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MEAN VALUE THEOREMS ASSOCIATED TO THE DIFFERENCES OF OPIAL–TYPE INEQUALITIES AND THEIR FRACTIONAL VERSIONS

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Abstract. In this paper we analyze the error estimations of Opial-type inequalities for convex functions. The error bounds of these inequalities are studied by using mean value theorems for generalized kernels. Further applications of these results are obtained in fractional calculus by letting appropriate kernels.

1. Introduction and preliminary results

Opial obtained the following integral inequality in 1960 [9].

THEOREM 1. *Let* $g \in C^1[0,h]$ *be such that* $g(0) = g(h) = 0$ *and* $g(t) > 0$ *for* $t \in (0,h)$. *Then we have*

$$\int_0^h |g(t)g^{'}(t)|dt \leqslant \frac{h}{4}\int_0^h (g'(t))^2 dt.$$

Here $\frac{h}{4}$ *is a best possibility constant.*

It is well known as Opial inequality and researchers have studied it's generalizations, extensions and fractional versions. Agarwal and Pang dedicated a book [1] to this inequality and its further consequences in a very nice way. They thoroughly studied year wise, its integral generalizations, extensions as well as discrete versions. Also a whole chapter is dedicated to the applications of Opial and related inequalities in differential equations. For a systematic and qualitative study of Opial inequalities readers are suggested this book [1]. Anastassiou gave Opial inequalities involving fractional derivatives of functions with applications to fractional differential equations [4, 5].

The aim of this research is to study the differences of generalized Opial type inequalities for convex functions. Moreover the findings are associated with fractional integral and differential operators. In the following first we state Opial type inequalities for convex functions, for this it is need to define some classes of functions [10]:

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Let $U_1(v,k)$ denotes the class of functions $u:[a,b]\to\mathbb{R}$ having representation

$$u(x)=\int_a^x k(x,t)v(t)dt,$$

where v is a continuous function and k is an arbitrary nonnegative kernel such that $k(x,t)=0$ for $t>x$, $v(t)>0$ implies $u(t)>0$ for every $x\in[a,b]$. Let $U_2(v,k)$ denotes the class of functions $u:[a,b]\to\mathbb{R}$ having representation

$$u(x)=\int_x^b k(x,t)v(t)dt,$$

where v is a continuous function and k is an arbitrary nonnegative kernel such that $k(x,t)=0$ for $t<x$, $v(t)>0$ implies $u(t)>0$ for every $x\in[a,b]$.

Mitrinović and Pečarić in [8] gave the following Opial-type inequalities for convex function.

THEOREM 2. *Let $\phi:[0,\infty)\to\mathbb{R}$ be a differentiable function such that for $q>1$, the function $\phi(x^{1/q})$ is convex and $\phi(0)=0$. Let $u\in U_1(v,k)$ where $\left(\int_a^x (k(x,t))^p dt\right)^{1/p}\leqslant M$ and $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_a^b |u(t)|^{1-q}\phi'(|u(t)|)|v(t)|^q dt\leqslant \frac{q}{M^q}\phi\left(M\left(\int_a^b |v(t)|^q dt\right)^{1/q}\right). \tag{1}$$

If the function $\phi(x^{1/q})$ is concave, then the reverse inequality holds.

THEOREM 3. *Let $\phi:[0,\infty)\to\mathbb{R}$ be a differentiable function such that for $q>1$, the function $\phi(x^{1/q})$ is convex and $\phi(0)=0$. Let $u\in U_1(v,k)$ where $\left(\int_a^x (k(x,t))^p dt\right)^{1/p}\leqslant M$ and $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_a^b |u(t)|^{1-q}\phi'(|u(t)|)|v(t)|^q dt\leqslant \frac{q}{M^q(b-a)}\int_a^b \phi((b-a)^{1/q}M|v(t)|)dt. \tag{2}$$

If the function $\phi(x^{1/q})$ is concave, then the reverse inequality holds.

Recently, Farid et al. gave the following generalized Opial type inequality [6]:

THEOREM 4. *Let $\phi,\ g:[0,\infty)\to\mathbb{R}$ be differentiable convex and increasing functions with $\phi(g(0))=0$. Also let $u\in U_1(g\circ v,k)$ and $|k(x,t)|\leqslant M$, where M is a constant. Then the following inequality holds*

$$\begin{aligned}\int_a^b \phi'(g(|u(t)|))g'(|u(t)|)|g\circ v(t)|dt &\leqslant \frac{1}{M}\phi\left(g\left(M\int_a^b |g\circ v(t)|dt\right)\right)\\ &\leqslant \frac{1}{M(b-a)}\int_a^b \phi(g(M(b-a)|g\circ v(t)|))dt.\end{aligned} \tag{3}$$

The aim of this paper is to study the nonnegative differences of Opial-type inequalities stated in aforementioned theorems. Further these differences are studied for fractional integral operators. Therefore, we define fractional calculus operators: Riemann-Liouville fractional integral, Caputo and Canavati fractional derivatives and their compositions identities as follows:

DEFINITION 1. [7] Let $f \in L_1[a,b]$. Then the left-sided and right-sided Riemann-Liouville fractional integrals of order $\alpha > 0$ with $a \geqslant 0$ are defined as:

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(x)dt, \quad x > a,$$

and

$$I_{b_-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(x)dt, \quad x < b,$$

where $\Gamma(.)$ is the Gamma function.

The following lemma summarizes conditions in the composition identity for the left-sided Riemann-Liouville fractional derivative.

LEMMA 1. [3] *Let $\beta > \alpha \geqslant 0$, $m = [\beta]+1$, $n = [\alpha]+1$. Then the composition identity*

$$D_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-t)^{\beta-\alpha-1} D_{a^+}^{\beta} f(x)dt, \ x \in [a,b] \tag{4}$$

is valid if one of the following conditions holds:

(i) $f \in I_{a+}^{\beta}(L_1[a,b]) = \{f : f = I_{a+}^{\beta}\varphi, \varphi \in L_1[a,b]\}$.

(ii) $I_{a+}^{m-\beta} f \in AC^m[a,b]$ *and* $D_{a+}^{\beta-k} f(a) = 0$ *for* $k = 1,\ldots,m$.

(iii) $D_{a+}^{\beta-1} f \in AC[a,b]$, $D_{a+}^{\beta-k} f \in C[a,b]$ *and* $D_{a+}^{\beta-k} f(a) = 0$ *for* $k = 1,\ldots,m$.

(iv) $f \in AC^m[a,b], D_{a+}^{\beta} f, D_{a+}^{\alpha} f \in L_1[a,b], \beta-\alpha \notin \mathbb{N}, D_{a+}^{\beta-k} f(a) = 0$ *for* $k = 1,\ldots,m$ *and* $D_{a+}^{\alpha-k} f(a) = 0$ *for* $k = 1,\ldots,n$.

(v) $f \in AC^m[a,b], D_{a+}^{\beta} f, D_{a+}^{\alpha} f \in L_1[a,b], \beta-\alpha = l \in \mathbb{N}, D_{a+}^{\beta-k} f(a) = 0$ *for* $k = 1,\ldots,l$.

(vi) $f \in AC^m[a,b], D_{a+}^{\beta} f, D_{a+}^{\alpha} f \in L_1[a,b]$ *and* $f^k(a) = 0$ *for* $k = 0,\ldots,m-2$.

(vii) $f \in AC^m[a,b], D_{a+}^{\beta} f, D_{a+}^{\alpha} f \in L_1[a,b], \beta \notin \mathbb{N}$ *and* $D_{a+}^{\beta-1} f$ *is bounded in a neighborhood of* a.

DEFINITION 2. Let $\alpha > 0$ and $\alpha \notin \{1,2,3,\ldots\}$, $n = [\alpha]+1$, $f \in AC^n[a,b]$. Then left-sided and right-sided Caputo fractional derivatives of order α are defined as follows:

$$\left({}^C D_{a^+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(x)}{(x-t)^{\alpha-n+1}} dt, \ x > a,$$

and

$$\left({}^{C}D^{\alpha}_{b_-}f\right)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}dt, \;\; x<b.$$

In the following lemmas composition identities for the Caputo fractional derivatives are given, [2].

LEMMA 2. *Let* $\beta > \alpha \geqslant 0$, $m = [\beta]+1$ *and* $n = [\alpha]+1$, *for* $\alpha, \beta \notin \mathbb{N}_0$*;* $n = [\alpha]$ *and* $m = [\beta]$, *for* $\alpha, \beta \in \mathbb{N}_0$. *Let* $f \in AC^m[a,b]$ *be such that* $f^{(i)}(a) = 0$, *for* $i = n, n+1, \ldots, m-1$. *Let* ${}^{C}D^{\beta}_{a+}f, {}^{C}D^{\beta}_{\alpha+}f \in L_1[a,b]$. *Then*

$$^{C}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\beta-\alpha)}\int_a^x (x-t)^{\beta-\alpha-1}\ {}^{C}D^{\beta}_{a+}f(x)dt, \qquad x \in [a,b].$$

LEMMA 3. *Let* $\beta > \alpha \geqslant 0$, $m = [\beta]+1$ *and* $n = [\alpha]+1$, *for* $\alpha, \beta \notin \mathbb{N}_0$*;* $n = [\alpha]$ *and* $m = [\beta]$, *for* $\alpha, \beta \in \mathbb{N}_0$. *Let* $f \in AC^m[a,b]$ *be such that* $f^{(i)}(b) = 0$ *for* $i = n, n+1, \ldots, m-1$. *Let* ${}^{C}D^{\beta}_{b_-}f, {}^{C}D^{\beta}_{b_-}f \in L_1[a,b]$. *Then*

$$^{C}D^{\alpha}_{b_-}f(x) = \frac{1}{\Gamma(\beta-\alpha)}\int_x^b (t-x)^{\beta-\alpha-1}\ {}^{C}D^{\beta}_{b_-}f(x)dt, \qquad x \in [a,b].$$

Next we consider the subspace $C^{\alpha}_{a+}[a,b]$, which is defined by

$$C^{\alpha}_{a+}[a,b] = \{f \in C^{n-1}[a,b] : J^{n-\alpha}_{a+}f^{(n-1)} \in C^1[a,b]\}.$$

DEFINITION 3. [7] Let $f \in C^{\alpha}_{a+}[a,b]$. Then the left-sided Canavati fractional derivative is given by

$$^{\tilde{C}}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d}{dt}\int_a^x (x-t)^{n-\alpha-1}f^{(n-1)}(x)dt = \frac{d}{dt}I^{n-\alpha}_{a+}f^{(n-1)}(x).$$

The following lemma is useful to give mean value theorems for differences of Opial-type inequalities.

LEMMA 4. *Let* $\phi \in C^2(I)$, *where* $I \subseteq (0,\infty)$, *and*

$$m_1 \leqslant \phi''(y) \leqslant M_1 \;\; for \;\; all \;\; y \in I. \tag{5}$$

Then the functions ϕ_1, ϕ_2 *defined as*

$$\phi_1(t) = \frac{M_1 t^2}{2} - \phi(t), \qquad \phi_2(t) = \phi(t) - \frac{m_1 t^2}{2}, \tag{6}$$

are convex functions. Further if $m_1 \leqslant \frac{\phi'(t)}{t} \leqslant M_1$, *then* $\phi_i, i = 1,2$ *are increasing.*

Rest of the paper is organized as follows:

In Section 2, nonnegative differences of generalized Opial-type inequalities for arbitrary kernels via convex functions given in Theorem 4 are analyzed. By using these differences mean value theorems are obtained. Furthermore, these mean value theorems are investigated for different specific kernels and results for various fractional integral and derivative operators are achieved.

2. Main results

We define linear functionals ${}_g\mathbb{F}_i^{\phi}(u,v;M)$ for $i=1,2$, from nonnegative differences of Opial-type inequalities for convex functions given in Theorem 4 as follows:

$$
{}_g\mathbb{F}_1^{\phi}(u,v;M) = \phi\left(g\left(M\int_a^b |g\circ v(t)|dt\right)\right) - M\int_a^b \phi'(g(|u(t)|))g'(|u(t)|)|g\circ v(t)|dt,
$$

$$
\begin{aligned}
{}_g\mathbb{F}_2^{\phi}(u,v;M) = &\int_a^b \phi(g(M(b-a)|g\circ v(t)|))dt \\
&-M(b-a)\int_a^b \phi'(g(|u(t)|))g'(|u(t)|)|g\circ v(t)|dt.
\end{aligned}
$$

REMARK 1. Under the assumptions of Theorem 4 it is clear that ${}_g\mathbb{F}_i^{\phi}(u,v;M) \geqslant 0$ for $i=1,2$.

The following results consist mean value theorems for linear functionals ${}_g\mathbb{F}_i^{\phi}(u,v;M)$, $i=1,2$.

THEOREM 5. *Let $\phi,\ g:[0,\infty)\to\mathbb{R}$ be functions with assumptions of Theorem 4. If $\phi\in C^2(I)$, where $I\subseteq(0,\infty)$ is compact interval and $m_1\leqslant\frac{\phi'(t)}{t}\leqslant M_1$, where $\inf_{x\in I}\phi''(t)=m_1$ and $\sup_{x\in I}\phi''(t)=M_1$, then there exists an $\xi_i\in I$, $i=1,2$, such that the following equation holds:*

$$
{}_g\mathbb{F}_i^{\phi}(u,v;M) = \frac{\phi''(\xi_i)}{2}\, {}_g\mathbb{F}_i^{x^2}(u,v;M), \qquad i=1,2. \tag{7}
$$

Proof. By using ϕ_1 from Lemma 4 instead of ϕ in (3), the following inequality holds:

$$
\begin{aligned}
&\int_a^b \left(M_1(g(|u(t)|)) - \phi'(g(|u(t)|))\right) g'(|u(t)|)|g\circ v(t)|dt \qquad (8)\\
&\leqslant \frac{1}{M}\left(\frac{M_1}{2}\left(g\left(M\int_a^b |g\circ v(t)|dt\right)\right)^2 - \phi\left(g\left(M\int_a^b |g\circ v(t)|dt\right)\right)\right)\\
&\leqslant \frac{1}{M(b-a)}\left(\frac{M_1}{2}\left(\int_a^b (g(M(b-a)|g\circ v|))\right)^2 dt - \int_a^b \phi(g(M(b-a)|g\circ v(t)|))dt\right).
\end{aligned}
$$

From first inequality after simplification one can obtain

$$
\frac{{}_g\mathbb{F}_1^{\phi}(u,v;M)}{{}_g\mathbb{F}_1^{x^2}(u,v;M)} \leqslant \frac{M_1}{2}. \tag{9}
$$

Similarly using ϕ_2 from Lemma 4 instead of ϕ in (3) one can obtain

$$
\frac{{}_g\mathbb{F}_1^{\phi}(u,v;M)}{{}_g\mathbb{F}_1^{x^2}(u,v;M)} \geqslant \frac{m_1}{2}. \tag{10}
$$

By combining inequalities (9) and (10), the following inequalities are obtained:

$$m_1 \leqslant \frac{2\left({}_g\mathbb{F}_1^{\phi}(u,v;M)\right)}{{}_g\mathbb{F}_1^{x^2}(u,v;M)} \leqslant M_1.$$

Therefore, there exists an $\xi_1 \in I$ such that the following equation is valid:

$$\phi''(\xi_1) = \frac{2\left({}_g\mathbb{F}_1^{\phi}(u,v;M)\right)}{{}_g\mathbb{F}_1^{x^2}(u,v;M)},$$

which gives

$${}_g\mathbb{F}_1^{\phi}(u,v;M) = \frac{\phi''(\xi_1)}{2}\ {}_g\mathbb{F}_1^{x^2}(u,v;M). \tag{11}$$

Similarly from (8) one can obtain the following inequality for second functional:

$$m_1 \leqslant \frac{2\left({}_g\mathbb{F}_2^{\phi}(u,v;M)\right)}{{}_g\mathbb{F}_2^{x^2}(u,v;M)} \leqslant M_1. \tag{12}$$

Therefore, there exists an $\xi_2 \in I$ such that the following equation holds:

$$\phi''(\xi_2) = \frac{2\left({}_g\mathbb{F}_2^{\phi}(u,v;M)\right)}{{}_g\mathbb{F}_2^{x^2}(u,v;M)},$$

which gives

$${}_g\mathbb{F}_2^{\phi}(u,v;M) = \frac{\phi''(\xi_2)}{2}\ {}_g\mathbb{F}_2^{x^2}(u,v;M). \tag{13}$$

Hence (7) holds simultaneously.

THEOREM 6. *Let ϕ_1,ϕ_2 and g be the functions with assumptions of Theorem 4. If $\phi_1,\phi_2 \in C^2(I)$, where $I \subseteq (0,\infty)$ is compact interval, ${}_g\mathbb{F}_i^{\phi_2}(u,v;M)\phi_1^{(n)}(t) \geqslant {}_g\mathbb{F}_i^{\phi_1}(u,v;M)\phi_2^{(n)}(t)$, $n=1,2$, $t \in I$ and ${}_g\mathbb{F}_i^{x^2}(u,v;M) \neq 0$, $i=1,2$, then there exists an $\xi_i \in I$, $i=1,2$, such that we have*

$$\frac{{}_g\mathbb{F}_i^{\phi_1}(u,v;M)}{{}_g\mathbb{F}_i^{\phi_2}(u,v;M)} = \frac{\phi_1''(\xi_i)}{\phi_2''(\xi_i)}, \quad i=1,2.$$

Provided denominators are not equal to zero.

Proof. Let a function $h \in C^2(I)$ be defined as $h = a\phi_1 - b\phi_2$, where a and b are defined by

$$a = {}_g\mathbb{F}_i^{\phi_2}(u,v;M), \quad i=1,2.$$

$$b = {}_g\mathbb{F}_i^{\phi_1}(u,v;M), \quad i=1,2.$$

The function h will be increasing and convex, by using Theorem 5 with $\phi = h$ we have

$$0 = \left(a\phi_1''(\xi) - b\phi_2''(\xi)\right) \ {}_g\mathbb{F}_i^{x^2}(u,v;M), \quad i = 1,2.$$

As ${}_g\mathbb{F}_i^{x^2}(u,v;M) \neq 0, i = 1,2$, therefore we have

$$\frac{b}{a} = \frac{\phi_1''(\xi_i)}{\phi_2''(\xi_i)}.$$

Hence the required equation is achieved.

In the upcoming section nonnegative differences of generalized Opial-type integral inequalities are studied in fractional calculus by defining particular kernels.

3. Fractional versions of differences of generalized integral Opial-type inequalities

Here nonnegative differences are presented for Riemann-Liouville fractional integrals, Caputo and Canavati fractional derivatives. Also their compositions identities are used to get further formations.

THEOREM 7. *Let $\phi,\ g : [0,\infty) \to \mathbb{R}$ be functions with assumptions of Theorem 4. If $\phi \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is compact interval and let $v \in L_1[a,b]$ has Riemann-Liouville fractional integral of order α and $\alpha > 1$, then there exists an $\xi_i \in I$, $i = 1,2$, such that the following result holds*

$$ {}_g\mathbb{F}_i^{\phi}(I_{a+}^{\alpha}v, v; M) = \frac{\phi''(\xi_i)}{2} \ {}_g\mathbb{F}_i^{x^2}(I_{a+}^{\alpha}v, v; M), \qquad i = 1,2, \tag{14}$$

where $M = \dfrac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}$.

Proof. Let us define for $x \in [a,b]$, the kernel $k(x,t)$ as follows:

$$k(x,t) = \begin{cases} \frac{1}{\Gamma(\alpha)}(x-t)^{\alpha-1}, & a \leqslant t \leqslant x, \\ 0, & x < t \leqslant b. \end{cases}$$

Also if u is defined by

$$u(x) = I_{a+}^{\alpha}v(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}v(x)dt. \tag{15}$$

Then we have

$$|k(x,t)| \leqslant \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}.$$

For $\alpha > 1$, $(x-a)^{\alpha-1}$ is increasing on $[a,b]$, therefore we have

$$|k(x,t)| \leqslant \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} = M.$$

Applying Theorem 5 for this particular kernel (14) can be achieved.

THEOREM 8. *Let ϕ_1, ϕ_2 and g be the functions with assumptions of Theorem 4. If $\phi_1, \phi_2 \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is compact interval and let $v \in L[a,b]$ has Riemann-Liouville fractional integral of order α. If $\alpha > 1$ and ${}_g\mathbb{F}_i^{x^2}(I_{a^+}^{\alpha}v, v; M) \neq 0$, $i = 1,2$, then there exists an $\xi_i \in I$, $i = 1,2$, such that we have*

$$\frac{{}_g\mathbb{F}_i^{\phi_1}(I_{a^+}^{\alpha}v, v; M)}{{}_g\mathbb{F}_i^{\phi_2}(I_{a^+}^{\alpha}v, v; M)} = \frac{\phi_1''(\xi_i)}{\phi_2''(\xi_i)}, \quad i = 1,2, \tag{16}$$

where $M = \dfrac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}$. Provided the denominators are not equal to zero.

Proof. It can easily be proved by using the kernel just defined in proof of aforementioned theorem, function defined in (15), and Theorem 6.

THEOREM 9. *Let ϕ, $g : [0,\infty) \to \mathbb{R}$ be functions with assumptions of Theorem 4. If $\phi \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is compact interval and let $v \in AC^n[a,b]$ has Caputo fractional derivative of order α. If $\alpha \leqslant n-1$, then there exists an $\xi_i \in I$, $i = 1,2$, such that the following result holds*

$${}_g\mathbb{F}_i^{\phi}({}^CD_{a^+}^{\alpha}v, v^n; M) = \frac{\phi''(\xi_i)}{2}\, {}_g\mathbb{F}_i^{x^2}({}^CD_{a^+}^{\alpha}v, v^n; M), \qquad i = 1,2, \tag{17}$$

where $M = \dfrac{(b-a)^{n-\alpha-1}}{\Gamma(n-\alpha)}$.

Proof. Let us define the kernel $k(x,t)$ for $x \in [a,b]$ as follows:

$$k(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}(x-t)^{n-\alpha-1}, & a \leqslant t \leqslant x, \\ 0, & x < t \leqslant b. \end{cases}$$

Also the function u is defined by

$$u(x) = {}^CD_{a^+}^{\alpha}v(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}v^{(n)}(x)dt. \tag{18}$$

Then we have

$$|k(x,t)| \leqslant \frac{(x-a)^{n-\alpha-1}}{\Gamma(n-\alpha)}.$$

It is easy to see that for $n > \alpha+1$, $(x-t)^{n-\alpha-1}$ is increasing on $[a,b]$, therefore

$$|k(x,t)| \leqslant \frac{(b-a)^{n-\alpha-1}}{\Gamma(n-\alpha)} = M.$$

Using the function defined in (18) and value of M, Theorem 5 gives required result.

THEOREM 10. *Let ϕ_1, ϕ_2 and g be the functions with assumptions of Theorem 4. If $\phi_1, \phi_2 \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is compact interval and let $v \in AC^n[a,b]$ has Caputo fractional derivative of order α. If $\alpha \leqslant n-1$ and ${}_g\mathbb{F}_i^{x^2}({}^CD_{a^+}^{\alpha}v, v^n; M) \neq 0$, $i = 1,2$, then there exists an $\xi_i \in I$, $i = 1,2$, such that we have*

$$\frac{{}_g\mathbb{F}_i^{\phi_1}({}^CD_{a^+}^{\alpha}v, v^n; M)}{{}_g\mathbb{F}_i^{\phi_2}({}^CD_{a^+}^{\alpha}v, v^n; M)} = \frac{\phi_1''(\xi_i)}{\phi_2''(\xi_i)}, \quad i = 1,2, \tag{19}$$

where $M = \dfrac{(b-a)^{n-\alpha-1}}{\Gamma(n-\alpha)}$. Provided the denominators are not equal to zero.

Proof. It is easy to prove by using Theorem 6.

THEOREM 11. *Let ϕ, $g : [0,\infty) \to \mathbb{R}$ be functions with assumptions of Theorem 4. Also let $m = [\beta] + 1$ and $n = [\alpha] + 1$, for $\alpha, \beta \notin \mathbb{N}_0$; $n = [\alpha]$ and $m = [\beta]$, for $\alpha, \beta \in \mathbb{N}_0$ and $v \in AC^m[a,b]$ such that $f^{(i)}(a) = 0$ for $i = n, n+1, \ldots, m-1$. Let ${}^CD_{a^+}^{\beta}v \in L_q[a,b]$ and ${}^CD_{a^+}^{\alpha}v \in L_1[a,b]$. Then the following result holds for $\alpha \leqslant \beta - 1$*

$${}_g\mathbb{F}_j^{\phi}({}^CD_{a^+}^{\alpha}v,\ {}^CD_{a^+}^{\beta}v; M) = \frac{\phi''(\xi_j)}{2}\ {}_g\mathbb{F}_j^{x^2}({}^CD_{a^+}^{\alpha}v,\ {}^CD_{a^+}^{\beta}v; M),\ j = 1,2, \tag{20}$$

where $M = \dfrac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}$.

Proof. Let us define the kernel $k(x,t)$ for $x \in [a,b]$ as follows:

$$k(x,t) = \begin{cases} \frac{1}{\Gamma(\beta-\alpha)}(x-t)^{\beta-\alpha-1}, & a \leqslant t \leqslant x, \\ 0, & x < t \leqslant b. \end{cases}$$

Also let us define the function u by

$$u(x) = {}^CD_{a^+}^{\alpha}v(x) = \frac{1}{\Gamma(\beta-\alpha)}\int_a^x (x-t)^{\beta-\alpha-1}\, {}^CD_{a^+}^{\beta}v(x)dt. \tag{21}$$

Then we have

$$|k(x,t)| \leqslant \frac{(x-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}.$$

For $\beta \geqslant \alpha+1$, $(x-a)^{\beta-\alpha-1}$ is increasing on $[a,b]$, therefore

$$|k(x,t)| \leqslant \frac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} = M.$$

Using function defined in (21) and value of M in Theorem 5, we get required result.

THEOREM 12. *Let ϕ_1,ϕ_2 and g be the functions with assumptions of Theorem 4. If $\phi_1,\phi_2 \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is compact interval. Also let $m=[\beta]+1$ and $n=[\alpha]+1$, for $\alpha,\beta \notin \mathbb{N}_0$; $n=[\alpha]$ and $m=[\beta]$, for $\alpha,\beta \in \mathbb{N}_0$ and $v \in AC^m[a,b]$ such that $f^{(i)}(a)=0$ for $i=n,n+1,\dots,m-1$. Let ${}^CD^{\beta}_{a^+}v \in L_q[a,b]$, ${}^CD^{\alpha}_{a^+}v \in L_1[a,b]$, $\alpha \leqslant \beta-1$ and ${}_g\mathbb{F}^{x^2}_j({}^CD^{\alpha}_{a^+}v,\ {}^CD^{\beta}_{a^+}v;M) \neq 0$, $j=1,2$. Then there exists an $\xi_j \in I$, $j=1,2$, such that we have*

$$\frac{{}_g\mathbb{F}^{\phi_1}_j({}^CD^{\alpha}_{a^+}v,\ {}^CD^{\beta}_{a^+}v;M)}{{}_g\mathbb{F}^{\phi_2}_j({}^CD^{\alpha}_{a^+}v,\ {}^CD^{\beta}_{a^+}v;M)} = \frac{\phi_1''(\xi_j)}{\phi_2''(\xi_j)}, \quad j=1,2, \tag{22}$$

where $M=\dfrac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}$. Provided the denominators are not equal to zero.

Proof. It is easy to prove by using function defined in (21) and using Theorem 6.

THEOREM 13. *Let $\phi,\ g:[0,\infty)\to\mathbb{R}$ be functions with assumptions of Theorem 4. Also let $\alpha>1$, $m=[\beta]+1$ and $n=[\alpha]+1$. Let $v\in C^{\beta}_{a+}[a,b]$ such that $f^{(i)}(a)=0$ for $i=n-1,n,\dots,m-2$. Let ${}^{\tilde{C}}D^{\beta}_{a^+}v \in L_q[a,b]$. Then for $\alpha \leqslant \beta-1$ the following result holds:*

$${}_g\mathbb{F}^{\phi}_j({}^{\tilde{C}}D^{\alpha}_{a^+}v,\ {}^{\tilde{C}}D^{\beta}_{a^+}v;M) = \frac{\phi''(\xi_j)}{2}\ {}_g\mathbb{F}^{x^2}_j({}^{\tilde{C}}D^{\alpha}_{a^+}v,\ {}^{\tilde{C}}D^{\beta}_{a^+}v;M), \qquad j=1,2,$$

where $M=\dfrac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}$.

Proof. Let us define the kernel $k(x,t)$ for $x\in[a,b]$ as

$$k(x,t)=\begin{cases}\frac{1}{\Gamma(\beta-\alpha)}(x-t)^{\beta-\alpha-1}, & a\leqslant t\leqslant x,\\ 0, & x<t\leqslant b.\end{cases}$$

Also let us define the function u by

$$u(x) = {}^{\tilde{C}}D^{\alpha}_{a^+}v(x) = \frac{1}{\Gamma(\beta-\alpha)}\int_a^x (x-t)^{\beta-\alpha-1}\,{}^{\tilde{C}}D^{\beta}_{a^+}v(x)dt. \tag{23}$$

Then we have

$$|k(x,t)| \leqslant \frac{(x-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}.$$

For $\beta\geqslant\alpha+1$, $(x-a)^{\beta-\alpha-1}$ is increasing on $[a,b]$, therefore

$$|k(x,t)| \leqslant \frac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} = M.$$

Using function defined in (23) and value of M in Theorem 5, we get required result.

THEOREM 14. *Let ϕ_1, ϕ_2 and g be the functions with assumptions of Theorem 4. If $\phi_1, \phi_2 \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is compact interval. Also let $\alpha > 1$, $m = [\beta]+1$ and $n = [\alpha]+1$. Let $v \in C^{\beta}_{a^+}[a,b]$ such that $f^{(i)}(a) = 0$ for $i = n-1, n, \dots, m-2$. Let ${}^{\tilde{C}}D^{\beta}_{a^+}v \in L_q[a,b]$ and ${}_g\mathbb{F}^{x^2}_j({}^{\tilde{C}}D^{\alpha}_{a^+}v,\ {}^{\tilde{C}}D^{\beta}_{a^+}v;M) \neq 0$, $j = 1,2$. Then there exists an $\xi_j \in I$, $j = 1,2$, such that for $\alpha \leqslant \beta - 1$ we have*

$$\frac{{}_g\mathbb{F}^{\phi_1}_j({}^{\tilde{C}}D^{\alpha}_{a^+}v,\ {}^{\tilde{C}}D^{\beta}_{a^+}v;M)}{{}_g\mathbb{F}^{\phi_2}_j({}^{\tilde{C}}D^{\alpha}_{a^+}v,\ {}^{\tilde{C}}D^{\beta}_{a^+}v;M)} = \frac{\phi_1''(\xi_j)}{\phi_2''(\xi_j)}, \quad j = 1,2, \tag{24}$$

where $M = \dfrac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}$. Provided the denominators are not equal to zero.

Proof. It is easy to prove by using function defined in (23) and Theorem 6.

THEOREM 15. *Let ϕ, $g : [0,\infty) \to \mathbb{R}$ be functions with assumptions of Theorem 4. Also let $\alpha > 1$, $m = [\beta]+1$ and $n = [\alpha]+1$. Suppose that one of the following conditions (i)–(vii) in Lemma 1 hold for $\{\beta, \alpha, v\}$ and let $D^{\beta}_{a^+}v \in L_q[a,b]$. Then there exists an $\xi_i \in I$, $i = 1,2$, such that the following result holds*

$${}_g\mathbb{F}^{\phi}_i(D^{\alpha}_{a^+}v, D^{\beta}_{a^+}v;M) = \frac{\phi''(\xi_i)}{2}\ {}_g\mathbb{F}^{x^2}_i(D^{\alpha}_{a^+}v, D^{\beta}_{a^+}v;M), \quad i = 1,2. \tag{25}$$

Proof. The proof is similar to the proof of Theorem 11. Also the value of M is same as in Theorem 11.

THEOREM 16. *Let ϕ_1, ϕ_2 and g be the functions with assumptions of Theorem 4. If $\phi_1, \phi_2 \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is compact interval. Also let $\alpha > 1$, $m = [\beta]+1$ and $n = [\alpha]+1$. Suppose that one of the following conditions (i)–(vii) in Lemma 1 hold for $\{\beta, \alpha, v\}$. If $D^{\beta}_{a^+}v \in L_q[a,b]$ and ${}_g\mathbb{F}^{x^2}_i(D^{\alpha}_{a^+}v, D^{\beta}_{a^+}v;M) \neq 0, i = 1,2$, then there exists an $\xi_i \in I$, $i = 1,2$, such that we have*

$$\frac{{}_g\mathbb{F}^{\phi_1}_i(D^{\alpha}_{a^+}v, D^{\beta}_{a^+}v;M)}{{}_g\mathbb{F}^{\phi_2}_i(D^{\alpha}_{a^+}v, D^{\beta}_{a^+}v;M)} = \frac{\phi_1''(\xi_i)}{\phi_2''(\xi_i)}, \quad i = 1,2. \tag{26}$$

Provided the denominators are not equal to zero. The value of M is same as in Theorem 11.

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