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POSITIVE SOLUTIONS OF M–POINT FRACTIONAL BOUNDARY VALUE PROBLEM ON THE HALF LINE

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Abstract. In this paper, six functionals fixed point theorem is used to investigate the existence of positive solutions for fractional-order nonlinear boundary value problems on the half line. As an application, an example is given to illustrate the main result.

1. Introduction

The first application of fractional calculus was due to Abel in his solution to the Tautochrone problem[11]. It also appears in many engineering and scientific disciplines as the mathematical models of systems and processes in the fields of aerodynamics, polymer rheology, physics, chemistry, etc. Recently, many authors have been dealing with the existence of solutions of nonlinear boundary value problems for fractional differential equations thanks to techniques of nonlinear analysis, for example, see [2], [5], [6], [7], [9], [10], [16], [19], [20] and [21]. For general results and backround on the fractional calculus, we refer the reader to [1], [4] and [14].

It should be noted that most of the papers and books on fractional calculus are devoted the solvability of fractional differential equations on finite interval. Due to the fact that an infinite interval is noncompact, the discussion about boundary value problem on the half line is more complicated. Very recently, there are some results in the literature for fractional boundary value problem on an infinite interval, see [17], [18] and [22]. In particular, for Riemann-Liouville fractional integral boundary value problem on infinite interval, few works were done, see [3] and [8].

In [18], Ge and Zhao considered the following fractional integral boundary value problem on an infinite interval:

$$\begin{cases} D_{0^+}^{\alpha}u(t)+a(t)f(t,u(t))=0, \ \ t\in(0,\infty), \ \ \alpha\in(1,2), \\ u(0)=0, \\ \lim\limits_{t\to\infty} D_{0^+}^{\alpha-1}u(t)=\beta u(\xi), \end{cases}$$

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where $0 < \xi < \infty$, D^{α}_{0+} is the standard Riemann-Liouville fractional derivative. They obtained the existence of the unique positive solution by using the Leray-Schauder Nonlinear Alternative theorem.

In [17], Liang and Zhang considered the following m-point fractional boundary value problem on an infinite interval:

$$\begin{cases} D^{\alpha}_{0+}u(t)+a(t)f(u(t))=0,\ 0<t<\infty, \\ u(0)=u'(0)=0, \\ D^{\alpha-1}_{0+}u(\infty)=\sum\limits_{i=1}^{m-2}\beta_i u(\xi_i), \end{cases}$$

where $2<\alpha<3$, D^{α}_{0+} is the standard Riemann-Liouville fractional derivative, $0<\xi_1<\xi_2<...<\xi_{m-2}<\infty$, $i=1,...,m-2$ satisfies $0<\sum\limits_{i=1}^{m-2}\beta_i\xi_i{}^{\alpha-1}<\Gamma(\alpha)$. They obtained the existence of three positive solutions by using the Legget-Williams fixed point theorem.

In [8], Zhang et al. studied the existence of nonnegative solutions for the following boundary value problem for fractional differential equations with nonlocal boundary conditions on unbounded domains:

$$\begin{cases} D^{\alpha}_{0+}u(t)+f(t,u(t))=0,\ \ 1<\alpha\leqslant 2,\ \ t\in[0,\infty), \\ D^{\alpha-2}_{0}u(0)=0, \\ \lim\limits_{t\to\infty}D^{\alpha-1}_{0+}u(t)=\beta I^{\alpha-1}_{0+}u(\eta), \end{cases}$$

where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative of order α, $f\in C([0,\infty)\times\mathbb{R},\mathbb{R}^+)$ and $0<\beta,\eta>\infty$. The Leray - Schauder nonlinear alternative is used.

In [22], Gholami considered the following fractional integral boundary value problem on an infinite interval:

$$\begin{cases} D^{\alpha}_{0+}u(t)+a(t)f(t,u(t),u'(t))=0;\ \ t\in(0,\infty),\ \ \alpha\in(2,3), \\ u(0)=u'(0)=0, \\ \lim\limits_{t\to\infty}D^{\alpha-1}_{0+}u(t)=\sum\limits_{i=1}^{m-2}\beta_i D^{\alpha-1}_{0+}u(t)\mid_{t=\xi_i}, \end{cases}$$

where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative, $0<\xi_1<\xi_2<...<\xi_{m-2}<\infty$, $i=1,...,m-2$, $\beta_i\in\mathbb{R}$. The author obtained the existence of a bounded solution by using the Leray-Schauder Nonlinear Alternative theorem.

In [3], Wang considered the following fractional integral boundary value problem on semi-infinite interval:

$$\begin{cases} D^{\alpha}u(t)+f(t,u(t))=0,\ 0<t<\infty, \\ u(0)=u'(0)=0, \\ D^{\alpha-1}u(\infty)=\xi I^{\beta}u(\eta),\ \ \beta>0. \end{cases}$$

The author obtained the existence of the unique solution by using the monotone iterative technique.

Motivated by the above works, we consider the following m-point Riemann-Liouville fractional integral boundary value problem (BVP).

$$\begin{cases} D_{0+}^{\nu}\vartheta(t)+a(t)f(t,\vartheta(t))=0, \ \ t\in[0,\infty), \\ \vartheta(0)=\vartheta'(0)=0, \\ D_{0+}^{\nu-1}\vartheta(\infty)=\sum_{i=1}^{m-2}\eta_i I_{0+}^{\kappa}\vartheta(\xi_i), \end{cases} \tag{1}$$

where $2<\nu\leqslant 3$, $\kappa>0$, D_{0+}^{ν} is the standard Riemann-Liouville fractional derivative, $0<\xi_1<\xi_2<...<\xi_{m-2}<\infty$, $i=1,...,m-2$, $\eta_i>0$. Throughout this paper we assume that following conditions hold:

$(H1)$ $0<\sum_{i=1}^{m-2}\eta_i\xi_i^{\nu+\kappa-1}<\Gamma(\nu+\kappa)$,

$(H2)$ $f:[0,\infty)\times\mathbb{R}\to[0,\infty)$ continuous and $0<\int_0^{\infty}a(s)ds<\infty$,

$(H3)$ $F(t,\vartheta)=f(t,(1+t^{\nu-1})\vartheta)$, $f\in C([0,\infty)\times\mathbb{R},[0,\infty))$, $f(0,\vartheta)\not\equiv 0$ on any subinterval of $(0,\infty)$ and when ϑ is bounded $f(t,(1+t^{\nu-1})\vartheta)$ is bounded on $[0,\infty)$.

By using the Six Functionals fixed point theorem in [12], we get the existence of at least three positive solutions for the BVP (1). To the best our knowledge, the existence of solutions for m-point Riemann-Liouville fractional integral boundary value problem on infinite interval is not investigated till now. Hence, this results can be considered as a contribution to this field.

This paper is organized as follows. In section 2, we provide some definitions and preliminary lemmas which are key tools for our main result. In section 3, we give and prove our main result. In section 4, we give an example to illustrate how the main result can be used in practice. Finally, conclusion part is established in section 5.

2. Preliminaries

In this section, to state the main result of this paper, we need the following lemmas and definitions.

DEFINITION 1. The *Riemann-Liouville* fractional integral of order $\nu>0$ of an integrable function $f:(0,\infty)\to\mathbb{R}$ is given by

$$I_{0+}^{\nu}f(t)=\frac{1}{\Gamma(\nu)}\int_0^t(t-s)^{\nu-1}f(s)ds.$$

DEFINITION 2. The *Riemann-Liouville* fractional derivative of order $\nu > 0$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\nu} f(t) = \frac{1}{\Gamma(n-\nu)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\nu-n+1}} ds,$$

where $n-1 \leqslant \nu < n$.

LEMMA 1. [1] *Let* $\nu > 0$ *and* $\vartheta \in C(0,1) \cap L(0,1)$. *Then the fractional differential equation*

$$D_{0^+}^{\nu} \vartheta(t) = 0$$

has $\vartheta(t) = c_1 t^{\nu-1} + c_2 t^{\nu-2} + c_3 t^{\nu-3} + \ldots + c_n t^{\nu-n}$, $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$, $n = [\nu]+1$ *as an unique solution.*

LEMMA 2. [1] *Assume that* $\vartheta \in C(0,1) \cap L(0,1)$ *with a fractional derivative of order* $\nu > 0$ *that belongs to* $C(0,1) \cap L(0,1)$. *Then*

$$I_{0^+}^{\nu} D_{0^+}^{\nu} \vartheta(t) = \vartheta(t) + c_1 t^{\nu-1} + c_2 t^{\nu-2} + c_3 t^{\nu-3} + \ldots + c_n t^{\nu-n},$$

for some $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$, $n = [\nu]+1$.

LEMMA 3. *Assume that the conditions* $(H1)-(H3)$ *are satisfied. If* $h \in C[0,\infty)$, *fractional boundary value problem*

$$\begin{cases} D_{0^+}^{\nu} \vartheta(t) + h(t) = 0, \ t \in [0,\infty), \ 2 < \nu \leqslant 3, \\ \vartheta(0) = \vartheta'(0) = 0, \\ D_{0^+}^{\nu-1} \vartheta(\infty) = \sum\limits_{i=1}^{m-2} \eta_i I_{0^+}^{\kappa} \vartheta(\xi_i), \end{cases} \tag{2}$$

has an integral expression

$$\vartheta(t) = \int_0^{\infty} G(t,s) h(s) ds, \ t \in [0,\infty), \tag{3}$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s), \tag{4}$$

here

$$G_1(t,s) = \frac{1}{\Gamma(\nu)} \begin{cases} t^{\nu-1} - (t-s)^{\nu-1}, & 0 \leqslant s \leqslant t < \infty, \\ t^{\nu-1}, & 0 \leqslant t \leqslant s < \infty. \end{cases} \tag{5}$$

and

$$G_2(t,s) = \frac{\sum\limits_{i=1}^{m-2} \eta_i t^{\nu-1}}{\Gamma(\nu) \left[\Gamma(\nu+\kappa) - \sum\limits_{i=1}^{m-2} \eta_i \xi_i^{\nu+\kappa-1} \right]} \begin{cases} \xi_i^{\nu+\kappa-1} - (\xi_i - s)^{\nu+\kappa-1}, & 0 \leqslant s \leqslant \xi_i < \infty, \\ \xi_i^{\nu+\kappa-1}, & 0 \leqslant \xi_i \leqslant s < \infty. \end{cases} \tag{6}$$

Proof. According to Lemma 2, we can obtain that $\vartheta(t) = -I_{0+}^{\nu}h(t) + c_1 t^{\nu-1} + c_2 t^{\nu-2} + c_3 t^{\nu-3}$. By the boundary conditions of problem (1), we have

$$c_1 = \frac{\Gamma(\nu+\kappa)\int_0^{\infty} h(s)ds - \sum_{i=1}^{m-2}\eta_i \int_0^{\xi_i}(\xi_i - s)^{\nu+\kappa-1}h(s)ds}{\Gamma(\nu)\left[\Gamma(\nu+\kappa) - \sum_{i=1}^{m-2}\eta_i \xi_i^{\nu+\kappa-1}\right]}, c_2 = 0, c_3 = 0.$$

Therefore, we obtain

$$\begin{aligned}\vartheta(t) &= -\frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}h(s)ds + \frac{\Gamma(\nu+\kappa)t^{\nu-1}}{\Gamma(\nu)\left[\Gamma(\nu+\kappa) - \sum_{i=1}^{m-2}\eta_i \xi_i^{\nu+\kappa-1}\right]}\int_0^{\infty} h(s)ds \\ &\quad - \frac{\sum_{i=1}^{m-2}\eta_i t^{\nu-1}}{\Gamma(\nu)\left[\Gamma(\nu+\kappa) - \sum_{i=1}^{m-2}\eta_i \xi_i^{\nu+\kappa-1}\right]}\int_0^{\xi_i}(\xi_i - s)^{\nu+\kappa-1}h(s)ds \\ &= \int_0^{\infty} G(t,s)h(s)ds, \quad t \in [0,\infty).\end{aligned}$$

LEMMA 4. [17] *The function $G_1(t,s)$ defined by (5) satisfies*

i) *$G_1(t,s)$ is continuous and $G_1(t,s) \geqslant 0$ for $(t,s) \in [0,\infty) \times [0,\infty)$,*

ii) *$G_1(t,s)$ is strictly increasing in the first variable,*

iii) *$G_1(t,s)$ is concave in the first variable for $0 < s < t < \infty$.*

Proof. This proof given clearly as Lemma 3.2 in [17].

LEMMA 5. [17] *If $k > 1$, then $G_1(t,s)$ defined by (5) has the following property*

$$\min_{\frac{1}{k}\leqslant t\leqslant k} \frac{G_1(t,s)}{1+t^{\nu-1}} \geqslant \frac{1}{4k^2(1+k^{\nu-1})} \max_{t\in[0,\infty)} \frac{G_1(t,s)}{1+t^{\nu-1}}.$$

LEMMA 6. *From the definition of $G_1(t,s)$, we have*

$$\frac{G_1(t,s)}{1+t^{\nu-1}} \leqslant \frac{1}{\Gamma(\nu)}, \qquad \frac{G(t,s)}{1+t^{\nu-1}} \leqslant L \ \ for \ \ (t,s) \in [0,\infty) \times [0,\infty),$$

where $L = \dfrac{\Gamma(\nu+\kappa)}{\Gamma(\nu)\left[\Gamma(\nu+\kappa) - \sum_{i=1}^{m-2}\eta_i \xi_i^{\nu+\kappa-1}\right]}.$

Proof. The functions $G(t,s)$, $G_1(t,s)$ and $G_2(t,s)$ are as defined in equations (4), (5) and (6) respectively. Let $s \leqslant t$. Using Lemma 4, we have
$$\frac{G_1(t,s)}{1+t^{\nu-1}} = \frac{t^{\nu-1}-(t-s)^{\nu-1}}{\Gamma(\nu)(1+t^{\nu-1})} \leqslant \frac{t^{\nu-1}}{\Gamma(\nu)(1+t^{\nu-1})} \leqslant \frac{1}{\Gamma(\nu)}.$$
Let $t \leqslant s$. From Lemma 4, we have
$$\frac{G_1(t,s)}{1+t^{\nu-1}} = \frac{t^{\nu-1}}{\Gamma(\nu)(1+t^{\nu-1})} \leqslant \frac{1}{\Gamma(\nu)}.$$
In both cases, we obtain
$$\frac{G_1(t,s)}{1+t^{\nu-1}} \leqslant \frac{1}{\Gamma(\nu)}.$$

Similarly, we can obtain an inequality for function $G_2(t,s)$. If $0 \leqslant s \leqslant \xi_i$, then
$$\begin{aligned}
\frac{G_2(t,s)}{1+t^{\nu-1}} &= \frac{\sum_{i=1}^{m-1}\eta_i t^{\nu-1}\left(\xi_i^{\nu+\kappa-1}-(\xi_i-s)^{\nu+\kappa-1}\right)}{(1+t^{\nu-1})\Gamma(\nu)\left[\Gamma(\nu+\kappa)-\sum_{i=1}^{m-2}\eta_i\xi_i^{\nu+\kappa-1}\right]} \\
&\leqslant \frac{\sum_{i=1}^{m-2}\eta_i t^{\nu-1}\xi_i^{\nu+\kappa-1}}{(1+t^{\nu-1})\Gamma(\nu)\left[\Gamma(\nu+\kappa)-\sum_{i=1}^{m-2}\eta_i\xi_i^{\nu+\kappa-1}\right]} \\
&\leqslant \frac{\sum_{i=1}^{m-2}\eta_i\xi_i^{\nu+\kappa-1}}{\Gamma(\nu)\left[\Gamma(\nu+\kappa)-\sum_{i=1}^{m-2}\eta_i\xi_i^{\nu+\kappa-1}\right]}.
\end{aligned}$$

On the other hand, if $0 \leqslant \xi_i \leqslant s$, then
$$\begin{aligned}
\frac{G_2(t,s)}{1+t^{\nu-1}} &= \frac{\sum_{i=1}^{m-1}\eta_i t^{\nu-1}\xi_i^{\nu+\kappa-1}}{(1+t^{\nu-1})\Gamma(\nu)\left[\Gamma(\nu+\kappa)-\sum_{i=1}^{m-2}\eta_i\xi_i^{\nu+\kappa-1}\right]} \\
&\leqslant \frac{\sum_{i=1}^{m-2}\eta_i\xi_i^{\nu+\kappa-1}}{\Gamma(\nu)\left[\Gamma(\nu+\kappa)-\sum_{i=1}^{m-2}\eta_i\xi_i^{\nu+\kappa-1}\right]}.
\end{aligned}$$

The following inequality is obtained from both cases:

$$\frac{G_2(t,s)}{1+t^{\nu-1}} \leqslant \frac{\sum_{i=1}^{m-2}\eta_i\xi_i^{\nu+\kappa-1}}{\Gamma(\nu)\left[\Gamma(\nu+\kappa)-\sum_{i=1}^{m-2}\eta_i\xi_i^{\nu+\kappa-1}\right]}.$$

Consequently, from (4), we get

$$\frac{G(t,s)}{1+t^{\nu-1}} \leqslant \frac{\Gamma(\nu+\kappa)}{\Gamma(\nu)\left[\Gamma(\nu+\kappa)-\sum_{i=1}^{m-2}\eta_i\xi_i^{\nu+\kappa-1}\right]}.$$

LEMMA 7. *If $k>1$, then $G_2(t,s)$ defined by (6) has the following property*

$$\min_{\frac{1}{k}\leqslant t\leqslant k}\frac{G_2(t,s)}{1+t^{\nu-1}} \geqslant \frac{1}{k^{\nu-1}(1+k^{\nu-1})}\max_{t\in[0,\infty)}\frac{G_2(t,s)}{1+t^{\nu-1}}.$$

Proof. Let $0\leqslant s\leqslant \xi_i$, we have,

$$\begin{aligned}
\min_{\frac{1}{k}\leqslant t\leqslant k}\frac{G_2(t,s)}{1+t^{\nu-1}} &= \min_{\frac{1}{k}\leqslant t\leqslant k}\frac{\sum_{i=1}^{m-2}\eta_i t^{\nu-1}\left(\xi_i^{\nu+\kappa-1}-(\xi_i-s)^{\nu+\kappa-1}\right)}{(1+t^{\nu-1})\Gamma(\nu)\left[\Gamma(\nu+\kappa)-\sum_{i=1}^{m-2}\eta_i\xi_i^{\nu+\kappa-1}\right]}\\
&\geqslant \frac{\sum_{i=1}^{m-2}\eta_i\left(\frac{1}{k}\right)^{\nu-1}\left(\xi_i^{\nu+\kappa-1}-(\xi_i-s)^{\nu+\kappa-1}\right)}{(1+k^{\nu-1})\Gamma(\nu)\left[\Gamma(\nu+\kappa)-\sum_{i=1}^{m-2}\eta_i\xi_i^{\nu+\kappa-1}\right]}\\
&= \frac{1}{k^{\nu-1}(1+k^{\nu-1})}\max_{t\in[0,\infty)}\frac{\sum_{i=1}^{m-2}\eta_i t^{\nu-1}\left(\xi_i^{\nu+\kappa-1}-(\xi_i-s)^{\nu+\kappa-1}\right)}{(1+t^{\nu-1})\Gamma(\nu)\left[\Gamma(\nu+\kappa)-\sum_{i=1}^{m-2}\eta_i\xi_i^{\nu+\kappa-1}\right]}.
\end{aligned}$$

Let $0\leqslant \xi_i\leqslant s$, then the following inequality is obtained:

$$\min_{\frac{1}{k}\leqslant t\leqslant k}\frac{G_2(t,s)}{1+t^{\nu-1}} \geqslant \frac{1}{k^{\nu-1}(1+k^{\nu-1})}\max_{t\in[0,\infty)}\frac{G_2(t,s)}{1+t^{\nu-1}}.$$

LEMMA 8. *For a fixed* $k>1$,

$$\min_{\frac{1}{k}\leqslant t\leqslant k}\frac{G(t,s)}{1+t^{\nu-1}}\geqslant\lambda(k)\max_{t\in[0,\infty)}\frac{G(t,s)}{1+t^{\nu-1}}$$

where

$$\lambda(k)=\min\left\{\frac{1}{4k^2(1+k^{\nu-1})},\frac{1}{k^{\nu-1}(1+k^{\nu-1})}\right\}.$$

Proof. This Lemma is obvious from Lemma 6 and Lemma 7.

Set

$$E=\left\{\vartheta\in\mathscr{C}[0,\infty):\max_{t\geqslant 0}\ \frac{|\vartheta(t)|}{1+t^{\nu-1}}<\infty\right\}.$$

Clearly, E is Banach space with the norm

$$\|\vartheta\|=\max_{0\leqslant t<\infty}\frac{|\vartheta(t)|}{1+t^{\nu-1}}<\infty.$$

LEMMA 9. *Assume that* $(H1)$-$(H3)$ *hold. Let* $\vartheta\in E$ *and* $k>1$. *Then,* $\vartheta(t)\geqslant 0$ *and* $\min_{\frac{1}{k}\leqslant t\leqslant k}\frac{|\vartheta(t)|}{1+t^{\alpha-1}}\geqslant\lambda(k)\|\vartheta\|$.

Proof. From Lemma 4, positivity of $G_2(t,s)$ and conditions $(H1)$-$(H3)$, we can obtain $\vartheta(t)\geqslant 0$. For a fixed $k>1$, from Lemma 8

$$\begin{aligned}\min_{\frac{1}{k}\leqslant t\leqslant k}\frac{\vartheta(t)}{1+t^{\nu-1}}&=\min_{\frac{1}{k}\leqslant t\leqslant k}\frac{1}{1+t^{\nu-1}}\int_0^\infty G(t,s)a(s)f(s,\vartheta(s))ds\\&\geqslant\int_0^\infty\min_{\frac{1}{k}\leqslant t\leqslant k}\frac{G(t,s)}{1+t^{\nu-1}}a(s)f(s,\vartheta(s))ds\\&\geqslant\lambda(k)\int_0^\infty\max_{t\geqslant 0}\frac{G(t,s)}{1+t^{\nu-1}}a(s)f(s,\vartheta(s))ds\\&\geqslant\lambda(k)\max_{t\geqslant 0}\int_0^\infty\frac{G(t,s)}{1+t^{\nu-1}}a(s)f(s,\vartheta(s))ds\\&\geqslant\lambda(k)\|\vartheta\|\end{aligned}$$

By using Lemma 9, we can define the cone $\mathscr{P}\subset E$ by

$$\mathscr{P}=\left\{\vartheta\in E:\vartheta(t)\geqslant 0,\ \min_{\frac{1}{k}\leqslant t\leqslant k}\frac{|\vartheta(t)|}{1+t^{\nu-1}}\geqslant\lambda(k)\|\vartheta\|\right\}.$$

Denote the operator $T:\mathscr{P}\to E$ by

$$T\vartheta(t)=\int_0^\infty G(t,s)a(s)f(s,\vartheta(s))ds.$$

LEMMA 10. *Assume that* $(H1)$-$(H3)$ *hold. Then* $T : \mathscr{P} \to \mathscr{P}$ *is completely continuous.*

Proof. Firstly, it is easy to check that $T : \mathscr{P} \to \mathscr{P}$ is well-defined. Now, we will show that T is a completely continuous operator in three steps.

Step 1. $T : \mathscr{P} \to \mathscr{P}$ is a continuous operator.

Let $\vartheta_n \in \mathscr{P}$, there exists a sequence $\vartheta_n \to \vartheta$, $n \to \infty$ in $\mathscr{P}$. Since the convergent sequences are bounded, there is a real number r_0 such that $\max\limits_{n\in\mathbb{N}\setminus\{0\}} \|\vartheta_n\| < r_0$. Denote the set

$$B_{r_0} = \max\{f(t,(1+t^{\nu}-1)\vartheta),\ \ (t,\vartheta) \in [0,\infty)\times[0,r_0]\}.$$

For all $(t,s) \in [0,\infty)$, by the Lebesgue Dominated Convergence theorem and Lemma 6, we obtain

$$\begin{aligned}\left|\frac{T\vartheta_n(t) - T\vartheta(t)}{1+t^{\nu-1}}\right| &= \left|\int_0^\infty \frac{G(t,s)}{1+t^{\nu-1}} a(s)[f(s,\vartheta_n(s)) - f(s,\vartheta(s))]ds\right| \\ &\leqslant \int_0^\infty \frac{G(t,s)}{1+t^{\nu-1}} a(s)\,|f(s,\vartheta_n(s)) - f(s,\vartheta(s))|\,ds \\ &\leqslant L\int_0^\infty a(s)\,|f(s,\vartheta_n(s)) - f(s,\vartheta(s))|\,ds \to 0, (n\to\infty).\end{aligned}$$

This yields that

$$\|T\vartheta_n(t) - T\vartheta_0(t)\| \to 0.$$

Hence, $T : \mathscr{P} \to \mathscr{P}$ is sequentially continuous. If T is sequentially continuous, then T is continuous.

Step 2. $T : \mathscr{P} \to \mathscr{P}$ is relatively compact operator.

Let Ω be any bounded subset of $\mathscr{P}$. Then there exists $r > 0$ is such that $\|\vartheta\| \leqslant r$ for all $\vartheta \in \Omega$. Therefore, from (H2) and Lemma 6, for all $\vartheta \in \Omega$,

$$\begin{aligned}\frac{T\vartheta(t)}{1+t^{\nu-1}} &\leqslant L\int_0^\infty a(s) f(s,\vartheta(s))ds \\ &= L\int_0^\infty a(s) f\left(s,(1+s^{\nu-1})\frac{\vartheta(s)}{1+s^{\nu-1}}\right)ds \\ &\leqslant LB_r\int_0^\infty a(s)ds \\ &< \infty.\end{aligned}$$

This implies that $\|T\vartheta(t)\| < \infty$. So $T\Omega$ is uniformly bounded. Next, we show that $T\Omega$ is equicontinuous on $[0,\infty)$. For any $a > 0$ and $t_1, t_2 \in [0,a]$, without loss of generality, we may assume that $t_2 > t_1$. For all $\vartheta \in \Omega$, we obtain

$$\begin{aligned}
\left|\frac{(T\vartheta)(t_1)}{1+t_1^{\nu-1}} - \frac{(T\vartheta)(t_2)}{1+t_2^{\nu-1}}\right| &\leqslant \int_0^\infty \left|\frac{G(t_1,s)}{1+t_1^{\nu-1}} - \frac{G(t_2,s)}{1+t_2^{\nu-1}}\right| a(s)f(s,\vartheta(s))ds \\
&\leqslant \int_0^\infty \left|\frac{G_1(t_1,s)}{1+t_1^{\nu-1}} - \frac{G_1(t_2,s)}{1+t_2^{\nu-1}}\right| a(s)f(s,\vartheta(s))ds \\
&\quad + \int_0^\infty \left|\frac{G_2(t_1,s)}{1+t_1^{\nu-1}} - \frac{G_2(t_2,s)}{1+t_2^{\nu-1}}\right| a(s)f(s,\vartheta(s))ds \\
&\leqslant \int_0^\infty \left|\frac{G_1(t_1,s)}{1+t_1^{\nu-1}} - \frac{G_1(t_2,s)}{1+t_2^{\nu-1}}\right| a(s)f(s,\vartheta(s))ds \\
&\quad + \frac{\sum_{i=1}^{m-2}\eta_i\xi_i^{\nu+\kappa-1}\left|\frac{t_1^{\nu-1}}{1+t_1^{\nu-1}} - \frac{t_2^{\nu-1}}{1+t_2^{\nu-1}}\right|}{\Gamma(\nu)[\Gamma(\nu+\kappa) - \sum_{i=1}^{m-2}\eta_i\xi_i^{\nu+\kappa-1}]} \int_0^\infty a(s)f(s,\vartheta(s))ds \\
&\leqslant \int_0^\infty \left|\frac{G_1(t_1,s)}{1+t_1^{\nu-1}} - \frac{G_1(t_2,s)}{1+t_1^{\nu-1}}\right| a(s)f(s,\vartheta(s))ds \\
&\quad + \int_0^\infty \left|\frac{G_1(t_2,s)}{1+t_1^{\nu-1}} - \frac{G_1(t_2,s)}{1+t_2^{\nu-1}}\right| a(s)f(s,\vartheta(s))ds \\
&\quad + \frac{\sum_{i=1}^{m-2}\eta_i\xi_i^{\nu+\kappa-1}\left|\frac{t_1^{\nu-1}}{1+t_1^{\nu-1}} - \frac{t_2^{\nu-1}}{1+t_2^{\nu-1}}\right|}{\Gamma(\nu)[\Gamma(\nu+\kappa) - \sum_{i=1}^{m-2}\eta_i\xi_i^{\nu+\kappa-1}]} \int_0^\infty a(s)f(s,\vartheta(s))ds.
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
&\int_0^\infty \left|\frac{G_1(t_1,s)}{1+t_1^{\nu-1}} - \frac{G_1(t_2,s)}{1+t_1^{\nu-1}}\right| a(s)f(s,\vartheta(s))ds \\
&= \int_0^{t_1} \left|\frac{G_1(t_1,s)}{1+t_1^{\nu-1}} - \frac{G_1(t_2,s)}{1+t_1^{\nu-1}}\right| a(s)f(s,\vartheta(s))ds \\
&\quad + \int_{t_1}^{t_2} \left|\frac{G_1(t_1,s)}{1+t_1^{\nu-1}} - \frac{G_1(t_2,s)}{1+t_1^{\nu-1}}\right| a(s)f(s,\vartheta(s))ds \\
&\quad + \int_{t_2}^\infty \left|\frac{G_1(t_1,s)}{1+t_1^{\nu-1}} - \frac{G_1(t_2,s)}{1+t_1^{\nu-1}}\right| a(s)f(s,\vartheta(s))ds \\
&\leqslant B_r \int_0^{t_1} \frac{(t_2^{\nu-1} - t_1^{\nu-1}) + ((t_2-s)^{\nu-1} - (t_1-s)^{\nu-1})}{1+t_1^{\nu-1}} a(s)ds
\end{aligned}$$

$$+B_r\int_{t_1}^{t_2}\frac{(t_2^{\nu-1}-t_1^{\nu-1})+((t_2-s)^{\nu-1})}{1+t_1^{\nu-1}}a(s)ds$$

$$+B_r\int_{t_2}^{\infty}\frac{(t_2^{\nu-1}-t_1^{\nu-1})}{1+t_1^{\nu-1}}a(s)ds \;\;\rightarrow 0 \;\; as \;\; t_1\rightarrow t_2.$$

In a similar way, one can see that

$$\int_0^{\infty}\left|\frac{G_1(t_2,s)}{1+t_1^{\nu-1}}-\frac{G_1(t_2,s)}{1+t_2^{\nu-1}}\right|a(s)f(s,\vartheta(s))ds\rightarrow 0 \;\; as \;\; t_1\rightarrow t_2.$$

Thus, we have

$$\left|\frac{T\vartheta(t_1)}{1+t_1^{\nu-1}}-\frac{T\vartheta(t_2)}{1+t_2^{\nu-1}}\right|\rightarrow 0 \;\; as \;\; t_1\rightarrow t_2.$$

So, $T\Omega$ is equicontinuous on $[0,\infty)$.
Step 3: $T:\mathscr{P}\rightarrow\mathscr{P}$ is equiconvergent at ∞.
For any $\vartheta\in\Omega$, we get

$$\int_0^{\infty}a(s)f(s,\vartheta(s))ds\leqslant B_r\int_0^{\infty}a(s)ds<\infty.$$

From Lemma 6,we obtain

$$\begin{aligned}\lim_{t\rightarrow\infty}\left|\frac{T\vartheta(t)}{1+t^{\nu-1}}\right| &= \lim_{t\rightarrow\infty}\left|\int_0^{\infty}\frac{G(t,s)}{1+t^{\nu-1}}a(s)f(s,\vartheta(s))ds\right|\\ &\leqslant L\lim_{t\rightarrow\infty}\left|\int_0^{\infty}a(s)f(s,\vartheta(s))ds\right|\\ &<\infty.\end{aligned}$$

Hence, $T\Omega$ is equiconvergent at ∞. Hence, from steps 1-3, $T:\mathscr{P}\rightarrow\mathscr{P}$ is a completely continuous operator.

3. Main Result

In this section, we discuss the existence of three positive solutions for the BVP (1) by using the Six functionals fixed point theorem in [12].

Let α be nonnegative continuous concave functional on $\mathscr{P}$ and β be nonnegative continuous convex functional on $\mathscr{P}$ then for positive numbers r and R, we define the sets:

$$Q(\beta,R)=\{\vartheta\in\mathscr{P}:\beta(\vartheta)\leqslant R\}$$

and

$$and \;\; Q(\alpha,\beta,r,R)=\{\vartheta\in\mathscr{P}:r\leqslant\alpha(\vartheta) \; and \beta(\vartheta)\leqslant R\}.$$

THEOREM 1. *(Six Functionals Fixed Point Theorem)*[12] *Suppose $\mathscr{P}$ is a cone in a real Banach space E. α, ψ and ζ are nonnegative continuous concave functionals on $\mathscr{P}$, β, θ and η are nonnegative continuous convex functionals on $\mathscr{P}$, and there exist nonnegative numbers l, l', r, r', R and R' such that*

$$A : Q(\beta,R) \to \mathscr{P}$$

is a completely continuous operator and

(a) $Q(\beta,R)$ *is a bounded set,*

(b) $Q(\eta,l)$ *and* $Q(\alpha,\beta,r,R)$ *are disjoint subsets of* $Q(\beta,R)$,

(c) $\{u \in \mathscr{P} : \theta(u) < r', r < \alpha(u), R' < \psi(u) \text{ and } \beta(u) < R\} \neq 0$,

(d) $\{u \in \mathscr{P} : l' < \zeta(u) \text{ and } \eta(u) < l\} \neq 0$, *and*

(e) $\{u \in \mathscr{P} : l < \eta(u) \text{ and } \alpha(u) < r\} \neq 0$.

Let the following properties be satisfied

(i) $\alpha(Au) > r$, *for all* $u \in \mathscr{P}$ *with* $\alpha(u) = r, \beta(u) \leqslant R$ *and* $r' < \theta(Au)$,

(ii) $\alpha(Au) > r$, *for all* $u \in \mathscr{P}$ *with* $\alpha(u) = r, \beta(u) \leqslant R$ *and* $\theta(u) \leqslant r'$,

(iii) $\beta(Au) < R$, *for all* $u \in \mathscr{P}$ *with* $r \leqslant \alpha(u), \beta(u) = R$ *and* $\psi(Au) < R'$,

(iv) $\beta(Au) < R$, *for all* $u \in \mathscr{P}$ *with* $r \leqslant \alpha(u), \beta(u) = R$ *and* $R' \leqslant \psi(u)$,

(v) $\eta(Au) < l$, *for all* $u \in \mathscr{P}$ *with* $\eta(u) = l$ *and* $\zeta(Au) < l'$, *and*

(vi) $\eta(Au) < l$, *for all* $u \in \mathscr{P}$ *with* $\eta(u) = l$ *and* $l' \leqslant \zeta(u)$.

Then A has at least three fixed points u_1, u_2 and u_3 in $Q(\beta,R)$ such that

$$\eta(u_1) \leqslant l, r \leqslant \alpha(u_2) \text{ with } \beta(u_2) \leqslant R \text{ and } l < \eta(u_3) \text{ with } \alpha(u_3) < r.$$

Define the concave functionals α, ψ *ve* ζ by

$$\alpha(\vartheta) := \min_{\frac{1}{k} \leqslant t \leqslant k} \frac{\vartheta(t)}{1+t^{\nu-1}}, \quad \psi(\vartheta) = \zeta(\vartheta) := \frac{2\tau k(1+k^{\nu-1})}{k^2-1} \int_{\frac{1}{k}}^{k} \frac{\vartheta(t)}{1+t^{\nu-1}} dt$$

and the convex functionals θ, β *ve* η

$$\theta(\vartheta) := \max_{t \in [0,\infty)} \frac{\vartheta(t)}{1+t^{\nu-1}}, \quad \beta(\vartheta) = \eta(\vartheta) := \int_0^{\infty} \frac{\vartheta(t)}{1+t^{\nu-1}} dt,$$

where $\tau = \displaystyle\int_0^{\infty} \frac{1}{1+t^{\nu-1}} dt = \frac{\pi cosec(\frac{\pi}{\nu-1})}{\nu-1} > 1$.

Let $k > 1$, for the convenience, we take the notations $N = \dfrac{\int_{\frac{1}{k}}^{k} a(s)ds}{(1+k^{\nu-1})\Gamma(\nu)k^{\nu-1}}$ and $M = L\displaystyle\int_0^{\infty} a(s)ds$, where L is defined by Lemma 6.

THEOREM 2. *Assume that* $(H1)-(H3)$ *hold. If there exist constants* r, r', R *and* R' *with* $r < \frac{N}{M}R$ *and* $\tau(1+k^{\nu-1})r < R' < r' < \frac{R}{\tau}$, *and suppose that* $F(t,\vartheta)$ *satisfies the following conditions:*

$(H4)$ $F(t,\vartheta(t)) > \frac{r}{N}$ *for all* $(t,\vartheta(t)) \in \left[\frac{1}{k},k\right] \times [r,r']$,

$(H5)$ $F(t,\vartheta(t)) < \frac{R}{M}$ *for all* $(t,\vartheta(t)) \in [0,\infty) \times [r,\infty)$.

Then the boundary value problem (1) *has at least three positive solutions* ϑ_1, ϑ_2 *and* $\vartheta_3 \in Q(\beta,R)$.

Proof. Let $r' = \frac{(1+k^{\nu-1})r}{\lambda(k)}$, $R' = \frac{R}{k+1}$, $l = \frac{r(k^2-1)\lambda(k)}{k^2}$ and $l' = \frac{l}{k+1}$. By Lemma 10, we have that

$$T : Q(\beta,R) \to \mathscr{P}$$

is a completely continuous operator. Applying a standard calculus argument, we have that the set $Q(\beta,R)$ is bounded, since if $\vartheta \in Q(\beta,R)$, then $\lambda(k)\|\vartheta\| \leqslant \alpha(\vartheta)$, and hence $\|\vartheta\| \leqslant \frac{kR}{(k^2-1)\lambda(k)}$.

Also, it can easily be shown that

$$\frac{1}{2}\left[r' + \frac{R}{(k+1)\tau}\right] \in \left\{\vartheta \in \mathscr{P} : \theta(\vartheta) < r', r < \alpha(\vartheta), R' < \psi(\vartheta) \text{ and } \beta(\vartheta) < R\right\},$$

$$\frac{l}{\tau k} \in \left\{\vartheta \in \mathscr{P} : l' < \zeta(\vartheta) \text{ and } \eta(\vartheta) < l\right\},$$

and

$$\frac{r}{\tau} \in \{\vartheta \in \mathscr{P} : l < \eta(\vartheta) \text{ and } \alpha(\vartheta) < r\}.$$

and hence the sets are nonempty. Furthermore, if $\vartheta \in Q(\eta,l)$, then $\lambda(k)\|\vartheta\| \leqslant \alpha(\vartheta)$, and hence $\|\vartheta\| \leqslant \frac{kl}{(k^2-1)\lambda(k)}$, and so, we get

$$\alpha(\vartheta) = \min_{\frac{1}{k} \leqslant t \leqslant k} \frac{\vartheta(t)}{1+t^{\nu-1}} \leqslant \max_{t \geqslant 0} \frac{\vartheta(t)}{1+t^{\nu-1}} = \|\vartheta\|$$

$$\leqslant \frac{kl}{(k^2-1)\lambda(k)} \leqslant \frac{kr(k^2-1)\lambda(k)}{k^2(k^2-1)\lambda(k)} \leqslant \frac{r}{k} < r.$$

This implies that $\vartheta \notin Q(\alpha,\beta,r,R)$. Hence, the set conditions of (a), (b), (c), (d) and (e) of Theorem 1 are obtained. Now, we satisfy the functional conditions.

Claim 1. $\alpha(T\vartheta) > r$, for all $\vartheta \in Q(\alpha,\beta,r,R)$ with $\alpha(\vartheta) = r$, $\beta(\vartheta) \leqslant R$ and $r' < \theta(T\vartheta)$.

Let $\vartheta \in Q(\alpha,\beta,r,R)$ with $\alpha(\vartheta) = r$ and $r' < \theta(T\vartheta)$. Then by Lemma 9, we have

$$\begin{aligned}\alpha(T\vartheta) &= \min_{\frac{1}{k}\leqslant t\leqslant k} \frac{T\vartheta(t)}{1+t^{\nu-1}} \geqslant \lambda(k)\|T\vartheta\| > \lambda(k)r' \\ &= \frac{\lambda(k)(1+k^{\nu-1})r}{\lambda(k)} = (1+k^{\nu-1})r > r.\end{aligned}$$

Claim 2. $\alpha(T\vartheta) > r$, for all $\vartheta \in Q(\alpha,\beta,r,R)$ with $\alpha(\vartheta) = r$, $\beta(\vartheta) \leqslant R$ and $\theta(\vartheta) \leqslant r'$.

Let $\vartheta \in Q(\alpha,\beta,r,R)$ with $\alpha(\vartheta) = r$ and $\theta(\vartheta) \leqslant r'$. By using $\vartheta \in Q(\alpha,\beta,r,R)$ and $\theta(\vartheta) \leqslant r'$, we obtain $r \leqslant \frac{\vartheta(s)}{1+s^{\nu-1}} \leqslant r'$ for $s \in \left[\frac{1}{k},k\right]$. Using (H4), $F\left(s,\frac{\vartheta(s)}{1+s^{\nu-1}}\right) > \frac{r}{N}$, $\left(s,\frac{\vartheta(s)}{1+s^{\nu-1}}\right) \in \left[\frac{1}{k},k\right] \times [r,r']$. We get

$$\begin{aligned}\alpha(T\vartheta(t)) &= \min_{\frac{1}{k}\leqslant t\leqslant k} \frac{T\vartheta(t)}{1+t^{\nu-1}} \\ &= \min_{\frac{1}{k}\leqslant t\leqslant k} \frac{1}{1+t^{\nu-1}} \int_0^\infty G(t,s)a(s)f(s,\vartheta(s))ds \\ &\geqslant \int_0^\infty \min_{\frac{1}{k}\leqslant t\leqslant k} \frac{G(t,s)}{1+t^{\nu-1}} a(s)f(s,\vartheta(s))ds \\ &\geqslant \int_{\frac{1}{k}}^k \frac{G_1(\frac{1}{k},s)}{1+k^{\nu-1}} a(s)f(s,\vartheta(s))ds \\ &\geqslant \frac{1}{1+k^{\nu-1}} \int_{\frac{1}{k}}^k G_1(\frac{1}{k},s)a(s)F\left(s,\frac{\vartheta(s)}{1+s^{\nu-1}}\right)ds \\ &> \frac{r}{N}\frac{1}{(1+k^{\nu-1})\Gamma(\nu)k^{\nu-1}} \int_{\frac{1}{k}}^k a(s)ds \\ &= r\end{aligned}$$

Claim 3. $\beta(T\vartheta) < R$, for all $\vartheta \in Q(\alpha,\beta,r,R$ with $r \leqslant \alpha(\vartheta)$, $\beta(\vartheta) = R$ and $\psi(T\vartheta) < R'$.

Let $\vartheta \in Q(\alpha,\beta,r,R), \beta(\vartheta) = R$ and $\psi(T\vartheta) < R'$. If $\vartheta \in Q(\alpha,\beta,r,R)$, then $r \leqslant \frac{\vartheta(s)}{1+s^{\nu-1}}$ for $s \in [0,\infty)$. Using (H5), $F\left(s,\frac{\vartheta(s)}{1+s^{\nu-1}}\right) < \frac{R}{M}$, $\left(s,\frac{\vartheta(s)}{1+s^{\nu-1}}\right) \in$

$[0,\infty)\times[r,\infty)$. We obtain

$$\begin{aligned}
\beta(T\vartheta(t)) &= \int_0^\infty \frac{T\vartheta(t)}{1+t^{\alpha-1}}ds\\
&= \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}}a(s)f(s,\vartheta(s))ds\\
&\leqslant L\int_0^\infty a(s)f(s,\vartheta(s))ds\\
&\leqslant L\int_0^\infty a(s)F\left(s,\frac{\vartheta(s)}{1+s^{\nu-1}}\right)ds\\
&< \frac{R}{M}L\int_0^\infty a(s)ds\\
&= R
\end{aligned}$$

Note, the same argument as in claim 3 can be used to verify that $\eta(T\vartheta)<l$, for all $\forall\vartheta\in Q(\eta,l)$, with $\eta(\vartheta)=l$ and $\zeta(T\vartheta)<l'$ by simply replacing R with l, R' with l' in the arguments.

Claim 4. $\beta(T\vartheta)<R$, $\forall\vartheta\in Q(\alpha,\beta,r,R)$ with $r\leqslant\alpha(\vartheta)$, $\beta(\vartheta)=R$ and $R'\leqslant\psi(\vartheta)$.

Let $\vartheta\in Q(\alpha,\beta,r,R)$ with $\beta(\vartheta)=R$ and $R'\leqslant\psi(\vartheta)$. If $\vartheta\in Q(\alpha,\beta,r,R)$, then $r\leqslant\dfrac{\vartheta(s)}{1+s^{\nu-1}}$ for $s\in[0,\infty)$. Using (H5), $F\left(s,\dfrac{\vartheta(s)}{1+s^{\nu-1}}\right)<\dfrac{R}{M}$, $\left(s,\dfrac{\vartheta(s)}{1+s^{\nu-1}}\right)\in[0,\infty)\times[r,\infty)$. We have

$$\begin{aligned}
\beta(T\vartheta(t)) &= \int_0^\infty \frac{T\vartheta(t)}{1+t^{\alpha-1}}ds\\
&= \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}}a(s)f(s,\vartheta(s))ds\\
&\leqslant L\int_0^\infty a(s)f(s,\vartheta(s))ds\\
&\leqslant L\int_0^\infty a(s)F\left(s,\frac{\vartheta(s)}{1+s^{\nu-1}}\right)ds\\
&< \frac{R}{M}L\int_0^\infty a(s)ds\\
&= R
\end{aligned}$$

Note, the same argument as in Claim 4 can be used to verify that $\eta(T\vartheta)<l$, for all $\forall\vartheta\in Q(\eta,l)$, with $\eta(\vartheta)=l$ and $l'\leqslant\zeta(\vartheta)$ by simply replacing R with l, R' with l' in the arguments. From the conditions (i) - (vi) and Theorem 1, T has at least three positive solutions ϑ_1,ϑ_2 and ϑ_3 belonging to $Q(\beta,R)$ of BVP (1) such that

$$\eta(\vartheta_1)\leqslant l, r\leqslant\alpha(\vartheta_2)\ \textit{with}\ \beta(\vartheta_2)\leqslant R\ \textit{and}\ l<\eta(\vartheta_3)\ \textit{with}\ \alpha(\vartheta_3)<r.$$

4. An Example

EXAMPLE 1. Consider the following boundary value problem:

$$\begin{cases} D_{0+}^{\frac{5}{2}}\vartheta(t)+16^{-t}ln16f(t,\vartheta(t))=0, \ \ t\in[0,\infty), \\ \vartheta(0)=\vartheta'(0)=0, \\ D_{0+}^{\frac{3}{2}}\vartheta(\infty)=\sum_{i=1}^{2}\eta_i I_{0+}^{\frac{1}{2}}\vartheta(\xi_i), \end{cases} \tag{7}$$

where

$$F(t,\vartheta(t))=\begin{cases} 192+\frac{1329}{5183}(\vartheta-1)+\frac{t}{10}, & (t,\vartheta)\in\left[\frac{1}{4},4\right]\times[1,5184], \\ 6180, & (t,\vartheta)\in[0,\infty)\times[1,\infty). \end{cases}$$

By simple calculations, we have

$$L=0,8753486872, \quad N=0,0052237687, M=2,4269818981,$$
$$\tau=\int_0^\infty \frac{1}{1+t^{\frac{3}{2}}}dt=\frac{4\pi}{3\sqrt{3}}.$$

If we choose $r=1$ and $R=15000$, then we get

$$1=r<\frac{N}{M}R=32,2855845$$

and

$$1=r<R'=3000<r'=5148<\frac{R}{\tau}=6202,4500738.$$

It can be easily seen that the conditions $(H1)-(H3)$ satisfied. Now we show that conditions $(H4)$ and $(H5)$ are satisfied.

$$F(s,\vartheta(s))\geqslant 192,025>\frac{1}{0,0052237687}=191,43267197=\frac{r}{N},$$
$$(s,\vartheta)\in\left[\frac{1}{4},4\right]\times[1,5184],$$
$$F(s,\vartheta(s))=6180<\frac{15000}{2,4269818981}=6180,5158134=\frac{R}{M},$$
$$(s,\vartheta)\in[0,\infty]\times[1,\infty].$$

So, all conditions of Theorem 2 hold. Thus by Theorem 2, the BVP (7) has at least three positive solutions ϑ_1,ϑ_2 and ϑ_3 belonging to $Q(\beta,15000)$ such that

$$\int_0^\infty \frac{\vartheta_1(t)}{1+t^{\frac{3}{2}}}dt\leqslant l, \ \ r\leqslant \min_{\frac{1}{4}\leqslant t\leqslant 4}\frac{\vartheta_2(t)}{1+t^{\frac{3}{2}}} \quad \text{with} \quad \int_0^\infty \frac{\vartheta_2(t)}{1+t^{\frac{3}{2}}}dt\leqslant R$$

and

$$l<\int_0^\infty \frac{\vartheta_3(t)}{1+t^{\frac{3}{2}}}dt \quad \text{with} \quad \min_{\frac{1}{4}\leqslant t\leqslant 4}\frac{\vartheta_3(t)}{1+t^{\frac{3}{2}}}<r.$$

5. Conclusion

In this paper, by applying the six functionals fixed point theorem [12], which is a generalization of the five functionals fixed point theorem [13] and Leggett-Williams fixed point theorem [15], we investigate the existence of at least three positive solutions for the m-point fractional boundary value problem on the half line. And then an appropriate example that support the theoretical results is provided.

REFERENCES

[1] A. A. KILBAS, H. M. SRIVASTAVA AND J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies 204 (2006).

[2] C. ZHAI AND L. WANG, *Some existence, uniqueness results on positive solutions for a fractional differential equation with infinite-point boundary conditions*, Nonlinear Anal. Model. Control **22(4)** (2017), 566–567.

[3] G. WANG, *Explicit iteration and unbounded solutions for fractional integral boundary value problem on an infinite interval*, Appl. Math. Lett. **47** (2015), 1–7.

[4] I. PODLUBNY, *Fractional Differential Equations*, Academic Press (1999).

[5] I. YASLAN AND M. GUNENDI, *Positive solutions of higher-order nonlinear multi-point fractional equations with integral boundary conditions*, Fract. Calc. Appl. Anal. **19(4)** (2016), 989–1009.

[6] J. GRAEF, L. KONG, Q. KONG AND M. WANG, *Uniqueness of positive solutions of fractional boundary value problems with non-homogeneous integral boundary conditions*, Fract. Calc. Appl. Anal. **15(3)** (2012), 509–528.

[7] K. ZHANG AND J. XU, *Unique positive solution for a fractional boundary value problem*, Fract. Calc. Appl. Anal. **16(4)** (2013), 937–948.

[8] L. ZHANG, B. AHMAD, G. WANG, R. AGARWAL, M. AL-YAMI AND W. SHAMMAKH, *Nonlocal integrodifferential boundary value problem for nonlinear fractional differential equations on an unbounded domain*, Abstr. Appl. Anal. (2013), Article ID 813903.

[9] M. DALIR AND M. BASHAUR, *Applications of fractional calculus*, Appl. Math. Sci. (Ruse) **4** (2010), 1021–1032.

[10] M. REHMAN AND R. A. KHAN, *Existence and uniqueness of solutions for multi-point boundary value problems for fractional differential equations*, Appl. Math. Lett. **23(9)** (2010), 1038–1044.

[11] N. ABEL, *Solution de quelques problèmes à I'aide d'intégrales définies*, Mag. Naturv **1(2)** (1823), 1–27.

[12] R. AVERY, J. HENDERSON AND D. O'REGAN, *Six functionals fixed point theorem*, Commun. Appl. Anal. **12(1)** (2008), 69–81.

[13] R. I. AVERY, *A generalization of the Leggett-Williams fixed point theorem*, Math. Sci. Res. HotLine **3(7)** (1999), 9–14.

[14] R. P. AGARWAL, D. O'REGAN AND M. MEEHAN, *Fixed Point Theory and Applications*, Found. Phys. Cambridge University Press (2004).

[15] R. W. LEGGETT AND L. R. WILLIAMS, *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana Univ. Math. J. **28(4)** (1979), 673–688.

[16] S. BANACH, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math **3(1)** (1922), 133–181.

[17] S. LIANG AND J. ZHANG, *Existence of Three Positive Solutions of m-point Boundary Value Problems for Some Nonlinear Fractional Differential Equations on an Infinite Interval*, Comput. Math. Appl. (2011), 3343–3354.

[18] W. GE AND X. ZHAO, *Unbounded Solutions for a Fractional Boundary Value Problems on the Infinite Interval*, Acta Appl. Math. **109** (2008), 495–505.

[19] X. LI, X. LIU, M. JIA AND L. ZHANG, *The positive solutions of infinite-point boundary value problem of fractional differential equations on the infinite interval*, Adv. Difference Equ. **126** (2017).

[20] X. SU, *Boundary value problem for a coupled system of nonlinear fractional differential equations*, Appl. Math. Lett. **22(1)** (2009), 64–69.

[21] X. YUE AND K. ZHANG, *Existence of solution for integral boundary value problems of fractional differential equations*, Bound. Value Probl. **151** (2018), 13 pp.
[22] Y. GHOLAMI, *Existence of an Unbounded Solution for Multi-Point Boundary Value Problems of Fractional Differential Equations on an Infinite Domain*, Fract. Differ. Calc. **4(2)** (2014), 125–136.

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