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ON FRACTIONAL MEAN VALUE THEOREMS ASSOCIATED WITH HADAMARD FRACTIONAL CALCULUS AND APPLICATION

LI MA, CHENGCHENG LIU, RUILIN LIU, BO WANG AND YIXUAN ZHU

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Abstract. This paper is mainly to establish generalized mean value theorems involved with left and right Hadamard fractional calculus. In light of suitable absolutely continuous spaces and auxiliary scaling function, the novel Taylor type mean value theorem and Cauchy type mean value theorem are demonstrated in the functional space generated by logarithmic basis, respectively. Additionally, several indispensable examples are given to verify the effectiveness of our theoretical results.

1. Introduction

In recent decades, considerable attention on fractional calculus could be found in applied mathematics, mechanics of materials, biophysics and other applied science, such as in anomalous transport [1], system control [2], stability and chaos in fractional systems [3, 4], fractional Brownian motion [5]. In [6], they extend the typical Gray-Scott model by using of variable-order fractional differential equations. In [7], the authors investigate the numerical solutions of a class of fractional partial differential equations with Riesz fractional settings. More impressive works on this topic, one may refer [8, 9, 10, 11] and the references cited therein.

As we all know, Rolle mean value theorem, Lagrange mean value theorem, Cauchy mean value theorem and Taylor mean value theorem play a vital role in classic calculus. In fact, mean value theorems could build a bridge between the mean value of the function and its derivative. It is worthy to be mentioned that there exist some literatures dealing with fractional mean value theorems and bring us many valuable ideas. The generalized Cauchy's mean value theorem of Riemann-Liouville fractional derivative is derived by Pečarić et al, and a general abstract method is also extracted by operator theory [12]. Diethelm proposes a generalized Taylor formula and generalizes the classical Nagumo theorem for first-order differential equations [13] shortly after the Riemann-Liouville type fractional order mean value theorem is proposed. Guo et al. successfully

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construct the generalized fractional mean value theorems in sense of Riemann-Liouville and Caputo [14]. By using of the Taylor series expansion, a new model for the Boussinesq equation of fractional order is well established [15]. Nwaeze establishes the Rolle mean value theorem of fractional order of Benkhettou-Hassani-Torres type, and concludes that when the fractional order is equal to 1, it can degenerate into the classic Rolle mean value theorem [16]. The above mentioned mean value theorems with fractional order are all about Riemann-Liouville or Caputo type.

In fact, there is another important fractional version called Hadamard fractional calculus, which is first proposed by Hadamard in 1892. Some fundamental properties/dynamic behaviours on Hadamard fractional calculus/Hadamard fractional differential equations have been established in [17, 18, 19, 20, 21, 22, 23], and the references cited therein. As be reported, Hadamard fractional calculus has been widely applied into the problems of many mechanics and engineering, one may refer to [24, 25]. To the best of our knowledge, there are no reports on the mean value theorems on the Hadamard fractional calculus.

In this paper, new fractional mean value theorems are constructed for solving the problems related to Hadamard fractional calculus. And the remaining parts of this paper is organized as follows. Section 2 reviews some basic definitions of Hadamard fractional calculus including left and right sides, also introduces some related conclusions. In Section 3, several vital mean value theorems are stated and proved. Illustrative examples are presented in Section 4 which could be verify our main results well. Besides, the last section is the conclusion of our paper.

2. Preliminaries

In the sequel, several fundamental concepts and conclusions on Hadamard fractional calculus are introduced firstly [9].

DEFINITION 2.1. The left-sided Hadamard fractional integral of $f(x)$ with order $\alpha > 0$ is defined by

$$ {}_HD_{a^+}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{\tau}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau}. \tag{1}$$

DEFINITION 2.2. The right-sided Hadamard fractional integral of $f(x)$ with order $\alpha > 0$ is defined by

$$ {}_HD_{b^-}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\log \frac{\tau}{x}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau}. \tag{2}$$

DEFINITION 2.3. The left-sided Hadamard fractional derivative of $f(x)$ with order $\alpha > 0$ is defined by

$$ {}_HD_{a^+}^{\alpha} f(x) = \delta^n({}_HD_{a^+}^{-(n-\alpha)} f(x)), \tag{3}$$

where $x > a$, $\delta = x\frac{d}{dx}$, $n-1 < \alpha \leqslant n \in Z^+$.

DEFINITION 2.4. The right-sided Hadamard fractional derivative of $f(x)$ with order $\alpha > 0$ is defined by

$$ {}_HD^{\alpha}_{b^-} f(x) = (-\delta)^n ({}_HD^{-(n-\alpha)}_{b^-} f(x)), \tag{4} $$

where $x > a$, $\delta = x\frac{d}{dx}$, $n-1 < \alpha \leqslant n \in Z^+$.

LEMMA 2.1. *Set $-\infty < a < b < \infty$, for a finite closed interval $[a,b]$, and let $AC[a,b]$ be the space of absolutely continuous functions f on $[a,b]$. One admits that $AC[a,b]$ coincides with the space of primitives of Lebesgue summable functions,*

$$ f(x) \in AC[a,b] \Leftrightarrow f(x) = c + \int_a^x \psi(t)dt, \tag{5} $$

where $\psi(t) \in L(a,b)$.

DEFINITION 2.5. Space $AC^n_\delta[a,b]$ is defined as

$$ AC^n_\delta[a,b] = \left\{ h : [a,b] \to C | \delta^{n-1}[h(x)] \in AC[a,b], \delta = x\frac{d}{dx} \right\}. \tag{6} $$

If $n = 1$, the space $AC^1_\delta[a,b]$ coincides with $AC[a,b]$.

LEMMA 2.2. *For $0 < a < b < \infty$, let $\alpha > 0$, $n = \lceil \alpha \rceil$. If $f(x) \in L(a,b)$ and ${}_HD^{-(n-\alpha)}_{a^+} f(x) \in AC^n_\delta[a,b]$, then*

$$ {}_HD^{-\alpha}_{a^+}\, {}_HD^{\alpha}_{a^+} f(x) = f(x) - \sum_{k=1}^{n} \frac{[\delta^{n-k}({}_HD^{-(n-a)}_{a^+} f(x))](a)}{\Gamma(\alpha-k+1)} \left(\log\frac{x}{a}\right)^{\alpha-k}. \tag{7} $$

3. New fractional mean value theorems

In this section, we will establish compatible mean value theorems for left and right sided Hadamard fractional calculus, respectively.

THEOREM 3.1. *For $0 < a < b < \infty$, suppose that $\alpha > 0$, $n = \lceil \alpha \rceil$, $f(x) \in L(a,b)$, and ${}_HD^{\alpha}_{a^+} f(x) \in AC^n_\delta[a,b] \cap C[a,b]$. Then*

$$ f(x) = \sum_{k=1}^{n} \frac{[\delta^{n-k}({}_HD^{-(n-\alpha)}_{a^+} f(x))](a)}{\Gamma(\alpha-k+1)} \left(\log\frac{x}{a}\right)^{\alpha-k} + \frac{[{}_HD^{\alpha}_{a^+} f(x)](\xi)}{\Gamma(\alpha+1)} \left(\log\frac{x}{a}\right)^{\alpha}, \tag{8} $$

where $a \leqslant \xi \leqslant x$.

Proof. According to Definitions 2.1 and 2.3, we have

$$
\begin{aligned}
{}_HD_{a^+}^{-\alpha}\,{}_HD_{a^+}^{\alpha}f(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x \left(\log\frac{x}{\tau}\right)^{\alpha-1}[{}_HD_{a^+}^{\alpha}f(\tau)]\frac{d\tau}{\tau} \\
&= \frac{[{}_HD_{a^+}^{\alpha}f(x)](\xi)}{\Gamma(\alpha)}\int_a^x \left(\log\frac{x}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau} \\
&= \frac{[{}_HD_{a^+}^{\alpha}f(x)](\xi)}{\Gamma(\alpha+1)}\left(\log\frac{x}{a}\right)^{\alpha},
\end{aligned} \tag{9}
$$

where $a \leqslant \xi \leqslant x$. Now from Lemma 2.2, one has

$$
{}_HD_{a^+}^{-\alpha}\,{}_HD_{a^+}^{\alpha}f(x) = f(x) - \sum_{k=1}^{n}\frac{[\delta^{n-k}({}_HD_{a^+}^{-(n-\alpha)}f(x))](a)}{\Gamma(\alpha-k+1)}\left(\log\frac{x}{a}\right)^{\alpha-k}. \tag{10}
$$

Through the above two formulas, we can get

$$
f(x) = \sum_{k=1}^{n}\frac{[\delta^{n-k}({}_HD_{a^+}^{-(n-\alpha)}f(x))](a)}{\Gamma(\alpha-k+1)}\left(\log\frac{x}{a}\right)^{\alpha-k} + \frac{[{}_HD_{a^+}^{\alpha}f(x)](\xi)}{\Gamma(\alpha+1)}\left(\log\frac{x}{a}\right)^{\alpha}. \tag{11}
$$

COROLLARY 3.1. *For $0 < a < b < \infty$, suppose that $\alpha > 0$, $n = \lceil\alpha\rceil$, $f(x) \in L(a,b)$, and ${}_HD_{a^+}^{\alpha}f(x) \in AC_{\delta}^{n}[a,b] \cap C[a,b]$. If*

$$
f(b) = \sum_{k=1}^{n}\frac{[\delta^{n-k}({}_HD_{a^+}^{-(n-\alpha)}f)](a)}{\Gamma(\alpha-k+1)}\left(\log\frac{x}{a}\right)^{\alpha-k}\Big|_{x=b}, \tag{12}
$$

then there is at least one point $\xi \in [a,b]$, which makes

$$
[{}_HD_{a^+}^{\alpha}f(x)](\xi) = 0. \tag{13}
$$

COROLLARY 3.2. *With the same conditions of Theorem 3.1, but let $\alpha \in (0,1]$. Then,*

$$
f(x) = \frac{[{}_HD_{a^+}^{\alpha-1}f(x)](a)}{\Gamma(\alpha)}\left(\log\frac{x}{a}\right)^{\alpha-1} + \frac{[{}_HD_{a^+}^{\alpha}f(x)](\xi)}{\Gamma(\alpha+1)}\left(\log\frac{x}{a}\right)^{\alpha}, \tag{14}
$$

where $x \in [a,b]$ and $a \leqslant \xi \leqslant x$.

Proof. $\alpha \in (0,1]$, so $n = 1$. From Theorem 3.1, one has

$$
\begin{aligned}
f(x) &= \sum_{k=1}^{1}\frac{[{}_HD_{a^+}^{-(1-\alpha)}f(x)](a)}{\Gamma(\alpha)}\left(\log\frac{x}{a}\right)^{\alpha-k} + \frac{[{}_HD_{a^+}^{\alpha}f(x)](\xi)}{\Gamma(\alpha+1)}\left(\log\frac{x}{a}\right)^{\alpha} \\
&= \frac{[{}_HD_{a^+}^{\alpha-1}f(x)](a)}{\Gamma(\alpha)}\left(\log\frac{x}{a}\right)^{\alpha-1} + \frac{[{}_HD_{a^+}^{\alpha}f(x)](\xi)}{\Gamma(\alpha+1)}\left(\log\frac{x}{a}\right)^{\alpha},
\end{aligned} \tag{15}
$$

where $x \in [a,b]$ and $a \leqslant \xi \leqslant x$.

COROLLARY 3.3. *Consider* $\alpha \in (0,1]$, *and* $g(x) \in L(a,b)$ *such that*

$$ {}_H D^{\alpha}_{a^+}\left[\left(\log\frac{x}{a}\right)^{\alpha-1} g(x)\right] \in AC^n_\delta[a,b], $$

then, for some ξ, *we obtain*

$$ g(x) = g(a) + \frac{[{}_H D^{\alpha}_{a^+}(\log\frac{x}{a})^{\alpha-1} g(x)](\xi)}{\Gamma(\alpha+1)} \log\frac{x}{a}, \tag{16} $$

where $a \leqslant \xi \leqslant x$.

Proof. In light of Corollary 3.2, one has

$$ \begin{aligned} g(x)\left(\log\frac{x}{a}\right)^{\alpha-1} = & \frac{[{}_H D^{\alpha-1}_{a^+}(g(x)(\log\frac{x}{a})^{\alpha-1})](a)}{\Gamma(\alpha)}\left(\log\frac{x}{a}\right)^{\alpha-1} \\ & + \frac{[{}_H D^{\alpha}_{a^+}(g(x)(\log\frac{x}{a})^{\alpha-1})](\xi)}{\Gamma(\alpha+1)}\left(\log\frac{x}{a}\right)^{\alpha}. \end{aligned} \tag{17} $$

Hence,

$$ {}_H D^{\alpha-1}_{a^+}\left(g(x)\left(\log\frac{x}{a}\right)^{\alpha-1}\right) \overset{a\leqslant\eta\leqslant x}{=} \frac{g(\eta)}{\Gamma(1-\alpha)}\int_a^x \left(\log\frac{x}{\tau}\right)^{-\alpha}\left(\log\frac{\tau}{a}\right)^{\alpha-1}\frac{d\tau}{\tau}. \tag{18} $$

Moreover, let $\rho = \frac{\log\frac{\tau}{a}}{\log\frac{x}{a}}$, we have

$$ 1-\rho = \frac{\log\frac{x}{\tau}}{\log\frac{x}{a}}, \quad d\rho = \frac{1}{\log\frac{x}{a}}\frac{d\tau}{\tau}. \tag{19} $$

So

$$ \begin{aligned} & {}_H D^{\alpha-1}_{a^+}\left(g(x)\left(\log\frac{x}{a}\right)^{\alpha-1}\right)(a) \\ & = \lim_{x\to a^+} {}_H D^{\alpha-1}_{a^+}\left(g(x)\left(\log\frac{x}{a}\right)^{\alpha-1}\right) \\ & = \lim_{x\to a^+,\eta\to a^+} \frac{g(\eta)}{\Gamma(1-\alpha)}\int_a^x (1-\rho)^{-\alpha}\left(\log\frac{x}{a}\right)^{-\alpha}\rho^{\alpha-1}\left(\log\frac{x}{a}\right)^{\alpha} d\rho \\ & = \lim_{\eta\to a^+} g(\eta)\frac{1}{\Gamma(1-\alpha)}B(-\alpha+1,\alpha) \\ & = g(a)\Gamma(\alpha). \end{aligned} \tag{20} $$

Thus

$$ g(x)\left(\log\frac{x}{a}\right)^{\alpha-1} = g(a)\left(\log\frac{x}{a}\right)^{\alpha-1} + \frac{[{}_H D^{\alpha}_{a^+}(\log\frac{x}{a})^{\alpha-1} g(x)](\xi)}{\Gamma(\alpha+1)}\left(\log\frac{x}{a}\right)^{\alpha}. \tag{21} $$

Consequently, Corollary 3.3 is valid.

THEOREM 3.2. *Let both $f(x)$ and $g(x) \in L(a,b)$, also let both ${}_HD^{\alpha}_{a+}f(x)$ and ${}_HD^{\alpha}_{a+}g(x) \in AC^n_{\delta}[a,b]$. Then there is at least one state $\xi \in [a,x]$ for any $x \in (a,b]$ which satisfies*

$$\frac{f(x) - \sum_{k=1}^{n} \frac{[\delta^{n-k}{}_HD^{-(n-\alpha)}_{a^+}f(x)](a)}{\Gamma(\alpha-k+1)}(\log\frac{x}{a})^{\alpha-k}}{g(x) - \sum_{k=1}^{n} \frac{[\delta^{n-k}{}_HD^{-(n-\alpha)}_{a^+}g(x)](a)}{\Gamma(\alpha-k+1)}(\log\frac{x}{a})^{\alpha-k}} = \frac{[{}_HD^{\alpha}_{a+}f(x)](\xi)}{[{}_HD^{\alpha}_{a+}g(x)](\xi)}. \tag{22}$$

Proof. Firstly, we define F and G for any fixed state x in $[a,b]$ as the follows,

$$F = f(x) - \sum_{k=1}^{n} \frac{[\delta^{n-k}{}_HD^{-(n-\alpha)}_{a^+}f(x)](a)}{\Gamma(\alpha-k+1)}\left(\log\frac{x}{a}\right)^{\alpha-k}, \tag{23}$$

$$G = g(x) - \sum_{k=1}^{n} \frac{[\delta^{n-k}{}_HD^{-(n-\alpha)}_{a^+}g(x)](a)}{\Gamma(\alpha-k+1)}\left(\log\frac{x}{a}\right)^{\alpha-k}. \tag{24}$$

Further, we ponder the following function,

$$Z(t) = G\cdot f(t) - F\cdot g(t). \tag{25}$$

Due to f and g admit the presence of Theorem 3.1, thus it yields

$$Z(t) - \sum_{k=1}^{n} \frac{[\delta^{n-k}{}_HD^{-(n-\alpha)}_{a+}Z(t)](a)}{\Gamma(\alpha-k+1)}\left(\log\frac{t}{a}\right)^{\alpha-k} = [{}_HD^{\alpha}_{a^+}Z(t)](\xi)\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}, \tag{26}$$

where $\xi \in [a,t]$, this gives

$$\begin{aligned} G\cdot &\left[f(t) - \sum_{k=1}^{n} \frac{[\delta^{n-k}{}_HD^{-(n-\alpha)}_{a^+}f(t)](a)}{\Gamma(\alpha-k+1)}\left(\log\frac{t}{a}\right)^{\alpha-k}\right] \\ &- F\cdot\left[g(t) - \sum_{k=1}^{n} \frac{[\delta^{n-k}{}_HD^{-(n-\alpha)}_{a^+}g(t)](a)}{\Gamma(\alpha-k+1)}\left(\log\frac{t}{a}\right)^{\alpha-k}\right] \\ &= \frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}\{G\cdot[{}_HD^{\alpha}_{a^+}f(t)](\xi) - F\cdot[{}_HD^{\alpha}_{a^+}g(t)](\xi)\}. \end{aligned} \tag{27}$$

Then let $t = x$, it yields to

$$G\cdot F - F\cdot G = \frac{(\log\frac{x}{a})^{\alpha}}{\Gamma(\alpha+1)}\{G\cdot[{}_HD^{\alpha}_{a^+}f(x)](\xi) - F\cdot[{}_HD^{\alpha}_{a^+}g(x)](\xi)\}. \tag{28}$$

Obviously, the left side of (28) vanishes, so

$$G\cdot[{}_HD^{\alpha}_{a^+}f(x)](\xi) - F\cdot[{}_HD^{\alpha}_{a^+}g(x)](\xi) = 0. \tag{29}$$

Accordingly, we end this proof.

COROLLARY 3.4. *For* $0<\alpha\leqslant 1$*, let both* $f(x)$ *and* $g(x)\in L(a,b)$*, and assume that* ${}_HD^{\alpha}_{a^+}[(\log\frac{x}{a})^{\alpha-1}f(x)]$ *and* ${}_HD^{\alpha}_{a^+}[(\log\frac{x}{a})^{\alpha-1}g(x)]\in AC[a,b]$*, meanwhile,* ${}_HD^{\alpha}_{a^+}[(\log\frac{x}{a})^{\alpha-1}g(x)]\neq 0$*. Then there is* $\xi\in[a,x]$ *for any* $x\in(a,b]$ *which satisfies*

$$\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{[{}_HD^{\alpha}_{a^+}(\log\frac{x}{a})^{\alpha-1}f(x)](\xi)}{[{}_HD^{\alpha}_{a^+}(\log\frac{x}{a})^{\alpha-1}g(x)](\xi)}. \tag{30}$$

Proof. From Theorem 3.2, one has

$$\begin{aligned}
&\frac{f(x)(\log\frac{x}{a})^{\alpha-1}-\frac{[{}_HD^{\alpha-1}_{a^+}f(x)(\log\frac{x}{a})^{\alpha-1}](a)}{\Gamma(\alpha)}(\log\frac{x}{a})^{\alpha-1}}{g(x)(\log\frac{x}{a})^{\alpha-1}-\frac{[{}_HD^{\alpha-1}_{a^+}g(x)(\log\frac{x}{a})^{\alpha-1}](a)}{\Gamma(\alpha)}(\log\frac{x}{a})^{\alpha-1}}\\
&=\frac{f(x)-\frac{[{}_HD^{\alpha-1}_{a^+}f(x)(\log\frac{x}{a})^{\alpha-1}](a)}{\Gamma(\alpha)}}{g(x)-\frac{[{}_HD^{\alpha-1}_{a^+}g(x)(\log\frac{x}{a})^{\alpha-1}](a)}{\Gamma(\alpha)}}\\
&=\frac{[{}_HD^{\alpha}_{a^+}(\log\frac{x}{a})^{\alpha-1}f(x)](\xi)}{[{}_HD^{\alpha}_{a^+}(\log\frac{x}{a})^{\alpha-1}g(x)](\xi)}
\end{aligned} \tag{31}$$

From Corollary 3.3, we have

$${}_HD^{\alpha-1}_{a^+}\left[\left(\log\frac{x}{a}\right)^{\alpha-1}g(x)\right](a)=g(a)\Gamma(\alpha). \tag{32}$$

Analogously, we have

$${}_HD^{\alpha-1}_{a^+}\left[\left(\log\frac{x}{a}\right)^{\alpha-1}f(x)\right](a)=f(a)\Gamma(\alpha). \tag{33}$$

It yields to

$$\frac{f(x)-\frac{[{}_HD^{\alpha-1}_{a^+}f(x)(\log\frac{x}{a})^{\alpha-1}](a)}{\Gamma(\alpha)}}{g(x)-\frac{[{}_HD^{\alpha-1}_{a^+}g(x)(\log\frac{x}{a})^{\alpha-1}](a)}{\Gamma(\alpha)}}=\frac{f(x)-f(a)}{g(x)-g(a)}. \tag{34}$$

Therefore, the result follows.

Motivated by the left side case, for right Hadamard fractional settings, we could establish the corresponding mean value theorems as follows. For brevity, we omit the proofs of them.

THEOREM 3.3. *Suppose that* $\alpha>0$*,* $n=\lceil\alpha\rceil$*,* $0<a<b<\infty$*,* $f(x)\in L(a,b)$*, and* ${}_HD^{\alpha}_{b^-}f(x)\in AC^n_{\delta}[a,b]\cap C[a,b]$*. Then*

$$f(x)=\sum_{i=1}^{n}\frac{[(-\delta)^{n-i}({}_HD^{-(n-\alpha)}_{b^-}f(x))](b)}{\Gamma(\alpha-i+1)}\left(\log\frac{b}{x}\right)^{\alpha-i}+\frac{[{}_HD^{\alpha}_{b^-}f(x)](\xi)}{\Gamma(\alpha+1)}\left(\log\frac{b}{x}\right)^{\alpha}, \tag{35}$$

where $x\leqslant\xi\leqslant b$*.*

COROLLARY 3.5. *Suppose that* $\alpha > 0$, $n = \lceil \alpha \rceil$, $0 < a < b < \infty$, $f(x) \in L(a,b)$, *and* ${}_HD^{\alpha}_{b^-} f(x) \in AC^n_\delta[a,b] \cap C[a,b]$. *If*

$$f(a) = \sum_{i=1}^{n} \frac{[(-\delta)^{n-i}({}_HD^{-(n-\alpha)}_{b^-} f)](b)}{\Gamma(\alpha - i + 1)} \Big(\log \frac{b}{x}\Big)^{\alpha - i}\Big|_{x=a}, \tag{36}$$

then there is at least one state $\xi \in [a,b]$, *such that* $[{}_HD^{\alpha}_{b^-} f(x)](\xi) = 0$.

COROLLARY 3.6. *With the same assumptions of Theorem 3.3, and let* $0 < \alpha \leqslant 1$. *Then*

$$f(x) = \frac{[{}_HD^{\alpha-1}_{b^-} f(x)](b)}{\Gamma(\alpha)} \Big(\log \frac{b}{x}\Big)^{\alpha-1} + \frac{[{}_HD^{\alpha}_{b-} f(x)](\xi)}{\Gamma(\alpha+1)} \Big(\log \frac{b}{x}\Big)^{\alpha}, \tag{37}$$

where $x \in [a,b]$, $x \leqslant \xi \leqslant b$.

COROLLARY 3.7. *Let* $0 < \alpha \leqslant 1$ *and* $g(x) \in L(a,b)$ *which makes*

$${}_HD^{\alpha}_{b^-}\Big[\Big(\log \frac{b}{x}\Big)^{\alpha-1} g(x)\Big] \in AC^n_\delta[a,b]. \tag{38}$$

Then, for some ξ, *one has*

$$g(x) = g(b) + \frac{[{}_HD^{\alpha}_{b^-}(\log \frac{b}{x})^{\alpha-1} g(x)](\xi)}{\Gamma(\alpha+1)} \log \frac{b}{x}, \tag{39}$$

where $x \leqslant \xi \leqslant b$.

THEOREM 3.4. *Let both* $f(x)$ *and* $g(x) \in L(a,b)$, *also let* ${}_HD^{\alpha}_{b^-} f(x)$ *and* ${}_HD^{\alpha}_{b^-} g(x)$ $\in AC^n_\delta[a,b]$. *Then there is at least one state* $\xi \in [x,b]$ *for any* $x \in [a,b)$, *such that*

$$\frac{f(x) - \sum\limits_{i=1}^{n} \frac{[(-\delta)^{n-i}{}_HD^{-(n-\alpha)}_{b^-} f(x)](b)}{\Gamma(\alpha-i+1)} (\log \frac{b}{x})^{\alpha-i}}{g(x) - \sum\limits_{i=1}^{n} \frac{[(-\delta)^{n-i}{}_HD^{-(n-\alpha)}_{b^-} g(x)](b)}{\Gamma(\alpha-i+1)} (\log \frac{b}{x})^{\alpha-i}} = \frac{[{}_HD^{\alpha}_{b^-} f(x)](\xi)}{[{}_HD^{\alpha}_{b^-} g(x)](\xi)}. \tag{40}$$

COROLLARY 3.8. *Consider* $0 < \alpha \leqslant 1$ *and both* $f(x)$, $g(x) \in L(a,b)$, *such that* ${}_HD^{\alpha}_{b^-}[(\log \frac{b}{x})^{\alpha-1} f(x)]$ *and* ${}_HD^{\alpha}_{b^-}[(\log \frac{b}{x})^{\alpha-1} f(x)] \in AC[a,b]$. *If*

$${}_HD^{\alpha}_{b^-}\Big[\Big(\log \frac{b}{x}\Big)^{\alpha-1} f(x)\Big] \neq 0,$$

then for any $x \in [a,b)$, *there exists* $\xi \in [x,b]$ *such that*

$$\frac{f(x) - f(b)}{g(x) - g(b)} = \frac{[{}_HD^{\alpha}_{b^-}(\log \frac{b}{x})^{\alpha-1} f(x)](\xi)}{[{}_HD^{\alpha}_{b^-}(\log \frac{b}{x})^{\alpha-1} g(x)](\xi)}. \tag{41}$$

4. Illustrative examples

In this section, two examples are provided to verify the effectiveness of our theoretical results well.

EXAMPLE 1. Suppose that $\alpha \in (1,2)$, $0<a<b<\infty$, $f(x)\in L(a,b)$ and both ${}_HD^{\alpha}_{a^+}f(x)$ and ${}_HD^{\alpha}_{b^-}f(x)\in AC^2_\delta[a,b]\cap C[a,b]$. If $[{}_HD^{\alpha-2}_{a^+}f(x)](a)=[{}_HD^{\alpha-2}_{b^-}f(x)](b)=0$, then there are ξ and η which satisfy $a\leqslant\xi\leqslant\sqrt{ab}\leqslant\eta\leqslant b$ such that

$$|\delta[{}_HD^{\alpha-2}_{a^+}f(x)](a)+\delta[{}_HD^{\alpha-2}_{b^-}f(x)](b)|\leqslant\frac{\log\frac{b}{a}}{2\alpha}(|[{}_HD^{\alpha}_{a^+}f(x)](\xi)|+|[{}_HD^{\alpha}_{b^-}f(x)](\eta)|).$$

Proof. By using of Theorem 3.1, one has

$$\begin{aligned}f(x)=&\frac{\delta[{}_HD^{\alpha-2}_{a^+}f(x)](a)}{\Gamma(\alpha)}\Big(\log\frac{x}{a}\Big)^{\alpha-1}+\frac{[{}_HD^{\alpha-2}_{a^+}f(x)](a)}{\Gamma(\alpha-1)}\Big(\log\frac{x}{a}\Big)^{\alpha-2}\\&+\frac{[{}_HD^{\alpha}_{a^+}f(x)](\xi)}{\Gamma(\alpha+1)}\Big(\log\frac{x}{a}\Big)^{\alpha},\end{aligned}\tag{42}$$

where $\xi\in[a,x]$.

Then using Theorem 3.3, one has

$$\begin{aligned}f(x)=&\frac{-\delta[{}_HD^{\alpha-2}_{b^-}f(x)](b)}{\Gamma(\alpha)}\Big(\log\frac{b}{x}\Big)^{\alpha-1}+\frac{[{}_HD^{\alpha-2}_{b^-}f(x)](b)}{\Gamma(\alpha-1)}\Big(\log\frac{b}{x}\Big)^{\alpha-2}\\&+\frac{[{}_HD^{\alpha}_{b^-}f(x)](\eta)}{\Gamma(\alpha+1)}\Big(\log\frac{b}{x}\Big)^{\alpha},\ x\leqslant\eta\leqslant b.\end{aligned}\tag{43}$$

Now choose $x=\sqrt{ab}$, it yields

$$f(\sqrt{ab})=\frac{\delta[{}_HD^{\alpha-2}_{a^+}f(x)](a)}{\Gamma(\alpha)}\Big(\frac{1}{2}\log\frac{b}{a}\Big)^{\alpha-1}+\frac{[{}_HD^{\alpha}_{a^+}f(x)](\xi)}{\Gamma(\alpha+1)}\Big(\frac{1}{2}\log\frac{b}{a}\Big)^{\alpha},a\leqslant\xi\leqslant\sqrt{ab}.\tag{44}$$

$$f(\sqrt{ab})=\frac{-\delta[{}_HD^{\alpha-2}_{b^-}f(x)](b)}{\Gamma(\alpha)}\Big(\frac{1}{2}\log\frac{b}{a}\Big)^{\alpha-1}+\frac{[{}_HD^{\alpha}_{b^-}f(x)](\eta)}{\Gamma(\alpha+1)}\Big(\frac{1}{2}\log\frac{b}{a}\Big)^{\alpha},\sqrt{ab}\leqslant\eta\leqslant b.\tag{45}$$

It immediately gets

$$\begin{aligned}&\frac{\delta[{}_HD^{\alpha-2}_{a^+}f(x)](a)+\delta[{}_HD^{\alpha-2}_{b^-}f(x)](b)}{\Gamma(\alpha)}\Big(\frac{1}{2}\log\frac{b}{a}\Big)^{\alpha-1}\\&=\frac{[{}_HD^{\alpha}_{b^-}f(x)](\eta)-[{}_HD^{\alpha}_{a^+}f(x)](\xi)}{\Gamma(\alpha+1)}\Big(\frac{1}{2}\log\frac{b}{a}\Big)^{\alpha}\end{aligned}\tag{46}$$

Thus

$$|\delta[{}_HD^{\alpha-2}_{a^+}f(x)](a)+\delta[{}_HD^{\alpha-2}_{b^-}f(x)](b)|=\frac{\log\frac{b}{a}}{2\alpha}|[{}_HD^{\alpha}_{b^-}f(x)](\eta)-[{}_HD^{\alpha}_{a^+}f(x)](\xi)|,\tag{47}$$

where $a \leqslant \xi \leqslant \sqrt{ab} \leqslant \eta \leqslant b$.

Consequently, we have

$$|\delta[{}_HD_{a^+}^{\alpha-2}f(x)](a)+\delta[{}_HD_{b^-}^{\alpha-2}f(x)](b)| \leqslant \frac{\log\frac{b}{a}}{2\alpha}(|[{}_HD_{a^+}^{\alpha}f(x)](\xi)|+|[{}_HD_{b^-}^{\alpha}f(x)](\eta)|), \tag{48}$$

where $a \leqslant \xi \leqslant \sqrt{ab} \leqslant \eta \leqslant b$.

EXAMPLE 2. Let $f(x)$ admits the conditions of Corollary 3.4 and Corollary 3.8, for $\alpha \in (0,1)$, $f(a) = f(b)$, then one has

$$\frac{[{}_HD_{a^+}^{\alpha}\left(\log\frac{x}{a}\right)^{\alpha-1}f(x)](\xi)}{(\log\frac{\xi}{a})^{-\alpha}} = \frac{[{}_HD_{b^-}^{\alpha}(\log\frac{b}{x})^{\alpha-1}f(x)](\eta)}{(\log\frac{b}{\eta})^{-\alpha}},$$

where $a \leqslant \xi \leqslant \sqrt{ab} \leqslant \eta \leqslant b$.

Proof. First we suppose $g(x) = (\log\frac{x}{a})^{1-\alpha}$, from Corollary 3.4, then it holds

$$\begin{aligned}&\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{[{}_HD_{a^+}^{\alpha}(\log\frac{x}{a})^{\alpha-1}f(x)](\xi)}{[{}_HD_{a^+}^{\alpha}(\log\frac{x}{a})^{\alpha-1}g(x)](\xi)}\\ \Rightarrow &\frac{f(x)-f(a)}{(\log\frac{x}{a})^{1-\alpha}} = \frac{[{}_HD_{a^+}^{\alpha}(\log\frac{x}{a})^{\alpha-1}f(x)](\xi)}{\frac{1}{\Gamma(1-\alpha)}(\log\frac{\xi}{a})^{-\alpha}},\end{aligned} \tag{49}$$

where $\xi \in [a,x]$.

Then we assume $h(x) = (\log\frac{b}{x})^{1-\alpha}$, from Corollary 3.8, it holds

$$\begin{aligned}&\frac{f(x)-f(b)}{h(x)-h(b)} = \frac{[{}_HD_{b^-}^{\alpha}(\log\frac{b}{x})^{\alpha-1}f(x)](\eta)}{[{}_HD_{b^-}^{\alpha}(\log\frac{b}{x})^{\alpha-1}h(x)](\eta)}\\ \Rightarrow &\frac{f(x)-f(b)}{(\log\frac{b}{x})^{1-\alpha}} = \frac{[{}_HD_{b^-}^{\alpha}(\log\frac{b}{x})^{\alpha-1}f(x)](\eta)}{\frac{1}{\Gamma(1-\alpha)}(\log\frac{b}{\eta})^{-\alpha}},\end{aligned} \tag{50}$$

where $\eta \in [x,b]$.

Due to $f(x)-f(a) = f(x)-f(b)$, we can get

$$\frac{[{}_HD_{a^+}^{\alpha}(\log\frac{x}{a})^{\alpha-1}f(x)](\xi)}{(\log\frac{\xi}{a})^{-\alpha}(\log\frac{b}{x})^{1-\alpha}} = \frac{[{}_HD_{b^-}^{\alpha}(\log\frac{b}{x})^{\alpha-1}f(x)](\eta)}{(\log\frac{b}{\eta})^{-\alpha}(\log\frac{x}{a})^{1-\alpha}}. \tag{51}$$

Now taking $x = \sqrt{ab}$, thus one has

$$\frac{[{}_HD_{a^+}^{\alpha}(\log\frac{x}{a})^{\alpha-1}f(x)](\xi)}{(\log\frac{\xi}{a})^{-\alpha}} = \frac{[{}_HD_{b^-}^{\alpha}(\log\frac{b}{x})^{\alpha-1}f(x)](\eta)}{(\log\frac{b}{\eta})^{-\alpha}}, \tag{52}$$

where $a \leqslant \xi \leqslant \sqrt{ab} \leqslant \eta \leqslant b$.

All this ends our proof.

REMARK 1. In effect, the above constructive examples only deal with the problem associated with both left and right sided Hadamard fractional calculus, so one may consider more complicated cases by using of our main theorems.

Conclusion. Our paper originates from the observation of fractional calculus and various typical mean value theorems, and establishes several extended mean value theorems involved with Hadamard fractional calculus. It is also found that the logarithmic series expansion is well compatible with Hadamard fractional operators well. Besides, our results could be applied into numerical and theoretical analysis of fractional differential equations associated with Hadamard settings conventionally, such as for the evaluation of the scale of solution to Hadamard fractional systems.

Conflict of interest. We declare that we don't have any associative or commercial interest that represents a conflict of interest in connection with the work submitted.

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Li Ma
School of Mathematics
Hefei University of Technology
China 230601
e-mail: mali@hfut.edu.cn

Chengcheng Liu
School of Mathematics
Hefei University of Technology
China 230601
e-mail: 2017214556@mail.hfut.edu.cn

Ruilin Liu
School of Mathematics
Hefei University of Technology
China 230601
e-mail: 2017214558@mail.hfut.edu.cn

Bo Wang
School of Mathematics
Hefei University of Technology
China 230601
e-mail: 2017214600@mail.hfut.edu.cn

Yixuan Zhu
School of Mathematics
Hefei University of Technology
China 230601
e-mail: 2017214522@mail.hfut.edu.cn