0, \tau_0)\}$$

with the norm $\| u \|_{S(\tau_0)} = \sup_{t \in (0, \tau_0)} \| u(t) \|$ and the operator T on $S(\tau_0)$ by

$$T(u(t)) = \mathcal{R}_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} \mathcal{S}_\alpha(t-s) f(s, u(s)) ds.$$

If $u \in S(\tau_0)$, we have $T(u(0)) = u_0$ and $T(u(t)) \in C((0, \tau_0), X)$. On the other hand, we have that

$$\begin{aligned} & \| T(u(t)) - u_0 \| \\ & \leq \| \mathcal{R}_\alpha(t)u_0 - u_0 \| \\ & \quad + \int_0^t (t-s)^{\alpha-1} \| \mathcal{S}_\alpha(t-s) \| (\| f(s, u(s)) - f(s, u_0) \| + \| f(s, u_0) \|) ds \\ & \leq \| \mathcal{R}_\alpha(t)u_0 - u_0 \| + \int_0^t (t-s)^{\alpha-1} NL \| u(s) - u_0 \| ds + \int_0^t (t-s)^{\alpha-1} NM ds \\ & \leq \| \mathcal{R}_\alpha(t)u_0 - u_0 \| + NLb \frac{t^\alpha}{\alpha} + NM \frac{t^\alpha}{\alpha} \\ & \leq \| \mathcal{R}_\alpha(t)u_0 - u_0 \| + \frac{N}{\alpha}(Lb + M)\tau_0^\alpha \leq \frac{b}{2} + \frac{b}{2} = b, \end{aligned}$$

for all $t \in [0, \tau_0]$, this show that $TS(\tau_0) \subset S(\tau_0)$. If $u, v \in S(\tau_0)$ we obtain

$$\begin{aligned} \|T(u(t)) - T(v(t))\| &\leq \int_0^t (t-s)^{\alpha-1} \|\mathcal{S}_\alpha(t-s)\| \|f(s, u(s)) - f(s, v(s))\| ds \\ &\leq \int_0^t (t-s)^{\alpha-1} NL \|u(s) - v(s)\| ds \\ &\leq \frac{NL\tau_0^\alpha}{\alpha} \sup_{s \in (0, \tau_0)} \|u(s) - v(s)\|. \end{aligned}$$

This implies,

$$\|T(u) - T(v)\|_{S(\tau_0)} \leq \frac{NL\tau_0^\alpha}{\alpha} \|u - v\|_{S(\tau_0)}.$$

From $\frac{NL\tau_0^\alpha}{\alpha} \leq \frac{1}{2}$ by the Banach contraction principle we have that T has a unique fixed point in $S(\tau_0)$. This prove that (26)-(27) has a unique mild solutions in $(0, \tau_0)$. \square

4. Applications

In this section we apply the abstract theory developed in the previous sections to two examples. We apply our α -resolvent theory in the existence of solutions of partial integro-differential fractional which arise in the theory of heat equation with memory. In what follows, we consider the initial boundary value problem

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x, t) = \frac{k_1}{\gamma} \frac{\partial^2 u(x, t)}{\partial x^2} + \frac{k_2}{\gamma_\tau} \int_0^t e^{-\frac{t-s}{\tau}} \frac{\partial^2 u(x, s)}{\partial x^2} ds, \quad t > 0, \tag{30}$$

$$u(x, 0) = u_0(x), \quad x \in (0, a), \tag{31}$$

$$u(0, t) = u(a, t) = 0. \tag{32}$$

In this system, $\alpha \in (0, 1)$, k_1, k_2, τ, γ and γ_τ are positive numbers and $\frac{\partial^\alpha}{\partial t^\alpha} = D_t^\alpha$. When $\alpha = 1$ the system (30)-(32) is call Jeffrey’s equation, see [3] for more details. To represent this system in the abstract form (1)-(2), we choose the space $X = L^2([0, a])$. In the sequel, $A : D(A) \subseteq X \rightarrow X$ is the operator given by $Ax = \frac{k_1}{\gamma} x''$ with domain $D(A) = \{x \in X : \frac{k_1}{\gamma} x'' \in X, x(0) = x(a) = 0\}$. It is well known that $\Delta x = x''$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t \geq 0}$ on X . Hence, A is sectorial of type and **(H1)** is satisfied. We also consider the operator $B(t) : D(A) \subseteq X \rightarrow X, t \geq 0, B(t)x = \frac{k_2}{\gamma} e^{-\frac{t}{\tau}} \Delta x$ for $x \in D(A)$. Moreover, it is easy to see that conditions **(H2)** and **(H3)** in Section 2 are satisfied with $b(t) = \frac{k_2}{\gamma} e^{-\frac{t}{\tau}}$ and $D = C_0^\infty([0, a])$, where $C_0^\infty([0, a])$ is the space of infinitely differentiable functions that vanish at $\xi = 0$ and $\xi = a$. Under the above conditions we can represent the system (30)-(32) in the abstract form (3)-(4). The next results is a consequence of Theorem 5.

PROPOSITION 1. *Assume that the above conditions are fulfilled. Then, there exists a mild solution of the system (30)-(32).*

To finish this paper, we study the existence of an α -resolvent operator and unique mild solution for the partial coupled integro-differential fractional system

$$\frac{\partial^\alpha u(t, x)}{\partial t^\alpha} = \frac{\partial^2 u(t, x)}{\partial x^2} + \int_0^t a(t-s) \frac{\partial^2 v(s, x)}{\partial x^2} ds + f(t, u(t, x)),$$

$$(t, x) \in (0, \infty) \times (0, L), \tag{33}$$

$$\frac{\partial^\alpha v(t, x)}{\partial t^\alpha} = \frac{\partial^2 v(t, x)}{\partial x^2} + \int_0^t b(t-s) \frac{\partial^2 u(s, x)}{\partial x^2} ds + g(t, v(t, x)),$$

$$(t, x) \in (0, \infty) \times (0, L), \tag{34}$$

$$u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad x \in (0, L), \tag{35}$$

$$u(t, 0) = u(t, L) = 0 \text{ and } v(t, 0) = v(t, L) = 0, \quad t \in (0, \infty), \tag{36}$$

where $\alpha \in (0, 1)$ and $\frac{\partial^\alpha}{\partial t^\alpha} = D_t^\alpha$. Here we consider that functions a and b in $L^1_{loc}(\mathbb{R}^+)$ and $f, g : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$.

Let $\mathcal{X} = \begin{pmatrix} u \\ v \end{pmatrix}$, from (33)-(36) we obtain that

$$\begin{aligned} & \frac{\partial^\alpha}{\partial t^\alpha} \begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} \\ &= \left(\frac{\partial^2 u(t, x)}{\partial x^2} + \int_0^t a(t-s) \frac{\partial^2 v(s, x)}{\partial x^2} ds \right) + \begin{pmatrix} f(t, u(t, x)) \\ g(t, v(t, x)) \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial^2}{\partial x^2} & 0 \\ 0 & \frac{\partial^2}{\partial x^2} \end{pmatrix} \begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} + \int_0^t \begin{pmatrix} 0 & a(t-s) \frac{\partial^2}{\partial x^2} \\ b(t-s) \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix} \begin{pmatrix} u(s, x) \\ v(s, x) \end{pmatrix} ds \\ &+ \begin{pmatrix} f(t, u(t, x)) \\ g(t, v(t, x)) \end{pmatrix}. \end{aligned} \tag{37}$$

Therefore, we can represent the system (33)-(36) in the abstract form

$$\frac{\partial^\alpha}{\partial t^\alpha} \mathcal{X}(t) = \mathcal{A} \mathcal{X}(t) + \int_0^t \mathcal{B}(t-s) \mathcal{X}(s) ds + \mathcal{F}(t, \mathcal{X}), \quad t \geq 0, \tag{38}$$

$$\mathcal{X}(0) = \mathcal{X}_0 \in X, \tag{39}$$

where

$$\mathcal{X}(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & 0 \\ 0 & \frac{\partial^2}{\partial x^2} \end{pmatrix}, \quad \mathcal{B}(t) = \begin{pmatrix} 0 & a(t) \frac{\partial^2}{\partial x^2} \\ b(t) \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix} \text{ and}$$

$$\mathcal{F}(t, \mathcal{X}) = \begin{pmatrix} f(t, u(t, x)) \\ g(t, v(t, x)) \end{pmatrix}.$$

In this problem the space X is defined by $X = L^2(0, L) \times L^2(0, L)$ under the norm

$$\| \begin{pmatrix} u \\ v \end{pmatrix} \| = \left(\int_0^L |u|^2 + |v|^2 dx \right)^{\frac{1}{2}},$$

it is easy to see that X is a Hilbert space. We define the $D(\mathcal{A})$ by

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in X : u, v \in H_0^1(0, L) \cap H^2(0, L) \right\}.$$

In the sequel we show the existence of α -resolvent operator associated to the system (33)-(36).

LEMMA 5. *The operator \mathcal{A} is a sectorial operator.*

Proof. First we prove that \mathcal{A} is a generator of C_0 -semigroup. Let

$$(\Lambda I - \mathcal{A}) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \Lambda u - u_{xx} \\ \Lambda v - v_{xx} \end{pmatrix}.$$

Is it easy to see that $D(\mathcal{A})$ is dense on X . We obtain for $\Lambda > 0$

$$\begin{aligned} \|(\Lambda I - \mathcal{A}) \begin{pmatrix} u \\ v \end{pmatrix}\|^2 &= \int_0^L |\Lambda u - u_{xx}|^2 + |\Lambda v - v_{xx}|^2 dx \\ &\geq \Lambda^2 \int_0^L |u|^2 + |v|^2 dx = \Lambda^2 \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2. \end{aligned}$$

Therefore

$$\|(\Lambda I - \mathcal{A}) \begin{pmatrix} u \\ v \end{pmatrix}\| \geq \Lambda \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|,$$

the previous fact shows that \mathcal{A} is a dissipative operator.

Now let us prove that $0 \in \rho(\mathcal{A})$. If $\Lambda = 0$, the problem

$$-\mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

is equivalent to the systems

$$-u_{xx} = f, \tag{40}$$

$$-v_{xx} = g. \tag{41}$$

From the standard theory in the linear elliptic equations we obtain a unique solution of (40) and (41) such that $u, v \in H^2(0, L) \cap H_0^1(0, L)$, therefore $\mathcal{X} = \begin{pmatrix} u \\ v \end{pmatrix} \in D(\mathcal{A})$ and $0 \in \rho(\mathcal{A})$, from [22, Theorem 1.2.3] we obtain that \mathcal{A} is a generator of C_0 -semigroup of contractions on X .

Now we show that \mathcal{A} is a generator of analytical semigroup. By the previous facts $0 \in \rho(\mathcal{A})$, it is easy to see that this implies that $i\mathbb{R} = \{i\beta : \beta \in \mathbb{R}\} \subset \rho(\mathcal{A})$. Now we proof that $\|\eta(i\eta I - \mathcal{A})^{-1}\| < C$, for all $\eta \in \mathbb{R}$. We observe that the equations $i\eta U - \mathcal{A}U = F$ is equivalent to problem

$$i\eta u - u_{xx} = f \in L^2(0, L), \tag{42}$$

$$i\eta v - v_{xx} = g \in L^2(0, L). \tag{43}$$

Multiplying the equation (42) by $\eta\bar{u}$ and integrating we obtain

$$i\eta^2 \int_0^L u\bar{u}dx - \eta \int_0^L u_{xx}\bar{u}dx = \eta \int_0^L f\bar{u}dx. \tag{44}$$

Multiplying the equation (43) by $i\eta\bar{v}$ and integrating we obtain

$$i\eta^2 \int_0^L v\bar{v}dx - \eta \int_0^L v_{xx}\bar{v}dx = \eta \int_0^L g\bar{v}dx. \tag{45}$$

From (44) and (45) we have

$$i\eta^2 \int_0^L |u|^2 + |v|^2 dx + \eta \int_0^L |u_x|^2 + |v_x|^2 dx = \eta \int_0^L \bar{u}f + \bar{v}gdx, \tag{46}$$

taking the imaginary part we obtain

$$\begin{aligned} & \eta^2 \int_0^L |u|^2 + |v|^2 dx \\ &= \eta \int_0^L \text{Im}(\bar{u}f + \bar{v}g)dx \\ &\leq \eta \int_0^L |\bar{u}f + \bar{v}g| dx \\ &\leq \eta \left(\int_0^L |\bar{u}f| dx + \int_0^L |\bar{v}g| dx \right) \\ &\leq \eta \left[\left(\int_0^L |u|^2 dx \right)^{\frac{1}{2}} \left(\int_0^L |f|^2 dx \right)^{\frac{1}{2}} + \left(\int_0^L |v|^2 dx \right)^{\frac{1}{2}} \left(\int_0^L |g|^2 dx \right)^{\frac{1}{2}} \right] \\ &\leq 2\eta \left[\left(\int_0^L |u|^2 + |v|^2 dx \right)^{\frac{1}{2}} \left(\int_0^L |f|^2 + |g|^2 dx \right)^{\frac{1}{2}} \right]. \end{aligned}$$

By the foregoing we obtain

$$\| 2\eta U \| \leq \| F \|,$$

this is equivalent to show that $\| \eta(i\eta I - \mathcal{A})^{-1} \| < C$, for all $\eta \in \mathbb{R}$. From [22, Theorem 1.3.3] we have that \mathcal{A} is a generator of analytic semigroup and from [25, Theorem 2.5.2] we obtain that \mathcal{A} is a sectorial operator. This prove is complete. \square

LEMMA 6. Assume that $a(\cdot)$ and $b(\cdot)$ are Laplace transformable absolutely convergent for $\text{Re}(\Lambda) > 0$ whith analytical extension to $\Sigma_{0, \vartheta}$, $\vartheta \in (\frac{\pi}{2}, \pi)$ and $|\hat{a}(\Lambda)| + |\hat{b}(\Lambda)| = O(\frac{1}{|\Lambda|})$ as $|\Lambda| \rightarrow \infty$. Then, the operator family $(\mathcal{B}(t))_{t \geq 0}$ satisfies the assumptions (\mathbf{H}_2) and (\mathbf{H}_3) .

Proof. From the definitions of $\mathcal{B}(t)$ we have for $U \in D(\mathcal{A})$

$$\begin{aligned} \|\mathcal{B}(t)U\|^2 &= \left\| \begin{pmatrix} 0 & a(t)\frac{\partial^2}{\partial x^2} \\ b(t)\frac{\partial^2}{\partial x^2} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2 \\ &\leq (a(t) + b(t))^2 \left(\int_0^L |v_{xx}|^2 dx + \int_0^L |u_{xx}|^2 dx \right) \\ &\leq (a(t) + b(t))^2 \left(\int_0^L |u_{xx}|^2 + |v_{xx}|^2 dx + \int_0^L |u|^2 + |v|^2 dx \right) \\ &\leq (a(t) + b(t))^2 (\|\mathcal{A}U\| + \|U\|)^2. \end{aligned}$$

Therefore

$$\|\mathcal{B}(t)U\| \leq k(t)(\|\mathcal{A}U\| + \|U\|), \quad (47)$$

where $k(t) = a(t) + b(t)$. From (47) we obtain the Laplace transform of $(\mathcal{B}(t))_{t \geq 0}$ is absolutely convergent for $\operatorname{Re}(\Lambda) > 0$, admits an analytical extension to $\Sigma_{0, \vartheta}$ and

$$\|\widehat{\mathcal{B}}(\Lambda)\|_{\mathcal{L}([D(\mathcal{A}), X])} = O\left(\frac{1}{|\Lambda|}\right), \text{ as } |\Lambda| \rightarrow \infty.$$

It is easy to see that (\mathbf{H}_3) is verified considering $D = C^\infty(0, L) \times C^\infty(0, L)$. The proof is finished. \square

Take $f(t, u) = \eta(t) \sin(u)$ and $g(t, v) = \gamma(t) \cos(v)$, where η and γ are bounded, continuous and positive functions on $[0, \infty)$. Then we have

$$\begin{aligned} &\|\mathcal{F}(t, \mathcal{X}_1) - \mathcal{F}(t, \mathcal{X}_2)\|^2 \\ &= \left\| \begin{pmatrix} f(t, u_1) \\ g(t, v_1) \end{pmatrix} - \begin{pmatrix} f(t, u_2) \\ g(t, v_2) \end{pmatrix} \right\|^2 \\ &= \left\| \begin{pmatrix} f(t, u_1) - f(t, u_2) \\ g(t, v_1) - g(t, v_2) \end{pmatrix} \right\|^2 \\ &= \int_0^L |f(t, u_1) - f(t, u_2)|^2 + |g(t, v_1) - g(t, v_2)|^2 dx \\ &= \int_0^L (\eta(t) |\sin(u_1) - \sin(u_2)|)^2 + (\gamma(t) |\cos(v_1) - \cos(v_2)|)^2 dx \\ &\leq (\eta(t) + \gamma(t))^2 \int_0^L |u_1 - u_2|^2 + |v_1 - v_2|^2 dx \\ &\leq (\eta(t) + \gamma(t))^2 \|\mathcal{X}_1 - \mathcal{X}_2\|^2. \end{aligned}$$

This implies there exists $L = \sup_{t \in [0, \infty)} (\eta(t) + \gamma(t)) > 0$ such that

$$\|\mathcal{F}(t, \mathcal{X}_1) - \mathcal{F}(t, \mathcal{X}_2)\| \leq L \|\mathcal{X}_1 - \mathcal{X}_2\|.$$

By the Theorem 3 and Theorem 6 we obtain the next result.

PROPOSITION 2. *Assume that the above conditions are fulfilled. Then there exist an analytical α -resolvent operator family and a mild solution defined on $(0, \tau_0)$ associated to system (33)-(36).*

Acknowledgements. The author would like to thank the reviewer for their thoughtful comments and efforts towards improving the manuscript.

REFERENCES

- [1] R. P. AGARWAL, J. P. C. DOS SANTOS, C. CUEVAS, *Analytic Resolvent Operator and Existence Results for Fractional Integro-Differential Equations*, Journal of Abstract Differential Equ. and Appl. **2**, (2012) 26–47.
- [2] B. ANDRADE, A. N. CARVALHO, P. M. CARVALHO NETO, P. MARÍN-RUBIO, *Semilinear fractional differential equations: global solutions, critical nonlinearities and comparison results*, Topol. Methods in Non. Analysis, **45**, (2014) 439–468.
- [3] A. ARAÚJO, J. A. FERREIRA, P. OLIVEIRA, *The effect of memory term in diffusion phenomena*, Journal of Computational Math. **24**, (1) (2006) 91–102.
- [4] W. ARENDT, C. BATTY, M. HIEBER, F. NEUBRANDER, *Vector-valued Laplace Transforms and Cauchy Problems*, Monogr. Math. **96**, Birkhäuser, Basel, 2011.
- [5] E. BAZHLEKOVA, *Fractional Evolution Equations in Banach Spaces*, Ph.D. Thesis, Eindhoven University of Technology, Eindhoven, 2001.
- [6] P. M. CARVALHO-NETO, *Fractional Differential Equations: a novel study of local and global solutions in Banach spaces*, Ph.D. thesis, Universidade de São Paulo, São Carlos, 2013.
- [7] G. DA PRATO, M. IANNELLI, *Existence and regularity for a class of integro-differential equations of parabolic type*, J. Math. Anal. Appl. **112**, (1) (1985) 36–55.
- [8] G. DA PRATO, A. LUNARDI, *Solvability on the real line of a class of linear Volterra integrodifferential equations of parabolic type*, Ann. Mat. Pura Appl. **4**, (150) (1988) 67–117.
- [9] J. P. C. DOS SANTOS, S. M. GUZZO, M. N. RABELO, *Asymptotically Almost Periodic Solutions for Abstract Partial Neutral Integro-Differential Equation*, Advances in Difference Equations, **2010**, (2010) 1–27.
- [10] J. P. C. DOS SANTOS, H. HENRÍQUEZ, *Existence of S -asymptotically ω -periodic solutions to abstract integro-differential equations*, Applied Math. and Comp. (256) (2015) 109–118.
- [11] J. P. C. DOS SANTOS, H. HENRÍQUEZ, E. HERNÁNDEZ, *Existence results for neutral integro-differential equations with unbounded delay*, Journal Integral Equ. and Appl., **23** (2) (2011) 289–330.
- [12] A. M. A. EL-SAYED, M. A. E. HERZALLAH, *Continuation and maximal regularity of an arbitrary (fractional) order evolutionary integro-Differential equation*, Applicable Anal. **84** (11) (2005) 1–14.
- [13] R. GRIMMER, F. KAPPEL, *Series expansions for resolvents of Volterra integrodifferential equations in Banach space*, Siam Journal of Math. Anal. **15**, (3) (1984) 595–604.
- [14] R. GRIMMER, A. PRITCHARD, *Analytic resolvent operators for integral equations in Banach space*, J. Diff. Equ. **50**, (2) (1983) 234–259.
- [15] R. GRIMMER, J. PRÜSS, *On linear Volterra equations in Banach spaces*, Comput. Math. with Appl. **11**, (1985) 189–205.
- [16] G. GRIPENBERG, S-O LONDEN, O. STAFFANS, *Volterra Integral and Functional Equations*, Cambridge University Press, Cambridge, 1990.
- [17] E. HERNÁNDEZ, D. O’ REGAN, K. BALACHANDRAM, *Existence results for abstract fractional differential equations with nonlocal conditions via resolvent operators*, Indagationes Mathematicae **24**, (2013) 68–82.
- [18] S. HONG-RUI, F. ZHAOSHENG, *Fractional abstract Cauchy problem with order $\alpha \in (1, 2)$* , Dynamics of Partial Differential Equ. **13**, (2) (2016) 155–177.
- [19] V. KEYANTUO, C. LIZAMA, M. WARNA, *Asymptotic behaviors of fractional-order semilinear evolution equations*, Differential Integral Equ. **26**, (2013) 757–780.
- [20] M. KOSTIĆ, *Abstract Volterra Integro-Differential Equations*, CRC Press, Boca Raton, FL, 2015.

- [21] Y. LI, H. SUN, Z. FENG, *Fractional abstract problem with order $\alpha \in (1, 2)$* , Dynamics of PDE. **2**, (2016) 155–177.
- [22] Z. LIU, S. ZHENG, *Semigroups Associated With Dissipative Systems*, π Research Notes Math. 398, Chapman&Hall/CRC, Boca Raton, 1999.
- [23] A. LUNARDI, *On the linear heat equation with fading memory*, SIAM J. Math. Anal. **21**, (5) (1990) 1213–1224.
- [24] A. LUNARDI, *Laplace transform method in integro-differential equations*, J. Integral Equations Appl. **10**, (1985) 185–211.
- [25] A. PAZY, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New-York, 1983.
- [26] R. PONCE, *Bounded mild solutions to fractional integro-differential equations in Banach spaces*, Semigroup Forum **87**, (2013) 377–392.
- [27] J. PRÜSS, *Evolutionary Integral Equations and Applications*, Monographs in Mathematics, **87**, Birkhäuser Verlag, Basel, 1993.
- [28] S. G. SAMKO, A. A. KILBAS, O. I. MARICHEV, *Fractional Derivatives and Integrals: Theory and Applications*, Gordon and Breach, New York, 1993.
- [29] D. SFORZA, *Parabolic Integrodifferential Equations with Singular Kernels*, J. Integral Equations Appl. **4**, (1991) 601–623.
- [30] R. N. WANG, D. H. CHEN, T. J. XIAO, *Abstract fractional Cauchy problems with almost sectorial operators*, Journal of Differential Equ. 252 (2012) 202–235.
- [31] Y. ZHOU, F. JIAO, *Nonlocal Cauchy problem for fractional evolution equations*, Nonlinear Analysis: Real World Appl. **5**, (2010), 4465–4475.

(Received February 6, 2019)

José Paulo Carvalho dos Santos
Department of Mathematics
Federal University of Alfenas
e-mail: zepaulo@unifal-mg.edu.br