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POSITIVE SOLUTIONS FOR SEMIPOSITONE SINGULAR α–ORDER ($2 < \alpha < 3$) FRACTIONAL BVPS ON THE HALF–LINE WITH D^{β}–DERIVATIVE DEPENDENCE

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Abstract. This article deals with existence of positive solutions to the fractional boundary value problem

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t), D^{\beta}u(t)) = 0, \ t > 0 \\ u(0) = D^{\alpha-2}u(0) = \lim_{t\to\infty} D^{\alpha-1}u(t) = 0 \end{cases}$$

where $\alpha \in (2,3)$, $\beta \in (0, \alpha-2]$, D^{α} is the standard Riemann-Liouville fractional derivative and the function $f : (0,+\infty)^3 \to \mathbb{R}$ is continuous semipositone and may exhibit singular at $u = 0$ and at $D^{\beta}u = 0$ The main existence result is obtained by means of Guo-Krasnoselskii's version of expansion and compression of a cone principal in a Banach space.

1. Introduction

In the last few decades, fractional differential equations have gained a considerable interest and importance, since they arise from many physical applications. Physical experimentation showed that the integral and derivative operators of fractional order do share some of the characteristics exhibited by the processes associated with complex systems having long-memory in time and fractional calculus provide an excellent framework to describe the hereditary properties of various materials and processes. For recent developments in the theory of fractional calculus and its applications, we refer to [1, 4, 8, 9, 12, 13, 14, 15, 16].

Often, for physical considerations, the positivity of the solution is required. This why existence of positive solutions for various classes of boundary value problems associated with fractional differential equations has been the subject of many papers, see, [2, 6, 7, 10, 11, 17, 18] and references therein. Because of a lack of compactness, the case where such bvps are posed on unbounded intervals and having a singular dependence on the variable space, is somewhat complicated, and to the best of our knowledge, there are no works considering existence of positive solutions for such a case. Thus, the purpose of this paper is to fill in the gap in this area.

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We are concerned in this paper with existence of positive solutions to the fractional boundary value problem (fbvp for short),

$$\begin{cases} D^{\alpha}u(t)+f(t,u(t),D^{\beta}u(t))=0,\ t\in\mathbb{I} \\ u(0)=D^{\alpha-2}u(0)=\lim_{t\to\infty}D^{\alpha-1}u(t)=0 \end{cases} \tag{1.1}$$

where $\mathbb{I}=(0,+\infty)$, $\alpha\in(2,3]$, $\beta\in(0,\alpha-2]$, for $\nu=\alpha$ or β, D^{ν} is the standard Riemann-Liouville fractional derivative and $f:\mathbb{I}^3\to\mathbb{R}$ is a continuous function.

Throughout, we assume that the nonlinearity f satisfies the following hypotheses:

$$\begin{cases} \text{There exists } q:\mathbb{R}^+\longrightarrow\mathbb{R}^+ \text{ continuous such that} \\ f(t,u,v)+q(t)>0 \ \text{ for all } \ t,u,v\in\mathbb{I} \\ \text{and } \int_0^{+\infty}sq(s)ds<\infty, \end{cases} \tag{1.2}$$

$$\begin{cases} \text{for all } \rho>0 \text{ there exists two functions } \omega_{\rho}:\mathbb{I}\to\mathbb{I} \\ \text{and } \Psi_{\rho}:\mathbb{I}^2\to\mathbb{I} \text{ such that } \Psi_{\rho} \text{ is nondecreasing} \\ \text{with respect to its variables,} \\ \left|f(t,(1+t)^{\alpha-1}w,(1+t)^{\alpha-\beta-1}z)\right|\leqslant\omega_{\rho}(t)\Psi_{\rho}(w,z) \\ \text{for all } t>0 \text{ and } w,z>0 \text{ with } |(w,z)|\leqslant\rho \text{ and} \\ \int_0^{+\infty}\omega_{\rho}(t)\Psi_{\rho}\left(r\widetilde{\gamma}_{\alpha}(t),r\widetilde{\gamma}_{\alpha-\beta}(t)\right)dt<\infty \ \text{ for all } \ r\in(0,\rho], \end{cases} \tag{1.3}$$

where for $\theta\geqslant 2$

$$\widetilde{\gamma}_{\theta}(t)=\frac{\gamma_{\theta}(t)}{(1+t)^{\theta-1}} \ \text{ and } \ \gamma_{\theta}(t)=\min\left(t^{\theta-1},\frac{t^{\theta-2}}{\theta-1}\right).$$

Notice that the nonlinearity f may exhibit singular at the solution and at its derivative. It is well known that the bvp (1.1) is called positone if $q(t)=0$ for all $t\in\mathbb{I}$, and semipositone if $q(t_0)>0$ for some $t_0\in\mathbb{I}$.

Our approach in this work is based on a fixed point formulation of the fbvp (1.1) and the main existence result in this work is then proved by the Guo-Krasnoselskii's version of expansion and compression of a cone principal in a Banach space.

The paper is organized as follows: Section 2 is devoted for preliminaries and in Section 3 we provide a fixed point formultion for the fbvp (1.1). In Section 4, we present the main result and its proof and we end the paper by an illustrative example.

2. Preliminaries

2.1. Abstract background

In this subsection we recall the Guo-Krasnoselskii's version of expansion and compression of a cone principal in a Banach space and the related absract background. Let $(E,||.||)$ be a real Banach space. A nonempty closed convex subset C in E is said to be a cone in E, if $C\cap(-C)=\{0_E\}$ and $tC\subset C$ for all $t\geqslant 0$.

Let Ω be a nonempty subset in E, a mapping $A:\Omega\to E$ is said to be compact if it is continuous and $A(\Omega)$ is relatively compact in E.

THEOREM 2.1. (Theorem 2.3.4 in [5]) *Let P be a cone in E and let Ω_1, Ω_2 be open bounded subsets of E such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. If $T : P \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to P$ is a compact operator such that, either*

1. $\|Tu\| \leqslant \|u\|$ *for* $u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \geqslant \|u\|$ *for* $u \in P \cap \partial\Omega_2$, *or*
2. $\|Tu\| \geqslant \|u\|$ *for* $u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \leqslant \|u\|$ *for* $u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.

2.2. Riemann-Liouville fractional derivative

In this subsection we recall some basic facts related to Riemann-Liouville fractional derivative. Let β be a positive real number, the Riemann-Liouville fractional integral of order β of a function $f : (0, +\infty) \to \mathbb{R}$ is defined by

$$I_{0^+}^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds,$$

where $\Gamma(\beta)$ is the gamma function, provided that the right side is pointwise defined on $(0, +\infty)$. For example, we have for any real $\sigma > -1$, $I_{0^+}^{\beta} t^{\sigma} = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma+\beta+1)} t^{\sigma+\beta}$.

The Riemann-Liouville fractional derivative of order β, of a continuous function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\beta-n+1}} ds,$$

where $n = [\beta] + 1$, $[\beta]$ denotes the integer part of the number β, provided that the right side is pointwise defined on $\mathbb{R}^+$.

As a basic example, we quote for $\sigma > \beta - 1$, $D_{0^+}^{\beta} t^{\sigma} = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\beta+1)} t^{\sigma-\beta}$. Thus, if $u \in C(0, +\infty) \cap \mathbb{L}^1(0, +\infty)$, then

$$D^{\beta} I^{\alpha} u(t) = \begin{cases} I^{\alpha-\beta} u(t) \text{ if } \alpha > \beta, \\ u(t) \text{ if } \alpha = \beta \end{cases}$$
$$I_{0^+}^{\beta} D_{0^+}^{\beta} u(t) = u(t) + \sum_{i=1}^{i=[\beta]+1} c_i t^{\beta-i}, \; c_i \in \mathbb{R}.$$

In particular, the fractional differential equation $D_{0^+}^{\beta} u(t) = 0$ has $u(t) = \sum_{i=1}^{i=[\beta]+1} c_i t^{\beta-i}$, $c_i \in \mathbb{R}$, as unique solution.

For a detailed presentation on fractional differential calculus, see [8] or [13].

3. Fixed point formulation

Firstly, we introduce the necessary framework for the fixed point formulation of the fbvp (1.1). Throughout, we let E be the linear space defined by

$$E = \left\{ u \in C\left(\mathbb{R}^+\right) : D^{\beta} u \in BC\left(\mathbb{R}^+\right), \text{ and } \lim_{t \to \infty} \frac{u(t)}{(1+t)^{\alpha-1}} = \lim_{t \to \infty} \frac{D^{\beta} u(t)}{(1+t)^{\alpha-1-\beta}} = 0 \right\}.$$

Equipped with the norm $\|\cdot\|$ where for all $u \in E$,

$$\begin{aligned} \|u\| &= \max\left(\|u\|_1, \|u\|_2\right), \\ \|u\|_1 &= \sup_{t>0} \tfrac{|u(t)|}{(1+t)^{\alpha-1}}, \\ \|u\|_2 &= \sup_{t>0} \tfrac{\left|D^{\beta} u(t)\right|}{(1+t)^{\alpha-1-\beta}}, \end{aligned}$$

E becomes a Banach space.

In all what follows P is the cone of E defined by

$$P = \left\{u \in E : u(t) \geqslant \gamma_{\alpha}(t) \|u\|_1 \text{ and } D^{\beta} u(t) \geqslant \gamma_{\alpha-\beta}(t) \|u\|_2 \text{ for all } t \geqslant 0\right\}.$$

Let for $\theta > 1$, $G_{\theta} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be the function given by

$$G_{\theta}(t,s) = \frac{1}{\Gamma(\theta)} \begin{cases} t^{\theta-1} - (t-s)^{\theta-1} & 0 \leqslant s \leqslant t < \infty \\ t^{\theta-1} & 0 \leqslant t \leqslant s < \infty. \end{cases}$$

LEMMA 3.1. *For all $\theta \geqslant 2$, the function G_{θ} is continuous and has the following properties:*

$$G_{\theta}(0,s) = 0 \text{ for all } s \geqslant 0, \tag{3.1}$$

$$0 < G_{\theta}(t,s) \leqslant \frac{t^{\theta-1}}{\Gamma(\theta)} \text{ for all } t,s \geqslant 0, \tag{3.2}$$

$$\lim_{t\to 0} \frac{G_{\theta}(t,s)}{t^{\theta-1}} = \frac{1}{\Gamma(\theta)}, \lim_{t\to+\infty} \frac{G_{\theta}(t,s)}{t^{\theta-1}} = 0 \text{ for all } s \geqslant 0, \tag{3.3}$$

$$\frac{\partial G}{\partial t}(t,s) > 0 \text{ for all } t,s > 0, \tag{3.4}$$

$$G_{\theta}(t,s) \geqslant \gamma_{\theta}(t) \frac{G_{\theta}(\tau,s)}{(1+\tau)^{\theta-1}} \text{ for all } t,\tau,s \geqslant 0. \tag{3.5}$$

Proof. Properties (3.1)–(3.4) are easy to check, so let us prove (3.5). Set for $\eta > 0$ and $s \in (0,\eta)$, $\varphi_{\eta}(s) = \eta^{\theta-1} - (\eta-s)^{\theta-1}$. The function φ_{η} has the following properties:

$$\varphi_{\eta}'(s) > 0 \text{ for all } s \in (0,\eta),$$

$$\lim_{s\to 0} \frac{\varphi_{\eta}(s)}{s} = (\theta-1)\eta^{\theta-2} \text{ and } \lim_{s\to\eta} \frac{\varphi_{\eta}(s)}{s} = \eta^{\theta-2},$$

$$(1+\eta)^{\alpha-1} \geqslant \varphi_{\eta}(s) \text{ for all } s \in (0,\eta), \tag{3.6}$$

$$\text{if } \eta < \xi \text{ then } \varphi_{\eta}(s) < \varphi_{\xi}(s) \text{ for all } s \in (0,\eta).$$

Moreover, we have

$$\left(\frac{\varphi_{\eta}(s)}{s}\right)' = \frac{h_{\eta}(s)}{s^2} \text{ for all } s \in (0,\eta),$$

where

$$\begin{aligned} &h_\eta(s) = (\eta - s)^{\theta-2}(\eta + (\theta-2)s) - \eta^{\theta-1} \\ &h_\eta(0) = 0, \qquad h_\eta(\eta) = -\eta^{\theta-1} \text{ and} \\ &h'_\eta(s) = -(\theta-1)(\theta-2)s(\eta-s)^{\theta-3} \leqslant 0. \end{aligned}$$

Therefore, we have

$$\left(\frac{\varphi_\eta(s)}{s}\right)' < 0 \text{ for all } s \in (0,\eta).$$

Notice that

$$G_\theta(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} \varphi_t(s) \text{ if } 0 \leqslant s \leqslant t < \infty \\ \varphi_t(t) \text{ if } 0 \leqslant t \leqslant s < \infty. \end{cases} \tag{3.7}$$

and we obtain from the above properties of the function φ_η that for all $t,\tau,s > 0$,

$$\frac{G_\theta(t,s)}{G_\theta(\tau,s)} = \begin{cases} \dfrac{\varphi_t(t)}{\varphi_\tau(\tau)} \geqslant \dfrac{t^{\theta-1}}{(\tau+1)^{\theta-1}} \geqslant \dfrac{\gamma_\theta(t)}{(1+\tau)^{\theta-1}} \text{ if } \tau,t \leqslant s \\ \dfrac{\varphi_t(s)}{\varphi_\tau(s)} = \dfrac{\varphi_t(s)/s}{\varphi_\tau(s)/s} \geqslant \dfrac{t^{\theta-2}}{(\theta-1)\tau^{\theta-2}} \geqslant \dfrac{\gamma_\theta(t)}{(1+\tau)^{\theta-1}} \text{ if } s \leqslant t,\tau \\ \dfrac{\varphi_t(t)}{\varphi_\tau(s)} \geqslant \dfrac{t^{\alpha-1}}{(1+\tau)^{\alpha-1}} \geqslant \dfrac{\gamma_\theta(t)}{(1+\tau)^{\alpha-1}} \text{ if } t \leqslant s \leqslant \tau \\ \dfrac{\varphi_t(s)}{\varphi_\tau(\tau)} \geqslant \dfrac{(\varphi_t(s)/s)s}{\tau^{\alpha-1}} \geqslant \dfrac{t^{\theta-2}s}{\tau^{\alpha-1}} \geqslant \dfrac{\gamma_\theta(t)}{\tau^{\alpha-2}} \geqslant \dfrac{\gamma_\theta(t)}{(1+\tau)^{\alpha-1}} \text{ if } \tau \leqslant s \leqslant t. \end{cases}$$

Ending the proof. □

LEMMA 3.2. *Assume that Hypothesis (1.2) holds and let ϕ be the function defined by $\phi(t) = \int_0^{+\infty} G(t,s)q(s)ds$. Then*

$$\phi^* = \max\left(\sup_{t>0}(\phi(t)/\gamma_\alpha(t)), \sup_{t>0}\left(D^\beta\phi(t)/\gamma_{\alpha-\beta}(t)\right)\right) < \infty.$$

Proof. Set for $\theta = \alpha$ or $\alpha-\beta$,

$$\phi_\theta = \begin{cases} \phi \text{ if } \theta = \alpha, \\ D^\beta\phi \text{ if } \theta = \alpha-\beta. \end{cases}$$

Properties (1.1) and (3.3) combined with Lebesgue dominated convergence theorem lead to

$$\lim_{t\to 0}\frac{\phi_\theta(t)}{\gamma_\theta(t)} = \lim_{t\to 0}\frac{\phi_\theta(t)}{t^{\theta-1}} = \lim_{t\to 0}\int_0^{+\infty}\frac{G_\theta(t,s)}{t^{\theta-1}}q(s)ds = \frac{1}{\Gamma(\theta)}\int_0^{+\infty} q(s)ds,$$

For t large and $\theta = \alpha$ or $\alpha-\beta$, we have by the mean value theorem

$$\begin{aligned} \frac{(\theta-1)\Gamma(\theta)\phi_\theta(t)}{\gamma_\theta(t)} &= \frac{\Gamma(\theta)\phi_\theta(t)}{t^{\theta-2}} \\ &= \int_0^t \frac{t^{\theta-1}-(t-s)^{\theta-1}}{t^{\theta-1}}q(s)ds + t\int_t^{+\infty} q(s)ds \\ &\leqslant (\theta-1)\int_0^t sq(s)ds + \int_t^{+\infty} sq(s)ds < \infty. \end{aligned}$$

The proof is complete. □

The following lemma is an adapted version to the case of the space E of Corduneanu's compactness criterion ([3], p. 62). It will be used in this work to prove that some operator is compact.

LEMMA 3.3. *A nonempty subset M of E is relatively compact if the following conditions hold:*

(a) M is bounded in E,

(b) the sets $\left\{u : u(t) = \dfrac{x(t)}{(1+t)^{\alpha-1}},\ x \in M\right\}$ and $\left\{u : u(t) = \dfrac{x(t)}{(1+t)^{\alpha-\beta-1}},\ x \in M\right\}$ are locally equicontinuous on $[0,+\infty)$ and

(c) the sets $\left\{u : u(t) = \dfrac{x(t)}{(1+t)^{\alpha-1}},\ x \in M\right\}$ and $\left\{u : u(t) = \dfrac{x(t)}{(1+t)^{\alpha-\beta-1}},\ x \in M\right\}$ are equiconvergent at $+\infty$.

LEMMA 3.4. *Assume that Hypotheses (1.2) and (1.3) hold. Then for all $r,R \in \mathbb{I}$ with $R > r > \phi^*$ there exists a compact operator $T_{r,R} : P \cap \left(\overline{B}(0,R) \setminus B(0,r)\right) \to P$ such that if v is a fixed point of $T_{r,R}$ then $u = v - \phi$ is a positive solution to the bvp (1.1).*

Proof. Let $r,R > 0$ be such that $R > r > \phi^*$ and set $\Omega = P \cap \left(\overline{B}(0,R) \setminus B(0,r)\right)$. In all this proof, we denote by Φ the function defined by

$$\Phi(s) = \omega_R(s)\Psi_R\left((r-\phi^*)\widetilde{\gamma}_\alpha(s), (r-\phi^*)\widetilde{\gamma}_{\alpha-\beta}(s)\right) + q(s),$$

where ω_R and Ψ_R are the functions given by Hypothesis (1.3) for $\rho = R$ and ϕ^* is the constant given by Lemma 3.2. The proof is divided into four steps.

Step 1. In this step we prove the existence of the operator $T_{r,R}$. We have from the definition of the cone P and Lemma 3.2 that, for all $v \in \Omega$ and all $t > 0$,

$$\begin{aligned}&v(t) - \phi(t) \geqslant (\|v\|_1 - \phi^*)\gamma_\alpha(t) \geqslant (r-\phi^*)\gamma_\alpha(t) > 0 \text{ and}\\ &D^\beta v(t) - D^\beta\phi(t) \geqslant (\|v\|_2 - \phi^*)\gamma_{\alpha-\beta}(t) \geqslant (r-\phi^*)\gamma_{\alpha-\beta}(t) > 0.\end{aligned}$$

Therefore, for all $v \in \Omega$ the expression

$$f_{r,R}v(t) = f\left(t, v(t) - \phi(t), D^\beta v(t) - D^\beta\phi(t)\right) + q(t) \tag{3.8}$$

is well defined.

Let $u \in \Omega$, for all $s > 0$ we have

$$\begin{aligned}f_{r,R}u(s) &= f\left(s, (1+s)^{\alpha-1}\frac{v(s)-\phi(s)}{(1+s)^{\alpha-1}}, (1+s)^{\alpha-\beta-1}\frac{D^\beta v(s) - D^\beta\phi(s)}{(1+s)^{\alpha-\beta-1}}\right) + q(s)\\ &\leqslant \Phi(s),\end{aligned}$$

leading to

$$\int_0^{+\infty} G_\alpha(t,s) f_{r,R} v(t)\, ds \leqslant \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{+\infty} f_{r,R} v(s)\, ds \leqslant \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{+\infty} \Phi(s) ds < \infty$$

and

$$\int_0^{+\infty} G_{\alpha-\beta}(t,s) f_{r,R} v(t)\, ds \leqslant \frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_0^{+\infty} f_{r,R} v(s)\, ds \leqslant \frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_0^{+\infty} \Phi(s) ds < \infty.$$

Let v and w be the functions defined by

$$v(t) = \int_0^{+\infty} G_\alpha(t,s) f_{r,R} u(s) ds \text{ and } w(t) = \int_0^{+\infty} G_{\alpha-\beta}(t,s) f_{r,R} u(s) ds.$$

Clearly, v and w are continuous and for all $t \geqslant 0$, we have from (1.1)

$$\begin{aligned} \frac{v(t)}{(1+t)^{\alpha-1}} &= \int_0^{+\infty} \frac{G_\alpha(t,s)}{(1+t)^{\alpha-1}} f_{r,R} u(s) ds \leqslant \frac{1}{\Gamma(\alpha)} \frac{t^{\alpha-1}}{(1+t)^{\alpha-1}} \left(\int_0^{+\infty} \Phi(s) ds \right) \\ &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \Phi(s) ds < \infty, \end{aligned}$$

$$\begin{aligned} \frac{w(t)}{(1+t)^{\alpha-\beta-1}} &= \int_0^{+\infty} \frac{G_{\alpha-\beta}(t,s)}{(1+t)^{\alpha-\beta-1}} f_{r,R} u(s) ds \\ &\leqslant \frac{1}{\Gamma(\alpha-\beta)} \frac{t^{\alpha-\beta-1}}{(1+t)^{\alpha-\beta-1}} \left(\int_0^{+\infty} \Phi(s) ds \right) \\ &\leqslant \frac{1}{\Gamma(\alpha-\beta)} \int_0^{+\infty} \Phi(s) ds < \infty \end{aligned}$$

and

$$\begin{aligned} D^\beta v(t) &= -D^\beta I^\alpha f_{r,R} u + D^\beta \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{+\infty} f_{r,R} u(s) ds \\ &= -I^{\alpha-\beta} f_{r,R} u + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \frac{t^{\alpha-\beta-1}}{\Gamma(\alpha)} \int_0^{+\infty} f_{r,R} u(s) ds \\ &= \int_0^{+\infty} G_{\alpha-\beta}(t,s) f_{r,R} u(s) ds = w(t). \end{aligned}$$

Moreover, it follows from (3.4) that for all $t, \tau \geqslant 0$

$$\begin{aligned} v(t) &= \int_0^{+\infty} G_\alpha(t,s) f_{r,R} u(s) ds \\ &\geqslant \frac{\gamma_\alpha(t)}{(1+\tau)^{\alpha-1}} \int_0^{+\infty} G_\alpha(\tau,s) f(s,u(s)) ds \\ &= \gamma_\alpha(t) \frac{v(\tau)}{(1+\tau)^{\alpha-1}} \end{aligned}$$

and

$$
\begin{aligned}
D^{\beta} v(t) = w(t) &= \int_0^{+\infty} G_{\alpha-\beta}(t,s) f_{r,R} u(s) ds \\
&\geqslant \frac{\gamma_{\alpha-\beta}(t)}{(1+\tau)^{\alpha-1}} \int_0^{+\infty} G_{\alpha-\beta} f_{r,R} u(s) ds \\
&= \gamma_{\alpha-\beta}(t) \frac{v(\tau)}{(1+\tau)^{\alpha-1}}.
\end{aligned}
$$

Passing to the supremum on τ, we obtain

$$
v(t) \geqslant \gamma_{\alpha}(t) \|v\|_1 \text{ and } D^{\beta} v(t) \geqslant \gamma_{\alpha-\beta}(t) \|v\|_2 \text{ for all } t \geqslant 0,
$$

that is $v \in P$.

Thus, we have proved that the oerator $T : \Omega \longrightarrow P$, where for $u \in \Omega$ and $t \geqslant 0$

$$
Tu(t) = \int_0^{+\infty} G(t,s) f_{r,R} u(s) ds,
$$

is well defined.

Step 2. In this step we prove that the operator $T_{r,R}$ is continuous. Let (u_n) be a sequence in Ω such that $\lim_{n\to\infty} u_n = u$ in E. For all $n \geqslant 1$, we have

$$
\begin{aligned}
\|T_{r,R} v_n - T_{r,R} v\|_1 &= \sup_{t>0} \left(\frac{|T_{r,R} u_n(t) - Tu(t)|}{(1+t)^{\alpha-1}} \right) \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} |f_{r,R} u_n(s) - f_{r,R} u(s)| ds
\end{aligned}
$$

and

$$
\begin{aligned}
\|T_{r,R} v_n - T_{r,R} v\|_2 &= \sup_{t>0} \left(\frac{\left|D^{\beta} T_{r,R} u_n(t) - D^{\beta} Tu(t)\right|}{(1+t)^{\alpha-\beta-1}} \right) \\
&\leqslant \frac{1}{\Gamma(\alpha-\beta)} \int_0^{+\infty} |f_{r,R} u_n(s) - f_{r,R} u(s)| ds.
\end{aligned}
$$

Because of

$$
|f_{r,R} v_n(s) - f_{r,R} v(s)| \to 0, \text{ as } n \to +\infty
$$

for all $s > 0$ and

$$
|f_{r,R} v_n(s) - f_{r,R} v(s)| \leqslant 2\Phi(s) \text{ with } \int_0^{+\infty} \Phi(s) ds < \infty,
$$

the Lebesgue dominated convergence theorem guarantees that $\lim_{n\to\infty} \|T_{r,R} v_n - T_{r,R} v\|_1 = \lim_{n\to\infty} \|T_{r,R} v_n - T_{r,R} v\|_2 = 0$. Hence, we have proved that T is continuous.

Step 3. In this step, we prove that $T_{r,R}$ is compact. For all $u \in \Omega$, we have

$$\begin{aligned}\|T_{r,R}u\|_1 &= \sup_{t>0} \frac{|T_{r,R}u(t)|}{(1+t)^{\alpha-1}} \leqslant \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \frac{G_\alpha(t,s)}{(1+t)^{\alpha-1}} f_{r,R}u(s)ds \\ &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \Phi(s)ds < \infty\end{aligned}$$

and

$$\begin{aligned}\|T_{r,R}u\|_2 &= \sup_{t>0} \frac{|D^\beta T_{r,R}u(t)|}{(1+t)^{\alpha-\beta-1}} \leqslant \frac{1}{\Gamma(\alpha-\beta)} \int_0^{+\infty} \frac{G_{\alpha-\beta}(t,s)}{(1+t)^{\alpha-1}} f_{r,R}u(s)ds \\ &\leqslant \frac{1}{\Gamma(\alpha-\beta)} \int_0^{+\infty} \Phi(s)ds < \infty.\end{aligned}$$

The above estimates show the condition (a) in Lemma 3.3 is satisfied.

Let $[\xi,\eta]$ be an interval of $\mathbb{R}^+$. For all $u \in \Omega$ and all $t_1,t_2 \in [\xi,\eta]$ with $0 < t_2 - t_1 < 1$, We have

$$\begin{aligned}&\left| \frac{T_{r,R}u(t_2)}{(1+t_2)^{\alpha-1}} - \frac{T_{r,R}u(t_1)}{(1+t_1)^{\alpha-1}} \right| \\ &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left| \left(\frac{t_2-s}{1+t_2}\right)^{\alpha-1} - \left(\frac{t_1-s}{1+t_1}\right)^{\alpha-1} \right| \Phi(s)ds \\ &\quad + \left(\frac{t_2-t_1}{1+t_2}\right)^{\alpha-1} |\Phi|_\alpha + \left| \left(\frac{t_2}{1+t_2}\right)^{\alpha-1} - \left(\frac{t_1}{1+t_1}\right)^{\alpha-1} \right| |\Phi|_\alpha,\end{aligned}$$

and

$$\begin{aligned}&\left| \frac{D^\beta T_{r,R}u(t_2)}{(1+t_2)^{\alpha-1}} - \frac{D^\beta T_{r,R}u(t_1)}{(1+t_1)^{\alpha-1}} \right| \\ &\leqslant \frac{1}{\Gamma(\alpha-\beta)} \int_0^{t_1} \left| \left(\frac{t_2-s}{1+t_2}\right)^{\alpha-\beta-1} - \left(\frac{t_1-s}{1+t_1}\right)^{\alpha-\beta-1} \right| \Phi(s)ds \\ &\quad + \left(\frac{t_2-t_1}{1+t_2}\right)^{\alpha-1} |\Phi|_{\alpha-\beta} + \left| \left(\frac{t_2}{1+t_2}\right)^{\alpha-1} - \left(\frac{t_1}{1+t_1}\right)^{\alpha-1} \right| |\Phi|_{\alpha-\beta},\end{aligned}$$

where for $\theta = \alpha$ or $\alpha - \beta$, $|\Phi|_\theta = \frac{1}{\Gamma(\theta)} \int_0^{+\infty} \Phi(s)ds$.

For $\theta = \alpha$ or $\alpha - \beta$, we have by the mean value theorem:

$$\begin{aligned}&\left| \left(\frac{t_2-s}{1+t_2}\right)^{\theta-1} - \left(\frac{t_1-s}{1+t_1}\right)^{\theta-1} \right| \\ &\leqslant (\theta-1)\left(\frac{\eta}{1+\eta}\right)^{\theta-2} \left| \frac{t_2-s}{1+t_2} - \frac{t_1-s}{1+t_1} \right| \leqslant (\theta-1)\left(\frac{\eta}{1+\eta}\right)^{\theta-2} \frac{(t_2-t_1)(1+s)}{(1+t_2)(1+t_1)} \\ &\leqslant (\theta-1)\left(\frac{\eta}{1+\eta}\right)^{\theta-2} (t_2-t_1)\end{aligned}$$

and

$$\left|\left(\frac{t_2}{1+t_2}\right)^{\theta-1}-\left(\frac{t_1}{1+t_1}\right)^{\theta-1}\right| \leqslant (\theta-1)\left(\frac{\eta}{1+\eta}\right)^{\theta-2}\left(\frac{t_2}{1+t_2}-\frac{t_1}{1+t_1}\right)$$
$$\leqslant (\theta-1)\left(\frac{\eta}{1+\eta}\right)^{\theta-2}(t_2-t_1).$$

The above calculations lead to

$$\left|\frac{T_{r,R}u(t_2)}{(1+t_2)^{\alpha-1}}-\frac{T_{r,R}u(t_1)}{(1+t_1)^{\alpha-1}}\right|$$
$$\leqslant 2(\alpha-1)\left(\frac{\eta}{1+\eta}\right)^{\alpha-2}|\Phi|_\alpha(t_2-t_1)+\left(\frac{t_2-t_1}{1+t_2}\right)^{\alpha-1}|\Phi|_\alpha$$
$$\leqslant \left(2(\alpha-1)\left(\frac{\eta}{1+\eta}\right)^{\alpha-2}+1\right)|\Phi|_\alpha(t_2-t_1)$$

and

$$\left|\frac{D^\beta T_{r,R}u(t_2)}{(1+t_2)^{\alpha-\beta-1}}-\frac{D^\beta T_{r,R}u(t_1)}{(1+t_1)^{\alpha-\beta-1}}\right|$$
$$\leqslant 2(\alpha-\beta-1)\left(\frac{\eta}{1+\eta}\right)^{\alpha-\beta-2}|\Phi|_{\alpha-\beta}(t_2-t_1)+\left(\frac{t_2-t_1}{1+t_2}\right)^{\alpha-\beta-1}|\Phi|_{\alpha-\beta}$$
$$\leqslant \left(2(\alpha-\beta-1)\left(\frac{\eta}{1+\eta}\right)^{\alpha-\beta-2}+1\right)|\Phi|_{\alpha-\beta}(t_2-t_1).$$

Thus, Condition (b) in Lemma 3.3 is satisfied.

We have for any u in Ω and $t \geqslant 0$

$$\left|\frac{T_{r,R}u(t)}{(1+t)^{\alpha-1}}\right| \leqslant \int_0^{+\infty}\frac{G_\alpha(t,s)}{(1+t)^{\alpha-1}}|f_{r,R}u(s)|\,ds$$
$$\leqslant \int_0^{+\infty}\frac{G_\alpha(t,s)}{(1+t)^{\alpha-1}}\Phi(s)ds = H_1(t)$$

and

$$\left|\frac{D^\beta T_{r,R}u(t)}{(1+t)^{\alpha-\beta-1}}\right| \leqslant \int_0^{+\infty}\frac{G_{\alpha-\beta}(t,s)}{(1+t)^{\alpha-\beta-1}}|f_{r,R}u(s)|\,ds$$
$$\leqslant \int_0^{+\infty}\frac{G_{\alpha-\beta}(t,s)}{(1+t)^{\alpha-\beta-1}}\Phi(s)ds = H_2(t)$$

Property (3.3) of the function G_θ and the dominated convergence theorem lead to $\lim_{t\to\infty} H_1(t) = \lim_{t\to\infty} H_2(t) = 0$, proving the equiconvergence. In view of Lemma 3.3 $T\Omega$ is relatively compact in E.

Step 4. In this step, we prove that if $v \in \Omega$ is a fixed point of $T_{r,R}$ then $u = v - \phi$ is a positive solution to the fbvp (1.1). Hence, we have

$$u(t) = \int_0^{+\infty} G_\alpha(t,s)\Big(f(s,u(s),D^\beta u(s)) + q(s)\Big)ds + \phi(t)$$
$$= -I^\alpha f(s,u(s),D^\beta u(s)) + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty f(s,u(s),D^\beta u(s))ds,$$

leading to

$$D^{\alpha-2}u(t) = D^{\alpha-2}u(t) = -\int_0^t (t-s)f(s,u(s),D^\beta u(s))ds + t\int_0^{+\infty} f(s,u(s),D^\beta u(s))ds.$$
$$D^{\alpha-1}u(t) = -\int_0^t f(s,u(s),D^\beta u(s))ds + \int_0^{+\infty} f(s,u(s),D^\beta u(s))ds$$
$$= \int_t^{+\infty} f(s,u(s),D^\beta u(s))ds.$$
$$D^\alpha u(t) = -f(t,u(t),D^\beta u(s)),$$
$$D^{\alpha-2}u(0) = \lim_{t\to+\infty} D^{\alpha-1}u(t) = 0$$

and we obtain from (3.1), $u(0) = \int_0^{+\infty} G(0,s)f(s,u(s))ds = 0.$

These show that u is a positive solution to the fbvp (1.1), ending the proof. □

4. Main result

The main result of this paper needs to introduce the following notations. For $m \in L^1(\mathbb{I})$ with $m(t) \geqslant 0$ a.e. $t > 0$ and $\sigma > 1$, we let

$$\Lambda(m) = \max\left(\sup_{t>0}\left(\int_0^{+\infty}\frac{G_\alpha(t,s)}{(1+t)^{\alpha-1}}m(s)\,ds\right), \sup_{t>0}\left(\int_0^{+\infty}\frac{G_{\alpha-\beta}(t,s)}{(1+t)^{\alpha-\beta-1}}m(s)\,ds\right)\right),$$
$$\Delta(m,\sigma) = \min\left(\sup_{t>0}\left(\int_{1/\sigma}^{\sigma}\frac{G_\alpha(t,s)}{(1+t)^{\alpha-1}}m(s)\,ds\right), \sup_{t>0}\left(\int_{1/\sigma}^{\sigma}\frac{G_{\alpha-\beta}(t,s)}{(1+t)^{\alpha-\beta-1}}m(s)\,ds\right)\right).$$

THEOREM 4.1. *Suppose that Hypotheses (1.2) and (1.3) hold,*

(a) *there exist a function $a \in L^1(\mathbb{I})$ and $R_1 > \max(\phi^*, \Lambda(a))$ such that*

$$f(t,(1+t)^{\alpha-1}u,(1+t)^{\alpha-\beta-1}v) + q(t) \leqslant a(t)$$

for a.e. $t \in \mathbb{I}$ and all $u, v \in \mathbb{I}$ with $|(u,v)| \leqslant R_1$,

(b) *there exist $\sigma > 1$, a function $b \in L^1(\mathbb{I})$ and a constant $R_2 \in (\phi^*, \Delta(b,\sigma))$ with $R_2 \neq R_1$ such that*

$$f(t,(1+t)^{\alpha-1}u,(1+t)^{\alpha-\beta-1}v) + q(t) \geqslant b(t),$$

for a.e. $t \in [1/\sigma, \sigma]$, all $u \in [\gamma_{\alpha,\sigma}(R_2 - \phi^*), R_2]$, and all $v \in [\gamma_{\alpha,\beta,\sigma}(R_2 - \phi^*), R_2]$, where $\gamma_{\alpha,\sigma} = \min_{s\in[1/\sigma,\sigma]}(\widetilde{\gamma}_\alpha(s))$ and $\gamma_{\alpha,\beta,\sigma} = \min_{s\in[1/\sigma,\sigma]}(\widetilde{\gamma}_{\alpha-\beta}(s))$.

Then, the fbvp (1.1) admits a positive solution u such that for all $\gamma\in[0,\alpha-2)$, $\lim_{t\to+\infty}\frac{u(t)+\phi(t)}{t^{\gamma}}=+\infty$. Moreover,

i) *if for all $\rho>0$*

$$\int_0^{+\infty} t\omega_\rho(t)\Psi_\rho\left(r\widetilde{\gamma}_\alpha(t), r\widetilde{\gamma}_{\alpha-\beta}(t)\right)dt<\infty \text{ for all } r\in(0,\rho]$$

then

$$\lim_{t\to+\infty}\frac{u(t)+\phi(t)}{t^{\alpha-2}}=l\in(0,+\infty), \begin{cases} 0 \text{ if } \delta\in(0,1), \\ \text{if } \delta=1, \end{cases}$$

ii) *if there is $\delta\in(0,1)$ such that for all $\rho>0$ and all $r\in(0,\rho]$*

$$\lim_{t\to+\infty} t^{\delta+1}\omega_\rho(t)\Psi_\rho\left(r\widetilde{\gamma}_\alpha(t), r\widetilde{\gamma}_{\alpha-\beta}(t)\right)=0$$
$$\int_0^{+\infty} t^{\delta}\omega_\rho(t)\Psi_\rho\left(r\widetilde{\gamma}_\alpha(t), r\widetilde{\gamma}_{\alpha-\beta}(t)\right)dt<\infty$$

then

$$\lim_{t\to+\infty}\frac{u(t)+\phi(t)}{t^{\alpha-1-\delta}}=0.$$

Proof. Without loss of generality, assume that $R_1<R_2$ and let $T=T_{R_1,R_2}$ be the operator given by Lemma 3.4 and for all $v\in P\cap\left(\overline{B}(0,R_2)\setminus B(0,R_1)\right)$

$$f_{R_1,R_2}v(s)=f(s,v(s)-\phi(s),D^{\beta}v(s)-D^{\beta}\phi(s))+q(s).$$

For all $v\in P\cap\partial B(0,R_1)$ and all $t\in\mathbb{I}$, the following estimates hold,

$$\begin{aligned}\frac{Tv(t)}{(1+t)^{\alpha-1}} &= \int_0^{+\infty}\frac{G_\alpha(t,s)}{(1+t)^{\alpha-1}}f_{R_1,R_2}v(s)ds \\ &\leqslant \sup_{t>0}\left(\int_0^{+\infty}\frac{G_\alpha(t,s)}{(1+t)^{\alpha-1}}a(s)\,ds\right)\end{aligned}$$

and

$$\begin{aligned}\frac{D^{\beta}Tv(t)}{(1+t)^{\alpha-\beta-1}} &= \int_0^{+\infty}\frac{G_{\alpha-\beta}(t,s)}{(1+t)^{\alpha-\beta-1}}f_{R_1,R_2}v(s)ds \\ &\leqslant \sup_{t>0}\left(\int_0^{+\infty}\frac{G_{\alpha-\beta}(t,s)}{(1+t)^{\alpha-\beta-1}}a(s)\,ds\right).\end{aligned}$$

Passing to the supremum in the above estimates, we get

$$\begin{aligned}&\|Tv\|_1\leqslant\sup_{t>0}\left(\int_0^{+\infty}\frac{G_\alpha(t,s)}{(1+t)^{\alpha-1}}a(s)\,ds\right) \text{ and} \\ &\|Tv\|_2\leqslant\sup_{t>0}\left(\int_0^{+\infty}\frac{G_{\alpha-\beta}(t,s)}{(1+t)^{\alpha-\beta-1}}a(s)\,ds\right),\end{aligned} \tag{4.1}$$

leadig to

$$\|Tv\| \leqslant \Lambda(a) \leqslant R_1 = \|v\|.$$

For all $v \in P \cap \partial B(0,R_2)$ and $s \in [1/\sigma, \sigma]$,

$$\begin{aligned} R_2 &\geqslant \frac{v(s) - \phi(s)}{(1+s)^{\alpha-1}} \geqslant (R_2 - \phi^*)\,\widetilde{\gamma}_\alpha(s) = (R_2 - \phi^*)\,\gamma_{\alpha,\sigma} \\ R_2 &\geqslant \frac{D^\beta v(t) - D^\beta \phi(s)}{(1+s)^{\alpha-\beta-1}} \geqslant (R_2 - \phi^*)\,\widetilde{\gamma}_{\alpha-\beta}(s) = (R_2 - \phi^*)\,\gamma_{\alpha,\beta,\sigma}. \end{aligned} \tag{4.2}$$

Assumption **(b)** and (4.2) lead to the following estimates

$$\begin{aligned} \|Tv\|_1 &\geqslant \sup_{t>0}\left(\int_{1/\sigma}^{\sigma} \frac{G_\alpha(t,s)}{(1+s)^{\alpha-1}} f_{R_1,R_2} v(s)ds\right) \\ &\geqslant \sup_{t>0}\left(\int_{1/\sigma}^{\sigma} \frac{G_\alpha(t,s)}{(1+s)^{\alpha-1}} b(s)\,ds\right) \end{aligned}$$

and

$$\begin{aligned} \|Tv\|_2 &\geqslant \sup_{t>0}\left(\int_{1/\sigma}^{\sigma} \frac{G_{\alpha-\beta}(t,s)}{(1+s)^{\alpha-\beta-1}} f_{R_1,R_2} v(s)ds\right) \\ &\geqslant \sup_{t>0}\left(\int_{1/\sigma}^{\sigma} \frac{G_{\alpha-\beta}(t,s)}{(1+s)^{\alpha-\beta-1}} b(s)\,ds\right). \end{aligned}$$

From the above estimates, we obtain

$$\|Tv\| = \max(\|Tv\|_2, \|Tv\|_2) \geqslant \Delta(b,\sigma) \geqslant R_2 = \|v\|.$$

Thus, it follows from Assertion 1 in Theorem 2.1 that T_{R_1,R_2} admits a fixed point v such that $R_1 \leqslant \|v\| \leqslant R_2$. Then by Lemma 3.4, $u = v - \phi$ is a positive solution to the bvp (1.1).

Now, $v = u + \phi$ satisfies

$$\begin{aligned} \frac{v(t)}{t^\gamma} &= t^\varepsilon\left(\frac{v(t)}{t^{\gamma+\varepsilon}}\right) = t^\varepsilon \int_0^{+\infty} \frac{G_\alpha(t,s)}{t^\gamma} f_{R_1,R_2} v(s)ds \\ &\geqslant t^\varepsilon \int_0^{+\infty} \widetilde{G}_{\alpha-\gamma}(t,s)\, f_{R_1,R_2} v(s)ds = t^\varepsilon w(t), \end{aligned}$$

where $\varepsilon \in (0, \alpha-\gamma-2)$ and

$$\widetilde{G}_{\alpha-\gamma}(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-\gamma-1} - (t-s)^{\alpha-\gamma-1} & s \leqslant t \\ t^{\alpha-\gamma-1} & t \leqslant s. \end{cases}$$

Since $\alpha - \gamma - 2 > 0$ and

$$\begin{aligned} w'(t) = \frac{\alpha-\gamma-1}{\Gamma(\alpha)} &\int_0^t \left(t^{\alpha-\gamma-2} - (t-s)^{\alpha-\gamma-2}\right) f_{R_1,R_2} v(s)ds \\ &+ t^{\alpha-\gamma-2}\int_t^{+\infty} f_{R_1,R_2} v(s)ds > 0, \end{aligned}$$

we have that $\lim_{t\to+\infty} w(t) = l \in (0,+\infty]$, leading to

$$\lim_{t\to+\infty} \frac{v(t)}{t^{\gamma}} = \lim_{t\to+\infty} (t^{\varepsilon} w(t)) = +\infty.$$

At this stage suppose that the condition in Assertion i) hold, then

$$\begin{aligned}\frac{u(t)+\phi(t)}{t^{\alpha-2}} &= \frac{v(t)}{t^{\alpha-2}} = \int_0^{+\infty} \frac{G_\alpha(t,s)}{t^{\alpha-2}} f_{R_1,R_2} v(s)ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{t^{\alpha-1}-(t-s)^{\alpha-1}}{t^{\alpha-2}} f_{R_1,R_2} v(s)ds + \frac{t}{\Gamma(\alpha)} \int_t^{+\infty} f_{R_1,R_2} v(s)ds \\ &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^t \frac{t^{\alpha-1}-(t-s)^{\alpha-1}}{t^{\alpha-2}} f_{R_1,R_2} v(s)ds + \frac{1}{\Gamma(\alpha)} \int_t^{+\infty} s f_{R_1,R_2} v(s)ds.\end{aligned}$$

Clearly, $\lim_{t\to+\infty} \int_t^{+\infty} s f_{R_1,R_2} v(s)ds = 0$ and we have by the mean value theorem

$$(\alpha-1)s\left(\frac{t-s}{t}\right)^{\alpha-2} \leqslant \frac{t^{\alpha-1}-(t-s)^{\alpha-1}}{t^{\alpha-2}} \leqslant (\alpha-1)s.$$

proving that for all $s>0$

$$\lim_{t\to+\infty} \frac{t^{\alpha-1}-(t-s)^{\alpha-1}}{t^{\alpha-2}} = (\alpha-1)s.$$

This with Lebesgue dominated convergence theorem lead to

$$\lim_{t\to+\infty} \frac{u(t)+\phi(t)}{t^{\alpha-2}} = \frac{(\alpha-1)}{\Gamma(\alpha)} \int_0^t s f_{R_1,R_2} v(s)ds.$$

At the end, suppose that the condition in ii) is satisfied, then

$$\begin{aligned}\frac{u(t)+\phi(t)}{t^{\alpha-\delta-1}} &= \frac{v(t)}{t^{\alpha-\delta-1}} = \int_0^{+\infty} \frac{G_\alpha(t,s)}{t^{\alpha-\delta-1}} f_{R_1,R_2} v(s)ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{t^{\alpha-1}-(t-s)^{\alpha-1}}{t^{\alpha-\delta-1}} f_{R_1,R_2} v(s)ds + \frac{t^{\delta}}{\Gamma(\alpha)} \int_t^{+\infty} f_{R_1,R_2} v(s)ds \\ &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^t \frac{t^{\alpha-1}-(t-s)^{\alpha-1}}{t^{\alpha-\delta-1}} f_{R_1,R_2} v(s)ds + \frac{1}{\Gamma(\alpha)} \int_t^{+\infty} s^{\delta} f_{R_1,R_2} v(s)ds.\end{aligned}$$

Clearly, $\lim_{t\to+\infty} \int_t^{+\infty} s^{\delta} f_{R_1,R_2} v(s)ds = 0$ and we have by L'Hopital's rule

$$\begin{aligned}\lim_{t\to+\infty} \frac{u(t)+\phi(t)}{t^{\alpha-\delta-1}} &= \frac{1}{\Gamma(\alpha)} \lim_{t\to+\infty} \int_0^t \frac{\left(t^{\alpha-1}-(t-s)^{\alpha-1}\right)}{t^{\alpha-\delta-1}} f_{R_1,R_2} v(s)ds \\ &\leqslant \frac{1}{\Gamma(\alpha)} \lim_{t\to+\infty} \int_0^t \frac{(\alpha-1)st^{\alpha-2}}{t^{\alpha-\delta-1}} f_{R_1,R_2} v(s)ds \\ &= \frac{(\alpha-1)}{\Gamma(\alpha)} \lim_{t\to+\infty} \int_0^t \left(\frac{s}{t}\right)^{1-\delta} s^{\delta} f_{R_1,R_2} v(s)ds.\end{aligned}$$

Thus, we conclude by Lebesgue dominated convergence theorem that

$$\lim_{t\to+\infty}\frac{u(t)+\phi(t)}{t^{\alpha-\delta-1}}=0.$$

The proof is complete. □

Set for $\sigma>1$

$$f_\sigma=\liminf_{|(w,z)|\to+\infty}\left(\min_{t\in[1/\sigma,\sigma]}\frac{f(t,(1+t)^{\alpha-1}w,(1+t)^{\alpha-\beta-1}z)}{w+z}\right).$$

We obtain from Theorem 4.1 the following corollary:

COROLLARY 4.1. *Suppose that Hypotheses (1.2) and (1.3) hold,*

(c) *there exists $R_1>\phi^*$ such that $\Lambda(a_1)<R_1$ where*

$$a_1(s)=\omega_{R_1}(s)\Psi_{R_1}\left((R_1-\phi^*)\widetilde{\gamma}_\alpha(s),(R_1-\phi^*)\widetilde{\gamma}_{\alpha-\beta}(s)\right)+q(s),$$

(d) *there exists $\sigma>1$, such that $f_\sigma\Delta(b_0,\sigma)>1$, where $b_0(s)=\inf\left(\widetilde{\gamma}_\alpha(s),\widetilde{\gamma}_{\alpha-\beta}(s)\right)$.*

Then, all the conclusions in Theorem 4.1 hold.

Proof. Clearly, Condition (a) in Theorem 4.1 is satisfied for $a=a_1$. We have to prove that Condition (b) is also satisfied. Let $\varepsilon>0$ be such that $(f_\sigma-\varepsilon)\Delta(b_0,\sigma)>1$. There exists R_∞ such that

$$f(t,(1+t)^{\alpha-1}w,(1+t)^{\alpha-\beta-1}z)>(f_\sigma-\varepsilon)(w+z)$$

for all $t\in[1/\sigma,\sigma]$ and all w,z with $|(w,z)|\geqslant R_\infty$. Let

$$R_2=1+\sup\left(R_1,\phi^*+\frac{R_\infty}{\gamma_\sigma},\frac{\phi^*(f_\sigma-\varepsilon)\Delta(b_0,\sigma)}{(f_\sigma-\varepsilon)\Delta(b_0,\sigma)-1}\right)$$

and

$$b(t)=(f_\sigma-\varepsilon)(R_2-\phi^*)\widetilde{\gamma}_\alpha(s)+q(s).$$

where $\gamma_\sigma=\min\left(\gamma_{\alpha,\sigma},\gamma_{\alpha,\beta,\sigma}\right)$ and notice that

$$(f_\sigma-\varepsilon)\Delta(b_0,\sigma)(R_2-\phi^*)>R_2.$$

We have then

$$\begin{aligned}\Delta(b,\sigma)=\min\Bigg(&\sup_{t>0}\left(\int_{1/\sigma}^{\sigma}\tfrac{G_\alpha(t,s)}{(1+t)^{\alpha-1}}\left((f_\sigma-\varepsilon)(R_2-\phi^*)\widetilde{\gamma}_\alpha(s)+q(s)\right)ds\right),\\ &\sup_{t>0}\left(\int_{1/\sigma}^{\sigma}\tfrac{G_{\alpha-\beta}(t,s)}{(1+t)^{\alpha-\beta-1}}\left((f_\sigma-\varepsilon)(R_2-\phi^*)\widetilde{\gamma}_\alpha(s)+q(s)\right)ds\right)\Bigg)\\ &\geqslant(f_\sigma-\varepsilon)\Delta(b_0,\sigma)(R_2-\phi^*)>R_2.\end{aligned}$$

The proof is complete. □

5. Example

In this example we consider the case of the fbvp (1.1) where

$$f(t,u,v) = e^{-t}(g(t,u,v))^p + ce^{-t}\frac{g(t,u,v)}{1+g(t,u,v)} - e^{-2t} \tag{5.1}$$

with

$$g(t,u,v) = \frac{u}{(1+t)^{\alpha-1}} + \frac{v}{(1+t)^{\alpha-\beta-1}},\ p<0 \text{ and } c \in \mathbb{I}.$$

We obtain from Theorem 4.1 the following corollary:

COROLLARY 5.1. *Assume that* $p(\alpha-\beta-1) > -1$ *and* $c\Delta(b_0,\sigma) > 1+\phi^*$ *for some* $\sigma > 1$ *and*

$$b_0(t) = \frac{\min\left(\widetilde{\gamma}_\alpha(t), \widetilde{\gamma}_{\alpha-\beta}(t)\right)}{1+\min\left(\widetilde{\gamma}_\alpha(t), \widetilde{\gamma}_{\alpha-\beta}(t)\right)}$$

where ϕ^* *is that in Lemma 3.2. Then the fbvp (1.1) within* f *given in (5.1), has a positive solution* u *such that* $\lim_{t\to+\infty}\dfrac{u(t)+\phi(t)}{t^{\alpha-1}} \in (0,+\infty)$.

Proof. We have to show that all assumptions of Theorem 4.1 are satisfied. Clearly, Hypothesis (1.2) is satisfied for $q(t) = e^{-2t}$ and we have

$$f(t,(1+t)^{\alpha-1}w,(1+t)^{\alpha-\beta-1}z) = e^{-t}\left((w+z)^p + \frac{c(w+z)}{1+(w+z)} - e^{-t}\right),$$

leading to

$$\begin{aligned}\left|f(t,e^{kt}w,e^{kt}z)\right| &= \left|e^{-t}\left((w+z)^p - e^{-t}\right)\right| \\ &\leqslant e^{-t}\left((w+z)^p + c + 1\right).\end{aligned}$$

Set for all $\rho > 0$

$$\omega_\rho(t) = e^{-t} \quad \text{and} \quad \Psi_\rho(w,z) = (w+z)^p + c + 1.$$

Then

$$\omega_\rho(s)\,\psi_\rho\left(\rho\widetilde{\gamma}_\alpha(s), \rho\widetilde{\gamma}_{\alpha-\beta}(s)\right) = e^{-s}\left(1+c+\rho^p\theta(s)\right)$$

where $\theta(s) = \left(\widetilde{\gamma}_\alpha(s) + \widetilde{\gamma}_{\alpha-\beta}(s)\right)^p$ satisfies

$$\theta(s) \simeq s^{(\alpha-\beta-1)p} \text{ at } 0 \text{ and } \theta(s) \simeq \frac{s^{(2-\alpha)p}}{(\alpha-1)} \text{ at } +\infty.$$

Since $(\alpha-\beta-1)p > -1$, we have

$$\int_0^{+\infty} \omega_\rho(s)\,\psi_\rho\left(\rho\widetilde{\gamma}_\alpha(s), \rho\widetilde{\gamma}_{\alpha-\beta}(s)\right)ds < \infty.$$

The above calculations show that Hypothesis (1.3) is fulfilled.

Now, for

$$a_1(t) = e^{-t}\left((R-\phi^*)^p\,\theta(t)+c+1\right)+e^{-2t}$$

straightforward computations lead to

$$\Lambda(a_1) \leqslant \Lambda(R) = \lambda_0 (R-\phi^*)^p + c + \frac{3}{2},$$

where $\lambda_0 = \int_0^{+\infty} e^{-s}\theta(s)\,ds$.

For R_1 large, we have

$$\Lambda(R_1+\phi^*) = \lambda_0 R_1^p + c + \frac{3}{2} < R_1$$

and Condition (a) in Theorem 4.1 is satisfied.

At this stage, for all $u \in P$ with $R = \|u\| > \phi^*$ and $t \in [1/\sigma, \sigma]$ we have

$$\begin{aligned} & f(t,u(t)-\phi(t),D^\beta u(t)-D^\beta\phi(t))+q(t) \\ \geqslant\ & ce^{-t}\frac{g\left(t,u(t)-\phi(t),D^\beta u(t)-D^\beta\phi(t)\right)}{1+g\left(t,u(t)-\phi(t),D^\beta u(t)-D^\beta\phi(t)\right)} \\ \geqslant\ & ce^{-t}\frac{(\|u\|-\phi^*)\min\left(\widetilde{\gamma}_\alpha(t),\widetilde{\gamma}_{\alpha-\beta}(t)\right)}{1+(\|u\|-\phi^*)\min\left(\widetilde{\gamma}_\alpha(t),\widetilde{\gamma}_{\alpha-\beta}(t)\right)} \end{aligned}$$

and set

$$b_1(t) = ce^{-t}\frac{\min\left(\widetilde{\gamma}_\alpha(t),\widetilde{\gamma}_{\alpha-\beta}(t)\right)}{1+\min\left(\widetilde{\gamma}_\alpha(t),\widetilde{\gamma}_{\alpha-\beta}(t)\right)} = cb_0(t).$$

Therefore, we have for $R = 1+\phi^*$

$$\Delta(b_1,\sigma) = c\Delta(b_0,\sigma) > 1+\phi^* = R$$

and Condition (b) in Theorem 4.1 is satisfied.

At the end, for all $\rho > 0$ and all $r \in (0,\rho]$ we have

$$\int_0^{+\infty} s\omega_\rho(s)\,\psi_\rho\left(r\widetilde{\gamma}_\alpha(s),r\widetilde{\gamma}_{\alpha-\beta}(s)\right)ds = \int_0^{+\infty} se^{-s}\left(1+c+r^p\theta(s)\right)ds < \infty.$$

We conclude from Assertion i) in Theorem 4.1 that

$$\lim_{t\to+\infty}\frac{u(t)+\phi(t)}{t^{\alpha-2}} = l \in (0,+\infty). \quad \square$$

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