STUDY OF A DISCRETIZED FRACTIONAL–ORDER ECO–EPIDEMIOLOGICAL MODEL WITH PREY INFECTION

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Abstract. In this paper, an attempt is made to understand the dynamics of a three-dimensional discrete fractional-order eco-epidemiological model with Holling type II functional response. We first discretize a fractional-order predator-prey-parasite system with piecewise constant arguments and then explore the system dynamics. Analytical conditions for the local stability of different fixed points have been determined using the Jury criterion. Several examples are given to substantiate the analytical results. Our analysis shows that stability of the discrete fractional order system strongly depends on the step-size and the fractional order. More specifically, the critical value of the step-size, where the switching of stability occurs, decreases as the order of the fractional derivative decreases. Simulation results explore that the discrete fractional-order system may also exhibit complex dynamics, like chaos, for higher step-size.

1. Introduction

The idea of fractional calculus has been known since the notion of fractional derivative was firstly introduced by Leibniz in a letter dated 30th September, 1695, where half-order derivative was mentioned. Since then it was subsequently developed by Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Riemann, Heaviside, Caputo along with many others [1]. Generally, fractional calculus deals with the study of fractional order integral and derivative operators over real or complex domains and their applications [2]. Specifically, it is a generalization of classical differential and integral calculus of integer order to arbitrary order. In recent times, fractional order derivatives and fractional order differential equations have been applied in several fields of science and engineering [3, 4, 5, 6, 7, 8, 9, 10], however, initially it was treated as a topic of interest of pure mathematicians only [11]. The reason is two-fold. First, fractional derivatives have an additional degree of freedom over its integer order counterpart due to the additional parameter that represents its order and more suitable for those systems having higher order dynamics and complex nonlinear phenomena [12, 13]. Secondly

and more importantly, fractional order derivatives not only depend on the local conditions but also on the history of the function [14] and, therefore, fractional derivatives have become an efficient tool for those systems, where the consideration of memory or hereditary properties of the function is essential to represent the system, e.g., in the case of biological systems. In the last two decades, fractional order calculus has found its many applications in biological sciences [15, 16, 17, 18, 19, 20, 21, 22, 23].

Discrete system of a continuous time model has significant applications in simulations and computations. Particularly, the population model can be better described in discrete time and it is especially applicable in populations having non-overlapping generations, e.g., insect populations. Such discrete models are known to show richer dynamics compare to its continuous counterpart [24, 25, 26, 27, 28, 29]. Thus, both from biological and dynamic point of views, study of discrete system is important. Though the number of discrete population models is numerous, the study of discrete fractional-order population models is rarely observed in the literature. Recently, a transformed method was employed to obtain solutions of a family of finite fractional difference equations in [30]. The global and local existence of solutions for the fractional difference equation are given in [31]. Numerical solution was adopted to synchronize chaos in a discrete fractional logistic map [32]. For an ecological system, Elsadany and Matouk [33] recently studied a two-dimensional fractional-order Lotka-Volterra predator prey model with its discretization and showed complex dynamics. The existence and uniqueness of solutions in two-dimensional discrete fractional Lotka-Volterra model was addressed in [34]. Selvam and Janagaraj [35] studied the dynamics of a two-dimensional discrete fractional order predator-prey system with type II functional response. A two-dimensional fractional-order predator-prey model with saturated harvesting was studied in [36]. Local stability analysis and bifurcation analysis have been done in a simple fractional-order SIR model by Salman [37]. Similar analysis was done for a fractional-order discrete SI epidemic model in [38]. Discretization has also been done in other fractional-order biological and physical systems [39, 40, 41, 42, 43, 44, 45]. These relatively simple models show that the fractional-order plays a critical role in breaking the stability. By considering a simple two-dimensional epidemic model, Raheem and Salman [46] have shown that the dynamic complexity depends also on the step-size of the discrete system. Analytical complexity, however, multiplies if the dimension of the system increases and/or strong nonliearity is there. In this paper, we construct a discrete version of a highly nonlinear three-dimensional fractional-order predator-prey-parasite (PPP) model and reveal its dynamics. So far our knowledge goes, no study has been done on a discrete fractional-order PPP system. We unveil the dynamic properties of this highly nonlinear model both from analytical and numerical points of view. We show analytically that the stability of different fixed points depends on both the fractional order and step-size. It also shows complex dynamics like chaos as shown by the continuous time integer order system for some parameter values.

For this study, we assume the following continuous-time PPP model studied by Chattopadhyay and Bairagi [47]:

$$\frac{dX}{dt} = rX\left(1 - \frac{X+Y}{K}\right) - \lambda YX,$$
$$\frac{dY}{dt} = \lambda YX - \frac{mYZ}{a+Y} - \mu Y, \tag{1}$$
$$\frac{dZ}{dt} = \frac{\theta YZ}{a+Y} - dZ.$$

It says that a prey species is divided into two sub classes: susceptible prey (X) and infected prey (Y) due to some micro-parasite infection. Susceptible prey can give birth and grow logistically to its maximum value K supported by the environment with intrinsic growth rate r. The infection spreads by contact following mass action law with λ as its force of infection. Infected prey is unable to reproduce but share resources with susceptible prey. Infected prey dies out at a rate μ, which considers both the natural and infection-related deaths. Predator population Z consumes infected preys only as they are weakened due to infection and can not escape predation. Here m is the maximum prey attack rate of predator and a is the half-saturation constant. The parameter θ $(0 < \theta < 1)$ measures the reproductive gain of predator and d measures its natural mortality. All parameters are assumed to be positive from biological point of view. Readers are referred to [47] for more discussion about the model.

Table 1.1. Parameter descriptions and their units

Parameters	Description	Units
X	Density of susceptible prey	Number per unit designated area
Y	Density of infected prey	Number per unit designated area
Z	Density of predator	Number per unit designated area
r	Intrinsic growth rate of prey	Per day
K	Carrying capacity	Number per unit designated area
a	Half-saturation constant	Number per unit designated area
λ	Force of infection	Per day
m	Search rate for infected prey	Per day
μ	Total (natural + disease-induced) death rate of infected prey	Per day
θ	Reproductive gain	Per day
d	Death rate of predator	Per day
α	Fractional order	$0 < \alpha < 1$

Considering the fractional derivative in caputo sense, Mondal et al. [23] extended the work of Chattopadhyay and Bairagi [47] and analyzed the following fractional order PPP model (1):

$$^{c}_{0}D^{\alpha}_{t}X = rX\left(1 - \frac{X+Y}{K}\right) - \lambda YX, \, X(0) > 0,$$
$$^{c}_{0}D^{\alpha}_{t}Y = \lambda YX - \frac{mYZ}{a+Y} - \mu Y, \, Y(0) > 0, \tag{2}$$

$$\,^c_0D^\alpha_t Z = \frac{\theta YZ}{a+Y} - dZ, \ Z(0) > 0,$$

where $\,^c_0D^\alpha_t$ is the Caputo fractional derivative with fractional order $\alpha \ (0 < \alpha < 1)$. It is worth mentioning that the systems (1) and (2) become identical for $\alpha = 1$. They have shown that the solutions of system (2) are positively invariant and uniformly bounded in R^3_+ under some restrictions. Existence and uniqueness of solution of system (2) have also been discussed. Local and global stability of all biologically feasible equilibrium points of system (2) were also proved. Here we discritize the model system (2) and explore its dynamics.

2. Discretized fractional-order model and its analysis

Following Elsadany and Matouk [33], discretization of model system (2) with piecewise constant arguments can be done in the following manner:

$$\,^c_0D^\alpha_t X = rX([t/s]s)\left(1 - \frac{X([t/s]s) + Y([t/s]s)}{K}\right) - \lambda X([t/s]s)Y([t/s]s),$$

$$\,^c_0D^\alpha_t Y = \lambda X([t/s]s)Y([t/s]s) - \frac{mY([t/s]s)Z([t/s]s)}{(a+Y([t/s]s))} - \mu Y([t/s]s),$$

$$\,^c_0D^\alpha_t Z = \frac{\theta Y([t/s]s)Z([t/s]s)}{(a+Y([t/s]s))} - dZ([t/s]s),$$

with initial condition $X(0) = X_0 > 0$, $Y(0) = Y_0 > 0, Z(0) = Z_0 > 0$ and $s(>0)$ is the step-size. Let $t \in [0,s)$, so that $t/s \in [0,1)$. In this case, we have

$$\,^c_0D^\alpha_t X = X_0\left(r(1 - \frac{X_0 + Y_0}{K}) - \lambda Y_0\right),$$

$$\,^c_0D^\alpha_t Y = Y_0\left(\lambda X_0 - \frac{mZ_0}{(a+Y_0)} - \mu\right),$$

$$\,^c_0D^\alpha_t Z = Z_0\left(\frac{\theta Y_0}{(a+Y_0)} - d\right),$$

and the solution of this fractional differential equation can be written as

$$X_1(t) = X_0 + J^\alpha_0\left(X_0\left(r(1 - \frac{X_0 + Y_0}{K}) - \lambda Y_0\right)\right)$$

$$= X_0 + \frac{t^\alpha}{\alpha\Gamma(\alpha)}\left(X_0\left(r(1 - \frac{X_0 + Y_0}{K}) - \lambda Y_0\right)\right),$$

$$Y_1(t) = Y_0 + J^\alpha_0\left(Y_0\left(\lambda X_0 - \frac{mZ_0}{(a+Y_0)} - \mu\right)\right)$$

$$= Y_0 + \frac{t^\alpha}{\alpha\Gamma(\alpha)}\left(Y_0\left(\lambda X_0 - \frac{mZ_0}{(a+Y_0)} - \mu\right)\right),$$

$$Z_1(t) = Z_0 + J^\alpha_0\left(Z_0\left(\frac{\theta Y_0}{(a+Y_0)} - d\right)\right)$$

$$= Z_0 + \frac{t^\alpha}{\alpha\Gamma(\alpha)}\left(Z_0\left(\frac{\theta Y_0}{(a+Y_0)} - d\right)\right),$$

where $J_0^\alpha = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}d\tau, \alpha > 0$.

In the second step, we assume $t \in [s, 2s)$ so that $t/s \in [1,2)$ and similarly obtain

$$_0^cD_t^\alpha X = X_1(s)\left(r(1 - \frac{X_1(s)+Y_1(s)}{K}) - \lambda Y_1(s)\right),$$

$$_0^cD_t^\alpha Y = Y_1(s)\left(\lambda X_1(s) - \frac{mZ_1(s)}{(a+Y_1(s))} - \mu\right),$$

$$_0^cD_t^\alpha Z = Z_1(s)\left(\frac{\theta Y_1(s)}{(a+Y_1(s))} - d\right).$$

The solution of this equation reads

$$X_2(t) = X_1(s) + J_s^\alpha\left(X_1(s)\left(r(1 - \frac{X_1(s)+Y_1(s)}{K}) - \lambda Y_1(s)\right)\right)$$

$$= X_1(s) + \frac{(t-s)^\alpha}{\alpha\Gamma(\alpha)}\left(X_1(s)\left(r(1 - \frac{X_1(s)+Y_1(s)}{K}) - \lambda Y_1(s)\right)\right),$$

$$Y_2(t) = Y_1(s) + J_s^\alpha\left(Y_1(s)\left(\lambda X_1(s) - \frac{mZ_1(s)}{(a+Y_1(s))} - \mu\right)\right)$$

$$= Y_1(s) + \frac{(t-s)^\alpha}{\alpha\Gamma(\alpha)}\left(Y_1(s)\left(\lambda X_1(s) - \frac{mZ_1(s)}{(a+Y_1(s))} - \mu\right)\right),$$

$$Z_2(t) = Z_1(s) + J_s^\alpha\left(Z_1(s)\left(\frac{\theta Y_1(s)}{(a+Y_1(s))} - d\right)\right)$$

$$= Z_1(s) + \frac{(t-s)^\alpha}{\alpha\Gamma(\alpha)}\left(Z_1(s)\left(\frac{\theta Y_1(s)}{(a+Y_1(s))} - d\right)\right),$$

where $J_s^\alpha = \frac{1}{\Gamma(\alpha)}\int_s^t (t-\tau)^{\alpha-1}d\tau, \alpha > 0$.

Repeating the discretization process n times, we have

$$X_{n+1}(t) = X_n(ns) + \frac{(t-ns)^\alpha}{\alpha\Gamma(\alpha)}\left(X_n(ns)\left(r(1 - \frac{X_n(ns)+Y_n(ns)}{K}) - \lambda Y_n(ns)\right)\right),$$

$$Y_{n+1}(t) = Y_n(ns) + \frac{(t-ns)^\alpha}{\alpha\Gamma(\alpha)}\left(Y_n(ns)\left(\lambda X_n(ns) - \frac{mZ_n(ns)}{(a+Y_n(ns))} - \mu\right)\right),$$

$$Z_{n+1}(t) = Z_n(ns) + \frac{(t-ns)^\alpha}{\alpha\Gamma(\alpha)}\left(Z_n(ns)\left(\frac{\theta Y_n(ns)}{(a+Y_n(ns))} - d\right)\right),$$

where $t \in [ns, (n+1)s)$.

Making $t \to (n+1)s$, the corresponding fractional discrete model of the continuous fractional model (2) is obtained as

$$X_{n+1} = X_n + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(X_n\left(r(1 - \frac{X_n+Y_n}{K}) - \lambda Y_n\right)\right),$$

$$Y_{n+1} = Y_n + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(Y_n\left(\lambda X_n - \frac{mZ_n}{(a+X_n)} - \mu\right)\right),$$

$$Z_{n+1} = Z_n + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(Z_n\left(\frac{\theta Y_n}{(a+Y_n)} - d\right)\right). \tag{3}$$

For $\alpha = 1$, one obtains the so called Euler forward discrete model as a special case of this generalized discrete model.

2.1. Existence and stability of fixed points

In the following, we investigate the dynamics of the discretized fractional order model (3). At the fixed point, we have $X_{n+1} = X_n$, $Y_{n+1} = Y_n$ and $Z_{n+1} = Z_n$. One can then easily compute that (3) has the same fixed points as in the integer and fractional order systems (1) and (2). The system (3) has four equilibrium points [23, 47], viz. $E_0 = (0,0,0)$ as the trivial equilibrium, $E_1 = (K,0,0)$ as the axial equilibrium, $E_2 = (X_1, Y_1, 0)$ as the planar equilibrium, where $X_1 = \frac{\mu}{\lambda}$ and $Y_1 = \frac{r(\lambda K - \mu)}{\lambda(r+\lambda K)}$ and $E^* = (X^*, Y^*, Z^*)$ as the interior equilibrium, where

$$X^* = K - \left(1 + \frac{\lambda K}{r}\right)Y^*, \quad Y^* = \frac{ad}{\theta - d}, \quad Z^* = \frac{(a+Y^*)(\lambda X^* - \mu)}{m}. \tag{4}$$

Note that the equilibria E_0 and E_1 always exist. The planar equilibrium point E_2 exists if $R_0 > 1$, where $R_0 = \frac{\lambda K}{\mu}$. The interior equilibrium E^* exists if (i) $R_0 > 1$ and (ii) $\theta > \theta_1$, where $\theta_1 = d + \frac{\lambda ad(r+\lambda K)}{r(\lambda K - \mu)}$.

The Jacobian matrix of system (3) at any arbitrary fixed point (X,Y,Z) reads

$$J(X,Y,Z) = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}, \tag{5}$$

where

$$a_{11} = 1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(r(1 - \frac{2X+Y}{K}) - \lambda Y\right),$$

$$a_{12} = -\frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(X(\lambda + \frac{r}{K})\right),$$

$$a_{21} = \frac{s^\alpha}{\alpha\Gamma(\alpha)}\lambda Y,$$

$$a_{22} = 1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\lambda X - \frac{mZ}{a+Y} - \mu\right) + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{mYZ}{(a+Y)^2},$$

$$a_{23} = -\frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{mY}{a+Y},$$

$$a_{32} = \frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{a\theta Z}{(a+Y)^2},$$

$$a_{33} = 1 + \frac{s^{\alpha}}{\alpha \Gamma(\alpha)} \left(\frac{\theta Y}{a+Y} - d \right).$$

Let ξ_1, ξ_2 and ξ_3 be the eigenvalues of the Jacobian matrix (5). Then we have the following definition and lemma.

DEFINITION 1. [37, 45] A fixed point (X, Y, Z) of system (3) is called a sink which is locally asymptotically stable if $|\xi_i| < 1$ $(i = 1, 2, 3)$ for all i. It is called a source which is unstable if $|\xi_i| > 1$ $(i = 1, 2, 3)$ for all i. It is called a saddle if any two eigenvalues follow opposite inequality, and a non-hyperbolic fixed point if $|\xi_i| = 1$ $(i = 1, 2, 3)$ for at least one i.

LEMMA 1. [48] *Suppose the characteristic polynomial $F(\xi)$ of a second order Jacobian matrix is given by $F(\xi) = \xi^2 - a_1 \xi + a_2$. Then the solutions ξ_i, $i = 1, 2$, of $F(\xi) = 0$ satisfy $|\xi_i| < 1$ for $i = 1, 2$, if the following conditions hold:*

$$(i) \ a_2 < 1, \ (ii) \ 1 + a_2 > |a_1|.$$

LEMMA 2. [48] *Suppose the characteristic polynomial $p(\xi)$ of Jacobian matrix (5) is given by $p(\xi) = \xi^3 + a_1 \xi^2 + a_2 \xi + a_3$. Then the solutions ξ_i, $i = 1, 2, 3$, of $p(\xi) = 0$ satisfy $|\xi_i| < 1$ for $i = 1, 2, 3$ if the following conditions hold:*
(i) $p(1) = 1 + a_1 + a_2 + a_3 > 0$, (ii) $(-1)^3 p(-1) = 1 - a_1 + a_2 - a_3 > 0$, and (iii) $1 - (a_3)^2 > |a_2 - a_3 a_1|$.

2.2. Main result

We now prove our main theorem in relation to the stability of the discrete fractional order system (3).

THEOREM 1. *(a) The fixed point E_0 is always unstable for any feasible values of α and s.*

(b) The fixed point E_1 is locally asymptotically stable for $0 < \alpha \leqslant 1$ if $R_0 = \frac{\lambda K}{\mu} < 1$, $s < \min\{s_2, s_3, s_4\}$, where $s_2 = \sqrt[\alpha]{\frac{2\alpha \Gamma(\alpha)}{d}}$, $s_3 = \sqrt[\alpha]{\frac{2\alpha \Gamma(\alpha)}{r}}$ and $s_4 = \sqrt[\alpha]{\frac{2\alpha \Gamma(\alpha)}{\mu - \lambda K}}$. When $R_0 > 1$, the fixed point E_1 is always unstable for $0 < \alpha \leqslant 1$ for any step size s.

(c) The fixed point E_2 is locally asymptotically stable for $0 < \alpha \leqslant 1$ if $R_0 = \frac{\lambda K}{\mu} > 1$, $s_5 < s < \min\{s_6, s_7\}$ with $d > d_1$, where $s_5 = \sqrt[\alpha]{\frac{\alpha \Gamma(\alpha)}{\lambda K - \mu}}$, $s_6 = \sqrt[\alpha]{\frac{2\alpha \Gamma(\alpha)}{d - d_1}}$, $s_7 = \sqrt[2\alpha]{\frac{\lambda K (\alpha \Gamma(\alpha))^2}{\mu r (\lambda K - \mu)}}$ and $d_1 = \frac{\theta r (\lambda K - \mu)}{a \lambda (\lambda K + r) + r(\lambda K - \mu)}$. When $d < d_1$, the fixed point E_2 is always unstable for $0 < \alpha \leqslant 1$.

(d) The fixed point E^* is locally asymptotically stable for $0 < \alpha \leqslant 1$ if $R_0 > 1$, $0 < s < \min\{s_8, s_9\}$ with $\theta > \theta_1$, where $s_8 = \sqrt[\alpha]{\frac{2K\alpha \Gamma(\alpha)}{rX^*}}$, $\theta_1 = d + \frac{\lambda a d(r + \lambda K)}{r(\lambda K - \mu)}$ and s_9 is such that (8) holds.

Proof. At the fixed point E_0, the Jacobian matrix $J(E_0)$ can be obtained from (5) and it is given by

$$J(E_0) = \begin{pmatrix} 1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}r & 0 & 0 \\ 0 & 1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}\mu & 0 \\ 0 & 0 & 1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}d \end{pmatrix}.$$

The eigenvalues are $\xi_1 = 1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}r$, $\xi_2 = 1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}\mu$ and $\xi_3 = 1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}d$. Since $|\xi_1| > 1$, E_0 is always unstable for $0 < \alpha \leqslant 1$ and any $s > 0$.

The Jacobian matrix $J(E_1)$ at the fixed point E_1 is evaluated as

$$J(E_1) = \begin{pmatrix} 1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}r & \frac{s^\alpha}{\alpha\Gamma(\alpha)}(\lambda K + r) & 0 \\ 0 & 1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}(\lambda K - \mu) & 0 \\ 0 & 0 & 1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}d \end{pmatrix}.$$

The eigenvalues of $J(E_1)$ are

$$\xi_1 = 1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}r, \; \xi_2 = 1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}(\lambda K - \mu), \; \xi_3 = 1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}d.$$

As $0 < \alpha \leqslant 1$, $|\xi_{1,2,3}| < 1$ hold if

$$s < \min\left\{ \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{r}}, \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{d}}, \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{\mu - \lambda K}} \right\}, \mu > \lambda k.$$

Therefore, E_1 is locally asymptotically stable for $0 < \alpha \leqslant 1$ if $R_0 = \frac{\lambda K}{\mu} < 1$, $s < \min\{s_2, s_3, s_4\}$, where $s_2 = \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{d}}$, $s_3 = \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{r}}$ and $s_4 = \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{\mu - \lambda K}}$. However, if $R_0 > 1$ then $|\xi_2| > 1$. So E_1 is always unstable for all $0 < \alpha \leqslant 1$ and for any step size s.

At the fixed point E_2, the Jacobian matrix $J(E_2)$ takes the form

$$J(E_2) = \begin{pmatrix} 1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{r\mu}{\lambda K} & -\frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{\mu(r+\lambda K)}{\lambda K} & 0 \\ \frac{r(\lambda K - \mu)}{(\lambda K + r)}\frac{s^\alpha}{\alpha\Gamma(\alpha)} & 0 & -\frac{mr(\lambda K - \mu)}{a\lambda(\lambda K + r) + r(\lambda K - \mu)}\frac{s^\alpha}{\alpha\Gamma(\alpha)} \\ 0 & 0 & 1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{\theta r(\lambda K - \mu)}{a\lambda(\lambda K + r) + r(\lambda K - \mu)} - d \right) \end{pmatrix}.$$

After some mathematical manipulations, one can have the following characteristic equation

$$\left(1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{\theta r(\lambda K - \mu)}{a\lambda(\lambda K + r) + r(\lambda K - \mu)} - d \right) - \xi \right)(\xi^2 + A\xi + B) = 0, \quad (6)$$

where $A = \frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{r\mu}{\lambda K} - 1$ and $B = \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2 \frac{r\mu(\lambda K - \mu)}{\lambda K}$. Therefore, one eigenvalue is

$\xi_1 = 1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}(d_1 - d)$, where $d_1 = \frac{\theta r(\lambda K - \mu)}{a\lambda(\lambda K + r) + r(\lambda K - \mu)}$ and the other two eigenvalues $\xi_{2,3}$ are the roots of $\xi^2 + A\xi + B = 0$. Following Lemma 1, we observe $|\xi_{2,3}| < 1$ if $\sqrt[\alpha]{\frac{\alpha\Gamma(\alpha)}{\lambda K - \mu}} < s < \sqrt[2\alpha]{\frac{\lambda K(\alpha\Gamma(\alpha))^2}{\mu r(\lambda K - \mu)}}$ with $R_0 > 1$. For ξ_1, we consider the following two cases:

Case I: If $d > d_1$ then $\xi_1 = 1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}(d - d_1)$ and $|\xi_1| < 1$ if $0 < s < \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{d - d_1}}$. Therefore, the equilibrium E_2 is locally asymptotically stable for $0 < \alpha \leqslant 1$ if $R_0 = \frac{\lambda K}{\mu} > 1$, $s_5 < s < \min\{s_6, s_7\}$ with $d > d_1$, where $s_5 = \sqrt[\alpha]{\frac{\alpha\Gamma(\alpha)}{\lambda K - \mu}}$, $s_6 = \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{d - d_1}}$, $s_7 = \sqrt[2\alpha]{\frac{\lambda K(\alpha\Gamma(\alpha))^2}{\mu r(\lambda K - \mu)}}$ and $d_1 = \frac{\theta r(\lambda K - \mu)}{a\lambda(\lambda K + r) + r(\lambda K - \mu)}$.

Case II: If $d < d_1$ then $\xi_1 = 1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}(d_1 - d) > 1$. So E_2 is always unstable for all $0 < \alpha \leqslant 1$ and for any step size s.

At the interior fixed point E^*, the Jacobian matrix $J(E^*)$ is evaluated as

$$J(E^*) = \begin{pmatrix} 1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{rX^*}{K} & -\frac{s^\alpha}{\alpha\Gamma(\alpha)}X^*(\frac{r}{K} + \lambda) & 0 \\ \frac{s^\alpha}{\alpha\Gamma(\alpha)}\lambda Y^* & 1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{mY^*Z^*}{(a+Y^*)^2} & -\frac{mY^*}{a+Y^*}\frac{s^\alpha}{\alpha\Gamma(\alpha)} \\ 0 & \frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{\theta aZ^*}{(a+Y^*)^2} & 1 \end{pmatrix}.$$

After some algebraic manipulation, the characteristic equation of $J(E^*)$ can be expressed as

$$p(\xi) = \xi^3 + A_1\xi^2 + A_2\xi + A_3 = 0, \tag{7}$$

where

$$A_1 = \frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{rX^*}{K} - \frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{mY^*Z^*}{(a+Y^*)^2} - 3,$$

$$A_2 = 3 + \frac{2s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{mY^*Z^*}{(a+Y^*)^2} - \frac{rX^*}{K}\right) + \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2\left(\frac{amdZ^*}{(a+Y^*)^2} - \frac{rmX^*Y^*Z^*}{K(a+Y)^2}\right)$$
$$+ \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2 X^*Y^*\left(\frac{r\lambda}{K} + \lambda^2\right),$$

$$A_3 = 1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{mY^*Z^*}{(a+Y^*)^2} - \frac{rX^*}{K}\right) + \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2\frac{amdZ^*}{(a+Y^*)^2}\left(1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{rX^*}{K}\right)$$
$$+ \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2\left(\frac{rmX^*Y^*Z^*}{K(a+Y^*)^2} + X^*Y^*\left(\frac{r\lambda}{K} + \lambda^2\right)\right).$$

We already know that the interior equilibrium E^* exists if (i) $R_0 > 1$ and (ii) $\theta > \theta_1$, where $\theta_1 = d + \frac{\lambda ad(r + \lambda K)}{r(\lambda K - \mu)}$. Using these conditions, we prove the conditions of Lemma 2:

(i)

$$p(1) = 1 + A_1 + A_2 + A_3 = \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^3 \left(\frac{adrmX^*Z^*}{K(a+Y^*)^2}\right) > 0 \text{ for any } s.$$

(ii)

$$(-1)^3 p(-1)$$
$$= 1 - A_1 + A_2 - A_3$$
$$= 4\left(2 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{rX^*}{K}\right) + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{mZ^*}{(a+I^*)^2}\left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}ad + 2Y^*\right)\left(2 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{rX^*}{K}\right)$$
$$+ \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2 2X^*Y^*\left(\frac{r\lambda}{K} + \lambda^2\right) > 0, \text{ if } s < \sqrt[\alpha]{\frac{2K\alpha\Gamma(\alpha)}{rX^*}}.$$

(iii)

$$(1 - A_3^2) - |A_2 - A_3 A_1|$$
$$= \left[1 - \left\{\left(1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{mZ^*}{(a+Y^*)^2}\left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}ad + Y^*\right)\right)\left(1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{rX^*}{K}\right)\right.\right.$$
$$\left.\left. + \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2 X^*Y^*\left(\frac{r\lambda}{K} + \lambda^2\right)\right\}^2\right]$$
$$- \left|\left\{3 + \frac{2s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{mY^*Z^*}{(a+Y^*)^2} - \frac{rX^*}{K}\right) + \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2\left(\frac{amdZ^*}{(a+Y^*)^2} - \frac{rmX^*Y^*Z^*}{K(a+Y^*)^2}\right)\right.\right.$$
$$\left.+ \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2 X^*Y^*\left(\frac{r\lambda}{K} + \lambda^2\right)\right\} - \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{rX^*}{K} - \frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{mY^*Z^*}{(a+Y^*)^2} - 3\right)$$
$$\left\{\left(1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{mZ^*}{(a+Y^*)^2}\left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}ad + Y^*\right)\right)\left(1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{rX^*}{K}\right) + \right.$$
$$\left.\left.\left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2 X^*Y^*\left(\frac{r\lambda}{K} + \lambda^2\right)\right\}\right|.$$

Due to the complexity of mathematical expression, it is difficult to find a restriction on the step-size s in terms of other system parameters so that $(1 - A_3^2) - |A_2 - A_3 A_1| > 0$. However, in the numerical section, we show that there exists a value s_9 (say) of s such that

$$(1 - A_3^2) - |A_2 - A_3 A_1| > 0, \ \forall s < s_9. \tag{8}$$

Thus, following Jury criteria, the fixed point E^* is locally asymptotically stable for $0 < \alpha \leqslant 1$ if $R_0 > 1$, $\theta > \theta_1$ and $0 < s < min\{s_8, s_9\}$, where $s_8 = \sqrt[\alpha]{\frac{2K\alpha\Gamma(\alpha)}{rX^*}}$, $\theta_1 = d + \frac{\lambda ad(r+\lambda K)}{r(\lambda K - \mu)}$ and s_9 is such that (8) holds. Hence the theorem.

3. Numerical simulations

In this section, an extensive numerical simulations are performed for the fractional-order discrete system (3) to validate our theoretical results for different fractional values of α $(0 < \alpha < 1)$. Stability of fixed points depends on the step-size s and fractional-order α (see Theorem 1). To demonstrate the effect of fractional order α and step-size s on the system dynamics, we cite different examples.

Table 3.1. Restrictions on the step-size, following Theorem 1(b,c), for the stability of fixed points E_1 and E_2 for different fractional-orders α. For E_1, we consider $K = 40$, $\lambda = 0.005$, $\theta = 0.189$ and for E_2 we consider $K = 200$, $\lambda = 0.015$, $\theta = 0.08$. Following parameters are common for both the equilibrium points: $r = 2.0$, $m = 0.52$, $\mu = 0.28$, $a = 15.0$, $d = 0.09$.

	E_1	E_2
Fractional order α	Step-size $s < \min(s_2, s_3, s_4)$	Step-size $s_5 < s < \min(s_6, s_7)$
$\alpha = 0.3$	$s_2 = 21513$ $s_3 = 0.6973$ $s_4 = 31857$	$s_5 = 0.0248$ $s_6 = 1835600$ $s_7 = 2.1578$
$\alpha = 0.4$	$s_2 = 1726.2$ $s_3 = 0.7415$ $s_4 = 2317.3$	$s_5 = 0.0608$ $s_6 = 48464$ $s_7 = 1.7302$
$\alpha = 0.6$	$s_2 = 145.5959$ $s_3 = 0.8289$ $s_4 = 177.1754$	$s_5 = 0.1564$ $s_6 = 1344.9$ $s_7 = 1.4582$
$\alpha = 0.8$	$s_2 = 44.1464$ $s_3 = 0.9150$ $s_4 = 51.1488$	$s_5 = 0.2619$ $s_6 = 233.9135$ $s_7 = 1.3976$
$\alpha = 0.95$	$s_2 = 25.6082$ $s_3 = 0.9788$ $s_4 = 28.9884$	$s_5 = 0.3414$ $s_6 = 104.2794$ $s_7 = 1.3984$

EXAMPLE 1. We consider the following parameter values from [49]: $r = 2.0$, $K = 40.0$, $\lambda = 0.005$, $m = 0.52$, $\mu = 0.28$, $a = 15.0$, $\theta = 0.189$, $d = 0.09$ and initial point $S(0) = 30, I(0) = 5, P(0) = 10$. With these parameter values, we calculate different limits s_i on the step-size for the stability of fixed points E_1 and E_2 for different values of fractional-order α. For example, we compute $s_2 = 44.1464$, $s_3 = 0.9150$ and $s_4 = 51.1488$ for $\alpha = 0.8$. The equilibrium E_1 will then be locally asymptotically stable, following Theorem 1(b), if $s < \min(44.1464, 0.9150, 51.1488) = 0.9150$. It is unstable if $s > 0.9150$. Similarly, E_2 will be locally asymptotically stable, following Theorem 1(c), if $0.2619 < s < \min(233.9135, 1.3976) = 1.3976$. Similar step-size restrictions for the stability of the above two equilibrium points for different fractional order α are tabulated in Table 3.1. Considering $s = 0.65(< 0.9150)$ and $s = 0.95(> 0.9150)$ corresponding to $\alpha = 0.8$, the behavior of the solution trajectories of the system (2) are plotted in Fig. 1. It shows that the predator-free equilibrium E_1 is stable for the step-size $s = 0.65$ and unstable for $s = 0.95$.
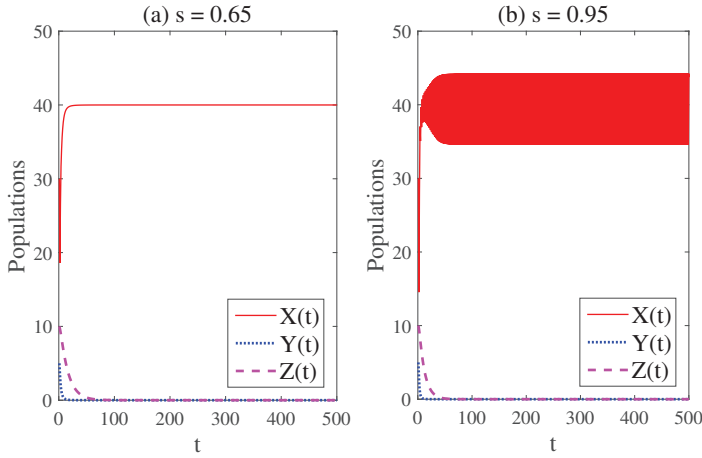
Figure 1: Stable (Fig. a) and unstable (Fig. b) behavior of the fixed point E_1 for the step-size $s = 0.65$ and $s = 0.95$ corresponding to the fraction order $\alpha = 0.8$. Parameter values are $r = 2.0$, $K = 40.0$, $\lambda = 0.005$, $m = 0.52$, $\mu = 0.28$, $a = 15.0$, $\theta = 0.189$, $d = 0.09$.
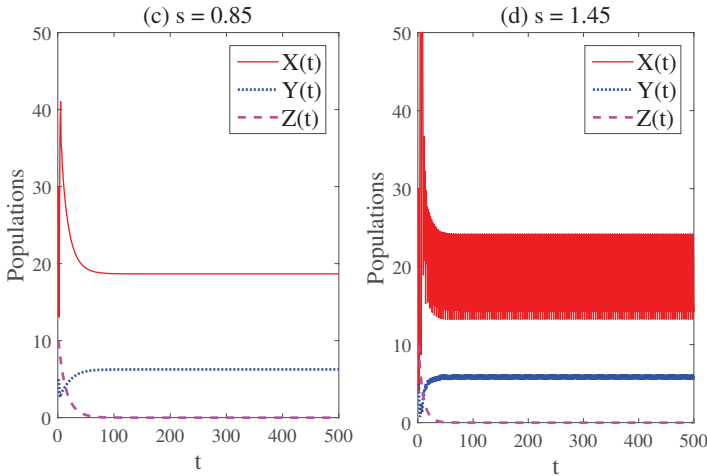

Figure 2: Stable (Fig. a) and unstable (Fig. b) behavior of the fixed point E_2 for the step-size $s = 0.85$ and $s = 1.45$ corresponding to the fraction order $\alpha = 0.8$. Parameter values are $\lambda = 0.005$, $K = 200$, $\theta = 0.08$. Other parameters and initial values are as in Fig. 1.

Similarly, we observe (Figure 2) that the predator-free fixed point E_2 of the system (2) is stable for $s = 0.85$ and unstable for $s = 1.45$ corresponding to order $\alpha = 0.8$ (see
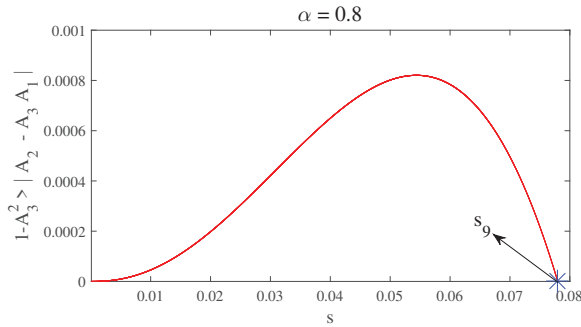
Theorem 1(c) and Table 3.1.).

Figure 3: Existence of the bound s_9 such that the condition $1 - A_3^2 > |A_2 - A_3 A_1|$ is hold. The bound is obtained as $s_9 = 0.07798$ for the parameter values $r = 15.0$, $K = 40.0$, $\lambda = 0.006$, $m = 14.5$, $\mu = 0.0019$, $a = 16.0$, $\theta = 11.1$, $d = 6.0$ and $\alpha = 0.8$.

Table 3.2. Restriction on the step-size, following Theorem 1(d), for the stability of fixed point E^* for different fractional-order α. Parameter values are as in Fig. 3.

	E^*
Fractional order α	Step-size $s < \min(s_8, s_9)$
$\alpha = 0.3$	$s_8 = 0.0074$
	$s_9 = 0.00098$
$\alpha = 0.4$	$s_8 = 0.0245$
	$s_9 = 0.00538$
$\alpha = 0.6$	$s_8 = 0.0853$
	$s_9 = 0.03108$
$\alpha = 0.8$	$s_8 = 0.1662$
	$s_9 = 0.07798$
$\alpha = 0.95$	$s_8 = 0.2327$
	$s_9 = 0.1073$

EXAMPLE 2. We consider another parameter set as in [47]: $r = 15.0$, $K = 40.0$, $\lambda = 0.006$, $m = 14.5$, $\mu = 0.0019$, $a = 16.0$, $\theta = 11.1$, $d = 6.0$ and initial point $S(0) = 30, I(0) = 5, P(0) = 10$. First we numerically determine the bound $s_9 = 0.07798$ (Fig. 3) for the stability of the interior fixed point E^* corresponding to a fractional order $\alpha = 0.8$ so that the condition $1 - A_3^2 > |A_2 - A_3 A_1|$ is satisfied (see Theorem 1(d)). As before, we construct Table 3.2 to show the bounds on step-size for the stability of the interior fixed point E^* for different values of fractional-order α. One can also compute that $R_0 = 126.3158 > 1$ and $\theta = 11.1 > \theta_1 = 8.4579$ for $\alpha = 0.8$. Following Theorem 1(d), the fixed point $E^*(20.8573, 18.8235, 0.2962)$ is then stable (Fig. 4a) for $s = 0.05 < \min(s_8, s_9) = 0.07798$ (see Table 3.2) and unstable for $s = 0.08 (> 0.07798)$ (Fig. 4b) for the fractional-order $\alpha = 0.8$.
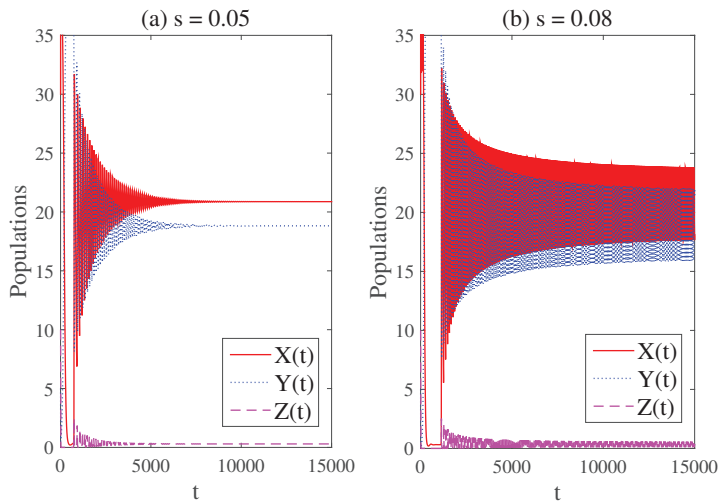
Figure 4: Stable and unstable behavior of the fixed point E^*, respectively, for the step-sizes $s = 0.05$ (Fig. a) and $s = 0.08$ (Fig. b) when $\alpha = 0.8$. Here all parameters are as in Figure 3.
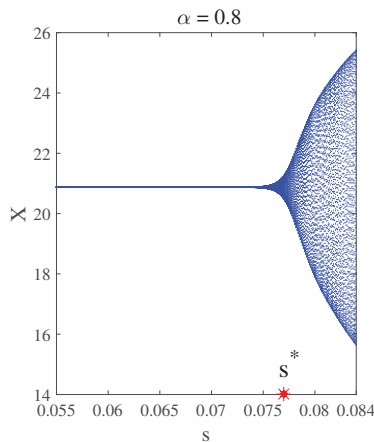

Figure 5: Bifurcation diagram of X population of system (3) in the range $[0.055, 0.085]$ with step-size s as the bifurcation parameter. System remains stable for $s < s^*$ and becomes unstable for $s > s^*$, where $s^* = 0.077$. Here $\alpha = 0.8$ and other parameters are as in Figure 3.

Bifurcation diagram of X population (Fig. 5) shows that population is stable for $s < s^*$ and unstable for $s > s^*$, where $s^* = 0.077$. These results show that stability of the discrete fractional order system (2) strongly depends on the step-size and the fractional

order.

EXAMPLE 3. In [50], the authors showed that the continuous system may show chaotic dynamics for some parameter values. Here we consider the same parameter set and initial value for which the system (1) shows chaotic behavior: $r = 22.0$, $K = 300.0$, $\lambda = 0.06$, $m = 15.5$, $\mu = 2.3$, $a = 15.0$, $\theta = 10$, $d = 8.3$ and $(S(0), I(0), P(0)) = (30, 5, 10)$. As before, we determine the bound $s_9 = 0.0150$ (see Fig. 6) for the stability of fixed point E^* when $\alpha = 0.85$ and tabulate (see Table 3.3) the bounds on step-size, following Theorem 1(d), for different values of fractional-order α. Table 3.3 shows that the critical value of the step-size, where the switching of stability occurs, decreases as the order of the fractional derivative decreases. In Figure 7, we present the stable (Fig. a) and unstable (Fig. b) behavior of the system around the interior fixed point $E^* = (166.8449, \ 73.2353, \ 43.8939)$ for the step-sizes $s = 0.01$ and $s = 0.04$, respectively, when $\alpha = 0.85$.
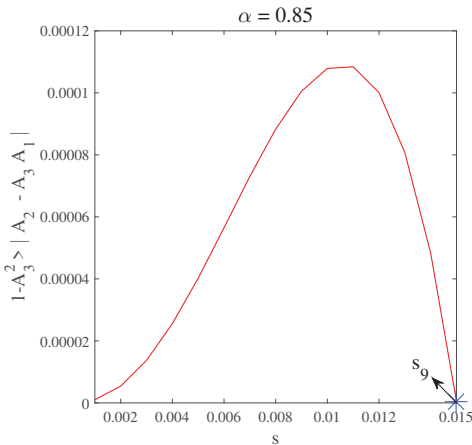

Figure 6: The bound $s_9 = 0.0150$ is determined such that the condition $1 - A_3^2 > |A_2 - A_3 A_1|$ holds. The parameter values are $r = 22.0$, $K = 300.0$, $\lambda = 0.06$, $m = 15.5$, $\mu = 2.3$, $a = 15.0$, $\theta = 10$, $d = 8.3$ and $\alpha = 0.85$.

Table 3.3. Restriction on the step-size for the stability of fixed point E^* for different fractional-orders α with the parameters of Fig. 6 .

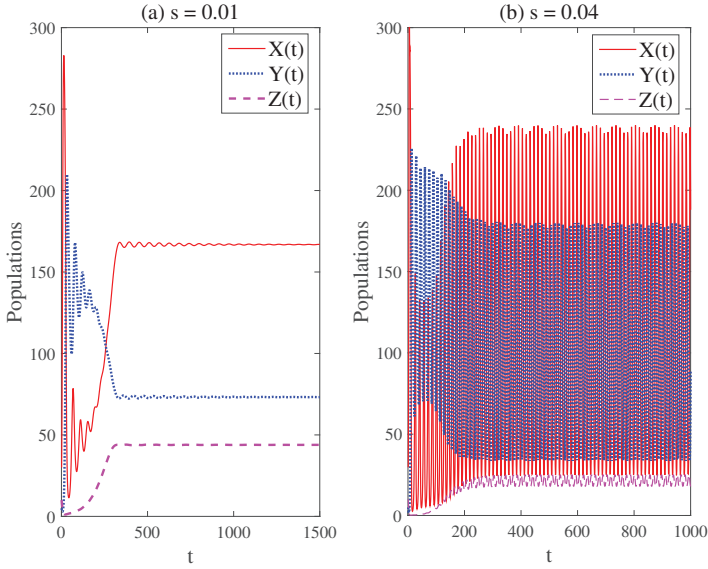| E^*	
Fractional order α	Step-size $s < \min(s_8, s_9)$
$\alpha = 0.95$	$s_8 = 0.1455, \ s_9 = 0.0242$
$\alpha = 0.85$	$s_8 = 0.1112, \ s_9 = 0.0150$
$\alpha = 0.6$	$s_8 = 0.0405, \ s_9 = 0.0023$
$\alpha = 0.55$	$s_8 = 0.0300 , \ s_9 = 0.0013$
$\alpha = 0.45$	$s_8 = 0.0136, \ s_9 = 0.0003$

Figure 7: Stable and unstable behavior of the fixed point $E^* = (166.8449, 73.2353, 43.8939)$ for the step-sizes $s = 0.01$ (Fig. a) and $s = 0.04$ (Fig. b) when $\alpha = 0.85$. Here all parameters are as in Fig. 6.
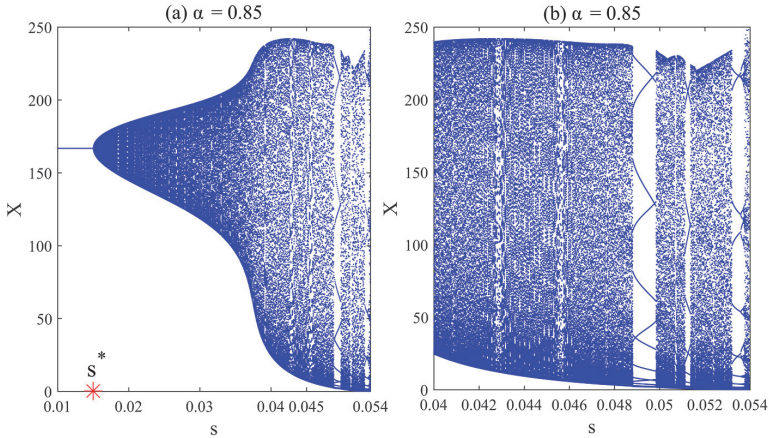

Figure 8: Bifurcation diagrams of X population of system (3) in the range $[0.01, 0.054]$ with respect to the step-size s. System is stable for $s < s^*$ and unstable for $s > s^*$. Exchange of stability occurs at $s^* = 0.015$ (Fig. 8a). A magnified view of the bifurcation diagram for $0.04 \leqslant s \leqslant 0.054$ is presented in Fig. 8b. All parameters are as in Fig. 6 with $\alpha = 0.85$.

Bifurcation diagram of system populations (Fig. 8a) shows that population is stable for $s < s^*$ and losts its stability for $s > s^*$ corresponding to fractional order $\alpha = 0.85$, where $s^* = 0.015$. As the step-size is further increased, the system shows complex dynamics like chaos. An enlarged view (Fig. 8b) of the bifurcation diagram for small range of higher step-size is presented for visualisation of the chaotic dynamics. Existence of such chaotic dynamics (see Fig. 9) may be verified by computing the largest Lyapunov exponent [38].
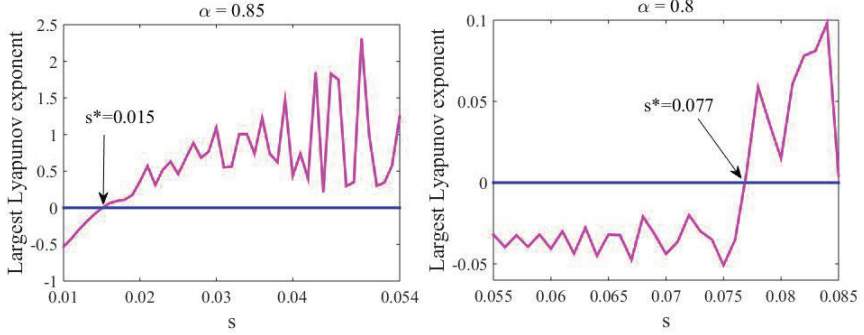

Figure 9: Largest Lyapunov exponent (LLE) drawn for two values of α. The left figure shows that LLE becomes positive for $s > s^*$, where $s^* = 0.015$, $\alpha = 0.85$ and other parameters are as in Fig. 8. The right figure shows that LLE becomes positive for $s > s^*$, where $s^* = 0.077$, $\alpha = 0.8$ and other parameters are as in Fig. 5.

4. Summary

The application of fractional calculus in the study of biological systems is important because it considers past history of the state variables. On the other hand, discretization of a continuous system is inevitable for numerical computations and for population models, particularly where species generation does not overlap. It is earlier shown that the stability of the discrete fractional-order system depends on its fractional-order [33, 34, 35, 36, 37, 38]. It was shown for a simpler two-dimensional epidemic model that the stability of a fractional-order discrete system also depends on the step-size [46]. It is worth mentioning that the determination of explict bounds would be difficult if the nonlinearity and dimension of the system are high. The novelty of this work is that we determined the analytical bounds on the step-size for a highly nonlinear three-dimensional system so that the stability of an equilibrium point is maintained. We considered a highly nonlinear three-dimensional system of fractional-order differential equations that represent the predator-prey interaction in the presence of infection. We revealed the dynamics of the corresponding discrete fractional system. Discretization of the fractional-order system was done with piecewise constant arguments and the corresponding dynamics were explored around different feasible fixed points. We used the Jury criterion for local stability of the discrete fractional-order systems. It is observed that stability of the system depends on both the step-size and fractional order. Different

examples were cited to illustrate the stability of predator-free, infection-free, and coexistence equilibrium points. These simulation results perfectly agree with the analytical results. More specifically, the critical value of the step-size, where the switching of stability occurs, decreases as the order of the fractional derivative decreases. Bifurcation diagrams with respect to step size show that the system is stable and all populations coexist in the stable state if the step-size does not exceed some threshold value, which is also dependent on the fractional order, and unstable if it exceeds. Our simulation result also shows that the system may show chaotic dynamics, as shown in the continuous-time case, if the step-size is large. This study thus reveals that the dynamics of a discrete fractional order system strongly depends on the step-size and the fractional order.

REFERENCES

[1] S. SAMKO, A. KILBAS, O. MARICHEV, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Yverdon, (1993).

[2] F.A. RIHAN, S. LAKSHMANAN, A.H. HASHISH, R. RAKKIYAPPAN, E. AHMED, *Fractional-order delayed predator-prey systems with Holling type-II functional response*, Nonlinear Dynamics, **80**, 1-2 (2015), 777–789.

[3] J. T. MACHADO, V. KIRYAKOVA, F. MAINARDI, *Recent history of fractional calculus*, Commun. Nonlinear Sci. Numer. Simulat., **16**, (2011), 1140–1153.

[4] I. MATYCHYN, V. ONYSHCHENKO, *Time-optimal control of fractional-order linear systems*, Fract. Calc. Appl. Anal. **18**, 3 (2015), 687–696.

[5] R. E. GUTIERREZ, J. M. ROSARIO, J. T. MACHADO, *Fractional Order Calculus: Basic Concepts and Engineering Applications*, Math. Prob. Eng. (2010); DOI: 10.1155/2010/375858.

[6] S. ABBAS, M. BENCHOHRA, G. M. N'GUEREKATA, *Topics in Fractional Differential Equations*, Springer Science & Business Media, (2012).

[7] S. DAS, *Introduction to Fractional Calculus for Scientists and Engineers*, Springer, (2011).

[8] M. L. RICHARD, *Fractional calculus in bioengineering*, Redding: Begell House, (2006).

[9] Y. LI AND Q. ZHANG, *Blow-up and global existence of solutions for a time fractional diffusion equation*, Fract. Calc. Appl. Anal. **21**, 6 (2018), 1619–1640.

[10] J. MUNKHAMMAR, *Chaos in a fractional order logistic map*, Fract. Calc. Appl. Anal. **16**, 3 (2013), 511–519.

[11] M. C. TRIPATHY, D. MONDAL, K. BISWAS, S. SEN, *Experimental studies on realization of fractional inductors and fractional order bandpass filters*, Int. J. Circuit Theory and Applications, **43**, 9 (2015), 1183–1196.

[12] P. J. TORVIK, R. L. BAGLEY, *On the appearance of the fractional derivative in the behaviour of real materials*, J. Appl. Mechanics, **51**, 2 (1984), 294–298.

[13] J. A. SABATIER, O. P. AGRAWAL, J. T. MACHADO, *Advances in Fractional Calculus*, Dordrecht: Springer, (2007).

[14] A. BOUKHOUIMA, K. HATTAF, N. YOUSFI, *Dynamics of a fractional order HIV infection model with specific functional response and cure rate*, Int. J. Diff. Equations, **2017**, (2017).

[15] E. AHMED, A. M. A. EL-SAYED, H. A. A. EL-SAKA, *Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models*, J. Math. Anal. Appl., **325**, (2007), 542–553.

[16] S. RANAA, S. BHATTACHARYAA, J. PAL, G. M. N'GUEREKATA, J. CHATTOPADHYAY, *Paradox of enrichment: A fractional differential approach with memory*, Physica A., **392**, (2013), 3610–3621.

[17] Z. CUI, Z. YANG, *Homotopy perturbation method applied to the solution of fractional Lotka-Volterra equations with variable coefficients*, J. Mod. Meth. Numer. Math., **5**, (2014), 1–9.

[18] S. MONDAL, N. BAIRAGI, G. M. N'GUEREKATA, *Global stability of a Leslie-Gower-type fractional order tritrophic food chain model*, Fractional Differential Calculus, **9**, 1 (2019), 149–161.

[19] S. MONDAL, N. BAIRAGI, A. LAHIRI, *A fractional calculus approach to Rosenzweig-MacArthur predator-prey model and its solution*, J. Mod. Meth. Numer. Math., **8**, 1-2 (2017), 66–76.

[20] J. HUO, H. ZHAO, L. ZHU, *The effect of vaccines on backward bifurcation in a fractional order HIV model*, Nonlinear Anal. RWA. **26**, (2015), 289–305.

[21] H. L. LI, L. ZHANG, C. HU, Y. L. JIANG, Z. TENG, *Dynamical analysis of a fractional-order predator-prey model incorporating a prey refuge*, J. Appl. Math. Comput. (2016); DOI: 10.1007/s12190-016-1017-8.

[22] C. VARGAS-DE-LEON, *Volterra-type Lyapunov functions for fractional-order epidemic systems*, Commun. Nonlinear Sci. Numer. Simul. **24**, (2015), 75–85.

[23] S. MONDAL, A. LAHIRI, N. BAIRAGI, *Analysis of a fractional order eco-epidemiological model with prey infection and type 2 functional response*, Math. Meth. Appl. Sci. **40**, 18 (2017), 6776–6789.

[24] K. HATTAF, A. A. LASHARI, B. EL BOUKARI, N. YOUSFI, *Effect of discretization on dynamical behavior in an epidemiological model*, Differential Equations and Dynamical Systems, **23**, 4 (2015), 403–413.

[25] M. SEKIGUCHI, E. ISHIWATA, *Global dynamics of a discretized SIRS epidemic model with time delay*, Journal of Mathematical Analysis and Applications, **371**, 1 (2010), 195–202.

[26] Q. CHEN, Z. TENG, L. WANG, H. JIANG, *The existence of codimension-two bifurcation in a discrete SIS epidemic model with standard incidence*, Nonlinear Dyn. **71**, (2013), 55–73.

[27] Z. HU, Z. TENG, L. WANG, H. JIANG, *Stability analysis in a class of discrete SIRS epidemic models*, Nonlinear Anal.: Real World Appl. **13**, (2012), 2017–2033.

[28] M. BISWAS, N. BAIRAGI, *Discretization of an eco-epidemiological model and its dynamic consistency*, Journal of Difference Equations and Applications, **23**, 5 (2017), 860–877.

[29] S. GHORAI, P. CHAKRABORTY, S. PORIA AND N. BAIRAGI, *Dispersal-induced pattern forming instabilities in host-parasitoid metapopulations*, Nonlinearity, (2020), DOI.org/10.1007/s11071-020-05505-w.

[30] F. M. ATICI, P. W. ELOE, *A transform method in discrete fractional calculus*, Int. J. Difference Equations, **2**, 2 (2007), 165–176.

[31] C. FULAI, L. XIANNAN, Z. YONG, *Existence results for nonlinear fractional difference equation*, Advances in Difference Equations, **2011**, (2011); DOI: 10.1155/2011/713201.

[32] G. C. WU, D. BALEANU, *Chaos synchronization of the discrete fractional logistic map*, Signal Processing, **102**, (2014), 96–99.

[33] A. A. ELSADANY, A. E. MATOUK, *Dynamical behaviors of fractional-order Lotka Volterra predator-prey model and its discretization*, J. Appl. Math. Comput. **49**, (2015), 269–283.

[34] J. ALZABUT, T. ABDELJAWAD, D. BALEANU, *Nonlinear delay fractional difference equations with applications on discrete fractional Lotka-Volterra competition model*, J. Comput. Anal. Appl. **25**, 5 (2018), 889–898.

[35] A.G. M. SELVAM, R. JANAGARAJ, *Numerical Analysis of a Fractional Order Discrete Prey-Predator System with Functional Response*, In. J. Engineering & Technology. **7**, 4.10 (2018), 681–684.

[36] Y. SHI, Q. MA AND X. DING, *Dynamical behaviors in a discrete fractional-order predator-prey system*, Filomat, **32**, 17 (2018), 5857–5874.

[37] S. M. SALMAN, *On a Discretized Fractional-Order SIR Model for Influenza*, Progress in Fractional Differentiation and Application, **3**, 2 (2017), 163–173.

[38] A. M. ABDELAZIZ, A. I. ISMAIL, F. A. ABDULLAH AND M. H. MOHD, *Bifurcations and chaos in a discrete SI epidemic model with fractional order*, Advances in Difference Equations. 1 (2018), 1–19.

[39] A. KHAN, J. F. GOMEZ-AGUILAR, T. S. KHAN, H. KHAN, *Stability analysis and numerical solutions of fractional order HIV/AIDS model*, Chaos, Solitons and Fractals, **122**, (2019), 119–128.

[40] A. KHAN, T. ABDELJAWAD, J. F. GOMEZ-AGUILAR, H. KHAN, *Dynamical study of fractional order mutualism parasitism food web module*, Chaos, Solitons and Fractals, **134**, (2020), 109685.

[41] S. MONDAL, M. BISWAS AND N. BAIRAGI, *Local and global dynamics of a fractional-order predator-prey system with habitat complexity and the corresponding discretized fractional-order system*, J. Applied Mathematics and Computing, (2020); DOI: https://doi.org/10.1007/s12190-020-01319-6.

[42] A. KHAN, J. F. GOMEZ-AGUILAR, T. ABDELJAWAD, H. KHAN, *Stability and numerical simulation of a fractional order plant-nectar-pollinator model*, Alexandria Engineering Journal, **59**, 1 (2020), 49–59.

[43] R. P. AGARWAL, A. M. A. EL-SAYED, S. M. SALMAN, *Fractional-order Chua's system: discretization, bifurcation and chaos*, Advances in Difference Equations, **1**, (2013), 320.

[44] A. M. A. El-Sayed, S. M. Salman, *On a discretization process of fractional-order Riccati differential equation*, J. Fract. Calc. Appl. **4**, 2 (2013), 251–259.

[45] A. G. M. Selvam, D. A. Vianny, *Behavior of a Discrete Fractional Order SIR Epidemic model*, International Journal of Engineering and Technology, **7**, 4.10 (2018), 675–680.

[46] Z. F. El Raheem, S. M. Salman, *On a discretization process of fractional-order logistic differential equation*, Journal of the Egyptian Mathematical Society, **22**, 3 (2014), 407–412.

[47] J. Chattopadhyay, N. Bairagi, *Pelicans at risk in Salton sea–an eco-epidemiological model*, Ecological Modelling, **136**, (2001), 103–112.

[48] S. N. Elaydi, *An Introduction to Difference Equations*, Springer, New York, USA, (1999).

[49] N. Bairagi, R. R. Sarkar, J. Chattopadhyay, *Impacts of incubation delay on the dynamics of an eco-epidemiological system–a theoretical study*, Bulletin of Mathematical Biology, **70**, 7 (2008), 2017–2038.

[50] R. K. Upadhyay, N. Bairagi, K. Kundu, J. Chattopadhyay, *Chaos in eco-epidemiological problem of the Salton Sea and its possible control*, Applied Mathematics and Computation, **196**, 1 (2008), 392–401.

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