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INITIAL–BOUNDARY VALUE AND INVERSE PROBLEMS FOR SUBDIFFUSION EQUATIONS IN $\mathbb{R}^N$

RAVSHAN ASHUROV AND RAKHIM ZUNNUNOV

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Abstract. An initial-boundary value problem for a subdiffusion equation with an elliptic operator $A(D)$ in $\mathbb{R}^N$ is considered. The existence and uniqueness theorems for a solution of this problem are proved by the Fourier method. Considering the order of the Caputo time-fractional derivative as an unknown parameter, the corresponding inverse problem of determining this order is studied. It is proved, that the Fourier transform of the solution $\hat{u}(\xi,t)$ at a fixed time instance recovers uniquely the unknown parameter. Further, a similar initial-boundary value problem is investigated in the case when operator $A(D)$ is replaced by its power A^σ. Finally, the existence and uniqueness theorems for a solution of the inverse problem of determining both the orders of fractional derivatives with respect to time and the degree σ are proved. We also note that when solving the inverse problems, a decrease in the parameter ρ of the Mittag-Leffler functions E_ρ has been proved.

1. Introduction and main results

The theory of differential equations with fractional derivatives has gained significant popularity and importance in the last few decades, mainly due to its applications in many seemingly distant fields of science and technology (see, for example, [1]–[6]).

One of the most important time-fractional equations is the subdiffusion equation, which models anomalous or slow diffusion processes. This equation is a partial integro-differential equation obtained from the classical heat equation by replacing the first-order derivative with a time-fractional derivative of order $\rho \in (0,1)$.

When considering the subdiffusion equation as a model equation in the analysis of anomalous diffusion processes, the order of the fractional derivative is often unknown and difficult to measure directly. To determine this parameter, it is necessary to investigate the inverse problems of identifying these physical quantities based on some indirectly observable information about solutions (see a survey paper Li, Liu and Yamamoto [7]).

In this paper, we investigate the existence and uniqueness of solutions to initial-boundary value problems for subdiffusion equations with the Caputo derivative and an elliptic operator $A(D)$ in $\mathbb{R}^N$, having constant coefficients. Inverse problems of

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determining the order of the fractional derivative with respect to time and with respect to the spatial variable will also be investigated.

Let us proceed to a rigorous formulation of the main results of this article.

1. Let $A(D) = \sum\limits_{|\alpha|=m} a_\alpha D^\alpha$ be a homogeneous symmetric elliptic differential expression of even order $m = 2l$, with constant coefficients, i.e. $A(\xi) > 0$, for all $\xi \neq 0$, where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ - multi-index and $D = (D_1, D_2, \dots, D_N)$, $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$, $i = \sqrt{-1}$.

The fractional integration in the Riemann-Liouville sense of order $\rho < 0$ of a function h defined on $[0, \infty)$ has the form

$$\partial_t^\rho h(t) = \frac{1}{\Gamma(-\rho)} \int\limits_0^t \frac{h(\xi)}{(t-\xi)^{\rho+1}} d\xi, \quad t > 0,$$

provided the right-hand side exists. Here $\Gamma(\rho)$ is Euler's gamma function. Using this definition one can define the Caputo fractional derivative of order ρ, $0 < \rho < 1$, as

$$D_t^\rho h(t) = \partial_t^{\rho-1} \frac{d}{dt} h(t).$$

Note that if $\rho = 1$, then fractional derivative coincides with the ordinary classical derivative of the first order: $D_t h(t) = \frac{d}{dt} h(t)$.

Let $\rho \in (0, 1]$ be a given number and $L_2^\tau(\mathbb{R}^N)$ stand for the Sobolev classes (see the definition in the next section). Consider the initial-boundary value problem: find a function $u(x,t) \in L_2^m(\mathbb{R}^N)$, $t \in [0, T)$, such that (note that this inclusion is considered as a boundary condition at infinity)

$$D_t^\rho u(x,t) + A(D) u(x,t) = 0, \quad x \in \mathbb{R}^N, \quad 0 < t < T, \tag{1}$$

$$u(x, 0) = \varphi(x), \quad x \in \mathbb{R}^N, \tag{2}$$

where $\varphi(x)$ is a given continuous function.

We call problem (1)–(2) *the forward problem.*

We draw attention to the fact, that in the statement of the forward problem the requirement $u(x,t) \in L_2^m(\mathbb{R}^N)$ is not caused by the merits. However, on the one hand, the uniqueness of just such a solution is proved quite simply, and on the other, the solution found by the Fourier method satisfies the above condition.

DEFINITION 1. A function $u(x,t) \in C(\mathbb{R}^N \times [0, T))$ with the properties

$$D_t^\rho u(x,t) \text{ and } A(D) u(x,t) \in C(\mathbb{R}^N \times (0, T))$$

and satisfying conditions (1)–(2) is called the classical solution (or simply, the solution) of the forward problem.

Let us denote by $E_\rho(t)$ the Mittag-Leffler function of the form

$$E_\rho(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\rho k+1)},$$

and denote by $\hat{f}(\xi)$ the Fourier transform of a function $f(x)\in L_2(\mathbb{R}^N)$:

$$\hat{f}(\xi)=(2\pi)^{-N}\int\limits_{\mathbb{R}^N} f(x)\,e^{-ix\xi}\,dx.$$

Now we can formulate the existence and uniqueness theorem for the forward problem.

THEOREM 1. *Let* $\tau>\frac{N}{2}$ *and* $\varphi\in L_2^{\tau}(\mathbb{R}^N)$. *Then the forward problem has a unique solution and this solution has the form*

$$u(x,t)=\int\limits_{\mathbb{R}^N} E_\rho(-A(\xi)t^\rho)\,\hat{\varphi}(\xi)e^{ix\xi}\,d\xi. \tag{3}$$

The integral uniformly and absolutely converges with respect to $x\in\mathbb{R}^N$ *and for each* $t\in[0,T)$. *Moreover, solution (3) has the property*

$$\lim_{|x|\to\infty} D^\alpha u(x,t)=0,\quad |\alpha|\leqslant m,\quad 0<t<T, \tag{4}$$

In recent years, many works by specialists have appeared in which various initial-boundary value problems for various subdiffusion equations are investigated. Let us mention only some of these works. Basically, the case of one spatial variable $x\in\mathbb{R}$ and subdiffusion equation with "the elliptical part" u_{xx} were considered (see, for example, handbook Machado, editor [1], book of A. A. Kilbas et al. [3] and monograph of A. V. Pskhu [8], and references in these works). The paper Gorenflo, Luchko and Yamamoto [9] is devoted to the study of subdiffusion equations in Sobelev spaces. In the paper by Kubica and Yamamoto [10], initial-boundary value problems for equations with time-dependent coefficients are considered. In the multidimensional case ($x\in\mathbb{R}^N$), instead of the differential expression u_{xx}, authors considered either the Laplace operator ([3], [11]–[13]) or pseudodifferential operators with constant coefficients in the whole space $\mathbb{R}^N$ (Umarov [14]). In the last work the initial function $\varphi\in L_p(\mathbb{R}^N)$ is such, that the Fourier transform $\hat{\varphi}$ is compactly supported. The authors of the recent paper [15] considered initial-boundary value problems for subdiffusion equations with arbitrary elliptic differential operators in bounded domains.

2. Determining the correct order of an equation in applied fractional modeling plays an important role. The corresponding inverse problem for subdiffusion equations has been considered by a number of authors (see a survey paper Li, Liu and Yamamoto [7] and references therein, [16]–[22]). Note that in all known works the subdiffusion equation was considered in a bounded domain $\Omega\subset\mathbb{R}^N$. In addition, it should be noted that in publications [16]–[19] the following relation was taken as an additional condition

$$u(x_0,t) = h(t),\ 0 < t < T, \tag{5}$$

at a monitoring point $x_0 \in \overline{\Omega}$. But this condition, as a rule (an exception is the work [19] by J. Janno, where both the uniqueness and existence are proved), can ensure only the uniqueness of the solution of the inverse problem [16]–[18]. The authors of the article Ashurov and Umarov [20] considered the value of the projection of the solution onto the first eigenfunction of the elliptic part of the subdiffusion equation as additional information. Note that the results from [20] are only applicable when the first eigenvalue is zero. The uniqueness and existence of an unknown order of the fractional derivative in the subdiffusion equation were proved in the recent work of Alimov and Ashurov [21]. In this case, the additional condition is $||u(x,t_0)||^2 = d_0$, and the boundary condition is not necessarily homogeneous. The authors of the article [22] investigated the inverse problem for the simultaneous determination of the order of the Riemann-Liouville time fractional derivative and the source function in the subdiffusion equations.

In what follows, we will assume that the initial function φ belongs to the class $L_2^\tau(\mathbb{R}^N)$ with $\tau > \frac{N}{2}$. Then, by Theorem 1, the forward problem has a unique solution of the form (3) for any $\rho \in (0,1]$.

Let us consider the order of fractional derivative ρ in equation (1) as an unknown parameter. To formulate our inverse problem we will additionally assume that $\varphi(x) \in L_1(\mathbb{R}^N)$. This implies that both functions $\hat{\varphi}(\xi)$ and $\hat{u}(\xi,t) = E_\rho(-A(\xi)t^\rho)\,\hat{\varphi}(\xi)$, $t \in [0,T)$, are continuous in the variable $\xi \in \mathbb{R}^N$. Let us fix a vector $\xi_0 \neq 0$, such that $\hat{\varphi}(\xi_0) \neq 0$ and put $\lambda_0 = A(\xi_0) > 0$. To determine the order ρ we use the following extra data:

$$U(t_0,\rho) \equiv |\hat{u}(\xi_0,t_0)| \;=\; d_0, \tag{6}$$

where t_0, $0 < t_0 < T$, is a fixed time instant.

The problem (1)–(2) together with extra condition (6) is called *the inverse problem*.

To solve the inverse problem fix the number $\rho_0 \in (0,1)$ and consider the problem for $\rho \in [\rho_0, 1]$.

DEFINITION 2. The pair $\{u(x,t),\rho\}$ of the solution $u(x,t)$ to the forward problem and the parameter $\rho \in [\rho_0, 1]$ is called the classical solution (or simply, the solution) of the inverse problem.

The following property of the Fourier transform $\hat{u}(\xi,t)$ of the forward problem's solution plays an important role in the solution of the inverse problem and, in our opinion, is of independent interest.

LEMMA 1. *For ρ_0 from the interval $0 < \rho_0 < 1$, there is a number $T_0 = T_0(\lambda_0,\rho_0)$ such that for all t_0, $T_0 \leqslant t_0 < T$, the function $U(t_0,\rho)$ decreases monotonically with respect to $\rho \in [\rho_0, 1]$.*

The result related to the inverse problem has the form.

THEOREM 2. *Let $T_0 \leqslant t_0 < T$. Then the inverse problem has a unique solution $\{u(x,t),\rho\}$ if and only if*

$$e^{-\lambda_0 t_0} \leqslant \frac{d_0}{|\hat{\varphi}(\xi_0)|} \leqslant E_{\rho_0}(-\lambda_0 t_0^{\rho_0}). \tag{7}$$

3. Finally, we will consider another inverse problem of determining both the orders of fractional derivatives with respect to time and the spatial derivatives in the subdiffusion equations.

For the best of our knowledge, only in the following two papers [23] and [24] such inverse problems were studied and only the uniqueness theorems ware proved (note that the uniqueness is a very important property of a solution from an application point of view). In the paper [23] by Tatar and Ulusoy it is considered the initial-boundary value problem for the differential equation

$$\partial_t^\rho u(t,x) = -(-\triangle)^\sigma u(t,x), \quad t>0,\, x\in(0,1),$$

where $\triangle^\sigma$ is the one-dimensional fractional Laplace operator, $\rho \in (0,1)$ and $\sigma \in (1/4,1)$. The authors have proved that if the initial function $\varphi(x)$ is sufficiently smooth and all its Fourier coefficients are positive, then the two-parameter inverse problem with additional information (5) may have only one solution. As for physical backgrounds for two-parameter differential equations, see, for example, [25].

In [24], M. Yamamoto proved the uniqueness theorem for the above two-parameter inverse problem in an N-dimensional bounded domain Ω with a smooth boundary $\partial\Omega$. The conditions for the initial function found in this work are less restrictive, for example, if φ is zero on $\partial\Omega$, $\varphi \in L_2^\tau(\Omega)$, $\tau > N/2$, $\varphi \geqslant 0$ in Ω and $\varphi(x_0) \neq 0$, then the uniqueness theorem is true.

Let us denote by A an operator in $L_2(\mathbb{R}^N)$ with the domain of definition $D(A) = C_0^\infty(\mathbb{R}^N)$, acting as $Af(x) = A(D)f(x)$. It is easy to verify that the closure $\hat{A}$ of operator A is nonnegative and selfadjoint. Therefore, by virtue of the von Neumann theorem, for any $\sigma > 0$, we can introduce the degree of the operator $\hat{A}$ as

$$\hat{A}^\sigma f(x) = \int_0^\infty \lambda^\sigma dP_\lambda f(x) = \int_{\mathbb{R}^N} A^\sigma(\xi)\hat{f}(\xi)e^{ix\xi}d\xi,$$

where projectors P_λ defined as

$$P_\lambda f(x) = \int_{A(\xi)<\lambda} \hat{f}(\xi)e^{ix\xi}d\xi.$$

The domain of definition of this operator is determined from the condition $\hat{A}^\sigma f(x) \in L_2(\mathbb{R}^N)$ and has the form

$$D(\hat{A}^\sigma) = \{f \in L_2(\mathbb{R}^N) : \int_{\mathbb{R}^N} A^{2\sigma}(\xi)|\hat{f}(\xi)|^2 d\xi < \infty\}.$$

Suppose that $\rho \in (0,1]$ and $\sigma \in (0,1]$ are given numbers and consider the initial-boundary value (*the second forward*) problem: find a function $v(x,t) \in D(\hat{A}^\sigma)$ such that (note that this inclusion is also considered as a boundary condition)

$$D_t^\rho v(x,t) + \hat{A}^\sigma v(x,t) = 0, \quad x \in \mathbb{R}^N, \quad 0 < t < T, \tag{8}$$

$$v(x,0) = \varphi(x), \quad x \in \mathbb{R}^N, \tag{9}$$

where $\varphi(x)$ is a given function and as mentioned above, we assume $\varphi \in L_2^\tau(\mathbb{R}^N)$ for some $\tau > \frac{N}{2}$.

The solution to this problem is defined similarly to the solution to problem (1)–(2) (see Definition 1). In exactly the same way as Theorem 1, it is proved that the unique solution of the second forward problem has the form

$$v(x,t) = \int\limits_{\mathbb{R}^N} E_\rho(-A^\sigma(\xi)t^\rho)\,\hat{\varphi}(\xi)\,e^{ix\xi}d\xi, \tag{10}$$

where the integral uniformly and absolutely converges in $x \in \mathbb{R}^N$ and for each $t \in [0,T)$.

Now let $\rho_0 > 0$ and $\sigma_0 > 0$ be fixed numbers and assume, that in the second forward problem, the parameters $\rho \in [\rho_0, 1]$ and $\sigma \in [\sigma_0, 1]$ are unknown. Since there are two unknown numbers, then one obviously needs two extra conditions. To formulate these conditions, we again assume that $\varphi \in L_1(\mathbb{R}^N)$. Then both functions $\hat{\varphi}(\xi)$ and $\hat{v}(\xi,t)$ are continuous in ξ. It should be noted, that the proposed in this paper method, for simultaneously finding both the order of fractional differentiation ρ and the power σ is applicable if there exists $\xi_0 \in \partial\Omega_A \equiv \{\xi \in \mathbb{R}^N; \quad A(\xi) = 1\}$, such that $\hat{\varphi}(\xi_0) \neq 0$. Note, that if $A(D)$ is the Laplace operator, then $\partial\Omega_A$ is the N-dimensional unit sphere. Let ξ_0 be one of such a vector. We consider the following information as additional conditions:

$$V(\xi_0, t_0, \rho, \sigma) = |\hat{v}(\xi_0, t_0)| = d_0, \quad t_0 \geqslant T_0(1, \rho_0), \tag{11}$$

$$V(\xi_1, t_1, \rho, \sigma) = |\hat{v}(\xi_1, t_1)| = d_1, \quad A(\xi_1) = \lambda_1(\neq 1) \geqslant \Lambda_1, \quad t_1 \geqslant 1, \tag{12}$$

where T_0 is defined in Lemma 1, ξ_1 is such that $\hat{\varphi}(\xi_1) \neq 0$ and Λ_1 is defined in (27).

We call the problem (8)–(9) together with extra conditions (11) and (12) *the second inverse problem.*

Note that since $\xi_0 \in \partial\Omega_A$, then $V(\xi_0, t_0, \rho, \sigma)$ is actually independent of σ:

$$V(\xi_0, t_0, \rho, \sigma) = |E_\rho(-A^\sigma(\xi_0)t_0^\rho)\,\hat{\varphi}(\xi_0)| = |E_\rho(-t^\rho)\,\hat{\varphi}(\xi_0)|.$$

Therefore, to solve the second inverse problem, we first find the unique $\rho^\star$ that satisfies the relation (11). Then, assuming that $\rho^\star$ is already known and using the relation (12), we find the second unknown parameter $\sigma^\star$. It should be noted that the number Λ_1 from condition (12) depends on σ_0 and $\rho^\star$.

THEOREM 3. *There is a unique* $\rho^\star \in [\rho_0, 1]$, *satisfying* (11), *if and only if* d_0 *satisfies the inequalities* (7) *with* $\lambda_0 = 1$. *For* $\sigma^\star \in [\sigma_0, 1]$ *to exist, it is necessary and sufficient that* d_1 *satisfy the inequalities*

$$E_{\rho^\star}(-\lambda_1 t_1^{\rho^\star}) \leqslant \frac{d_1}{|\hat{\varphi}(\xi_1)|} \leqslant E_{\rho^\star}(-\lambda_1^{\sigma_0} t_1^{\rho^\star}). \tag{13}$$

REMARK 1. As Theorems 2 and 3 show, in order for the inverse problems to have solutions, the domain $(0,T)$, where the equations are satisfied, must be large enough.

In conclusion, note that the theory and applications of various inverse problems, on determining the coefficients of the equation, the right-hand side, and also on determining the initial or boundary functions for differential equations of integer order are discussed in Kabanikhin [26] (see also references therein) Similar inverse problems for fractional-order equations were considered, for example, in the works [27]–[31].

2. Forward problems

In the present section we prove Theorems 1 and the equation (10).

The class of functions $L_2(\mathbb{R}^N)$ which for a given fixed number $a > 0$ make the norm

$$||f||^2_{L_2^a(\mathbb{R}^N)} = \Big|\Big| \int_{\mathbb{R}^N} (1+|\xi|^2)^{\frac{a}{2}} \hat{f}(\xi) e^{ix\xi} d\xi \Big|\Big|^2_{L_2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (1+|\xi|^2)^a |\hat{f}(\xi)|^2 d\xi$$

finite is termed the Sobolev class $L_2^a(\mathbb{R}^N)$. Since for $\tau > 0$ and some constants c_1 and c_2 one has the inequality

$$c_1(1+|\xi|^2)^{\tau m} \leqslant 1 + A^{2\tau}(\xi) \leqslant c_2(1+|\xi|^2)^{\tau m}, \tag{14}$$

then $D(\hat{A}^\tau) = L_2^{\tau m}(\mathbb{R}^N)$.

Let I be the identity operator in $L_2(\mathbb{R}^N)$. Operator $(\hat{A}+I)^\nu$ is defined in the same way as operator $\hat{A}^\sigma$.

Proof of Theorem 1. The existence of a solution to the forward problem is based on the following lemma (see M. A. Krasnoselski et al. [32], p. 453); for the operator $\hat{A}$ this lemma is a simple consequence of the Sobolev embedding theorem.

LEMMA 2. *Let a multi-index* α *be such that* $|\alpha| \leqslant m$ *and* $\nu > \frac{|\alpha|}{m} + \frac{N}{2m}$. *Then the operator* $D^\alpha(\hat{A}+I)^{-\nu}$ *continuously maps from* $L_2(\mathbb{R}^N)$ *into* $C(\mathbb{R}^N)$ *and moreover the following estimate holds true*

$$||D^\alpha(\hat{A}+I)^{-\nu} f||_{C(\mathbb{R}^N)} \leqslant C||f||_{L_2(\mathbb{R}^N)}. \tag{15}$$

Proof. For any $a > N/2$ one has the Sobolev embedding theorem: $L_2^a(\mathbb{R}^N) \to C(\mathbb{R}^N)$, that is

$$||D^\alpha(\hat{A}+I)^{-\nu} f||_{C(\mathbb{R}^N)} \leqslant C||D^\alpha(\hat{A}+I)^{-\nu} f||_{L_2^a(\mathbb{R}^N)}.$$

Therefore, it is sufficient to prove the inequality

$$||D^{\alpha}(\hat{A}+I)^{-\nu}f||_{L_2^a(\mathbb{R}^N)} \leqslant C||f||_{L_2(\mathbb{R}^N)}.$$

But this is a consequence of the estimate

$$\int\limits_{\mathbb{R}^N} |\hat{f}(\xi)|^2|\xi|^{2|\alpha|}(1+A(\xi))^{-2\nu}(1+|\xi|^2)^a d\xi \leqslant C\int\limits_{\mathbb{R}^N} |\hat{f}(\xi)|^2 d\xi,$$

that is valid for $\frac{N}{2} < a \leqslant \nu m - |\alpha|$.

To prove the existence of the forward problem's solution we remind the following estimate of the Mittag-Leffler function with a negative argument (see, for example, [6], p. 29)

$$|E_\rho(-t)| \leqslant \frac{C}{1+t}, \quad t>0. \tag{16}$$

Let a sequence $\{\Omega_k\}_{k=1}^{\infty}$ of domains $\Omega_k \subset \mathbb{R}^N$ have the following two properties:
1) closure $\overline{\Omega_k} = \Omega_k \cup \partial\Omega_k$ *of* Ω_k *is contained in* Ω_{k+1}:

$$\overline{\Omega_k} \subset \Omega_{k+1};$$

2) the union of all Ω_k *fills the entire space* $\mathbb{R}^N$:

$$\bigcup_{k=1}^{\infty} \Omega_k = \mathbb{R}^N.$$

Consider the truncated integral

$$S_k(x,t) = \int\limits_{\Omega_k} E_\rho(-A(\xi)t^\rho)\,\hat{\varphi}(\xi)\,e^{ix\xi}d\xi. \tag{17}$$

Step 1. It is not difficult to verify that for any k function $S_k(x,t)$ satisfies equation (1) and the initial condition (2) (see, for example, [6], page 173 and [33]). From the Sobolev embedding theorem and the condition $\varphi \in L_2^{\tau}(\mathbb{R}^N)$, $\tau > \frac{N}{2}$, it follows that $\varphi \in C(\mathbb{R}^N)$.

Step 2. In accordance with Definition 1, we will show that for the function (3) one has $A(D)u(x,t) \in C(\mathbb{R}^N \times (0,T))$.

Let $|\alpha| \leqslant m$, $\tau > \frac{N}{2}$ and $\nu = 1 + \frac{\tau}{m} > \frac{|\alpha|}{m} + \frac{N}{2m}$. Then

$$S_k(x,t) = (\hat{A}+I)^{-\tau/m-1}\int\limits_{\Omega_k}(A(\xi)+1)^{\tau/m+1}E_\rho(-A(\xi)t^\rho)\hat{\varphi}(\xi)\,e^{ix\xi}d\xi.$$

Therefore by virtue of Lemma 2 one has

$$\begin{aligned}
&||D^{\alpha}S_k(x,t)||^2_{C(\mathbb{R}^N)} \\
&= ||D^{\alpha}(\hat{A}+I)^{-\tau/m-1}\int\limits_{\Omega_k}(A(\xi)+1)^{\tau/m+1}E_\rho(-A(\xi)t^\rho)\hat{\varphi}(\xi)\,e^{ix\xi}d\xi||^2_{C(\mathbb{R}^N)} \\
&\leqslant C\left|\left|\int\limits_{\Omega_k}(A(\xi)+1)^{\tau/m+1}E_\rho(-A(\xi)t^\rho)\hat{\varphi}(\xi)\,e^{ix\xi}d\xi\right|\right|_{L_2(\mathbb{R}^N)}.
\end{aligned}$$

Using the Parseval equality, we will have

$$||D^\alpha S_k(x,t)||^2_{C(\mathbb{R}^N)} \leqslant C\int\limits_{\Omega_k} \left|(A(\xi)+1)^{\tau/m+1}E_\rho(-A(\xi)t^\rho)\hat{\varphi}(\xi)\right|^2 d\xi.$$

Applying the inequality (16) gives $|(A(\xi)+1)E_\rho(-A(\xi)t^\rho)| \leqslant C(1+t^{-\rho})$. Therefore,

$$||D^\alpha S_k(x,t)||^2_{C(\mathbb{R}^N)} \leqslant C(1+t^{-\rho})^2 \int\limits_{\Omega_k} \left|(A(\xi)+1)^{\tau/m}\hat{\varphi}(\xi)\right|^2 d\xi \leqslant C(1+t^{-\rho})^2 ||\varphi||^2_{L_2^\tau(\mathbb{R}^N)}.$$

This implies uniform (and absolute) in $x\in\mathbb{R}^N$ convergence of the differentiated integral (3) in the variables x_j for each $t\in(0,T)$.

Step 3. If $\alpha=0$, then taking $\nu=\frac{\tau}{m}$ and applying the inequality (16), we establish uniform (and absolute) convergence of the integral (3) (hence, the continuity of the solution) in the domain $t\in[0,T)$:

$$||S_k(x,t)||^2_{C(\mathbb{R}^N)} \leqslant C\int\limits_{\Omega_k} \left|(A(\xi)+1)^{\tau/m}E_\rho(-A(\xi)t^\rho)\hat{\varphi}(\xi)\right|^2 d\xi \leqslant$$

$$\leqslant C\int\limits_{\Omega_k} \left|(A(\xi)+1)^{\tau/m}\hat{\varphi}(\xi)\right|^2 d\xi \leqslant C||\varphi||^2_{L_2^\tau(\mathbb{R}^N)}.$$

Step 4. Further, from equation (1) we get $D_t^\rho S_k(x,t) = -A(D)S_k(x,t)$. Therefore, proceeding the above reasoning, we arrive at $D_t^\rho u(x,t)\in C(\mathbb{R}^N\times(0.T))$.

Step 5. The inclusion $u(x,t)\in L_2^m(\mathbb{R}^N)$ for all $t\in(0,T)$, is a consequence of the condition $\varphi\in L_2(\mathbb{R}^N)$. Indeed, using inequalities (14) and (16) we arrive at

$$||D^\alpha S_k(x,t)||^2_{L_2(\mathbb{R}^N)} = \int\limits_{\Omega_k} \left|\xi^\alpha E_\rho(-A(\xi)t^\rho)\hat{\varphi}(\xi)\right|^2 d\xi \leqslant$$

$$\leqslant C\int\limits_{\Omega_k} \left|A(\xi)E_\rho(-A(\xi)t^\rho)\hat{\varphi}(\xi)\right|^2 d\xi \leqslant C_T t^{-2\rho}||\varphi||^2_{L_2(\mathbb{R}^N)}.$$

Step 6. Let us show the property (4) of the solution (3). To do this, note first that the inclusion $\varphi\in L_2^\tau(\mathbb{R}^N), \tau>N/2$, implies $\hat{\varphi}\in L_1(\mathbb{R}^N)$. Indeed, application of the Hölder inequality gives

$$\int\limits_{\mathbb{R}^N} |\hat{\varphi}(\xi)|d\xi = \int\limits_{\mathbb{R}^N} |\hat{\varphi}(\xi)|(1+|\xi|^2)^{\tau/2}(1+|\xi|^2)^{-\tau/2}d\xi \leqslant C_\tau||\varphi||_{L_2^\tau(\mathbb{R}^N)}.$$

Therefore, by virtue of inequality (16), one has $E_\rho(-A(\xi)t^\rho)\hat{\varphi}(\xi)\in L_1(\mathbb{R}^N)$. Similarly, inequalities (14) and (16) imply

$$|\xi^\alpha E_\rho(-A(\xi)t^\rho)\hat{\varphi}(\xi)| \leqslant C|A(\xi)E_\rho(-A(\xi)t^\rho)\hat{\varphi}(\xi)| \in L_1(\mathbb{R}^N)$$

for all $|\alpha|\leqslant m$. Hence, $D^\alpha u(x,t)$, as a function of x, is the Fourier transform of a L_1-function. Obviously, this implies the property (4).

Step 7. Let us prove the uniqueness of the forward problem's solution.

Suppose that problem (1)–(2) has two solutions $u_1(x,t)$ and $u_2(x,t)$. Our aim is to prove that $u(x,t) = u_1(x,t) - u_2(x,t) \equiv 0$. Since the problem is linear, then we have the following homogenous problem for $u(x,t) \in L_2^m(\mathbb{R}^N)$:

$$D_t^\rho u(x,t) + A(D)u(x,t) = 0, \quad x \in \mathbb{R}^N, \quad 0 < t < T; \tag{18}$$

$$u(x,0) = 0, \quad x \in \mathbb{R}^N. \tag{19}$$

Let $u(x,t)$ be a solution of problem (18)–(19) and $\omega(x)$ be an arbitrary function with properties $\omega(x) \geqslant 0$ and $\omega \in C_0^\infty(\mathbb{R}^N)$. Obviously $\hat{\omega}(\xi) \in L_2(\mathbb{R}^N)$, and since $\hat{u}(\xi,t) \in L_2(\mathbb{R}^N)$, then $\hat{\omega}(\xi)\hat{u}(\xi,t) \in L_1(\mathbb{R}^N)$. Therefore, by virtue of Fubini's theorem, the following function of $t \in [0,T)$ exists for almost all λ:

$$w_\lambda(t) = \int\limits_{A(\xi)=\lambda} e^{iy\xi}\hat{\omega}(\xi)\hat{u}(\xi,t)d\sigma_\lambda(\xi), \tag{20}$$

where $d\sigma_\lambda(\xi)$ is the corresponding surface element and $y \in \mathbb{R}^N$.

Taking into account that $u(x,t)$ is a solution of equation (18) we have (note, $A(D)u(x,t) \in L_2(\mathbb{R}^N)$)

$$D_t^\rho w_\lambda(t) = -(2\pi)^{-N} \int\limits_{A(\xi)=\lambda} e^{iy\xi}\hat{\omega}(\xi) \int\limits_{\mathbb{R}^N} A(D)u(x,t)e^{-ix\xi}dx\,d\sigma_\lambda(\xi).$$

The inner integral exists as the Fourier transform of the L_2-function. From the equation

$$A(D)u(x,t) = \int\limits_{\mathbb{R}^N} A(\eta)\hat{u}(\eta,t)e^{ix\eta}d\eta,$$

one has

$$D_t^\rho w_\lambda(t) = -\int\limits_{A(\xi)=\lambda} e^{iy\xi}\hat{\omega}(\xi)A(\xi)\hat{u}(\xi,t)\,d\sigma_\lambda(\xi) = -\lambda w_\lambda(t).$$

Therefore, we have the following Cauchy problem for $w_\lambda(t)$:

$$D_t^\rho w_\lambda(t) + \lambda w_\lambda(t) = 0, \quad t > 0; \quad w_\lambda(0) = 0.$$

This problem has the unique solution; hence, the function defined by (20), is identically zero (see, for example, [6], p. 173 and [33]): $w_\lambda(t) \equiv 0$ for almost all $\lambda > 0$. Integrating the equation (20) with respect to λ over the domain $(0,+\infty)$ we obtain, that

$$\int\limits_{\mathbb{R}^N} e^{iy\xi}\hat{\omega}(\xi)\hat{u}(\xi,t)d\xi = \int\limits_{\mathbb{R}^N} \omega(y-x)u(x,t)dx = 0,$$

for almost all y and since both functions $\omega(\cdot)$ and $u(\cdot,t)$ are continuous, then for all $y \in \mathbb{R}^N$ and $t \in [0,T)$. Taking into account that the function $\omega(x)$ is arbitrary with the above properties, then from the last equality we have $u(x,t) \equiv 0$.

Thus Theorem 1 is proved.

The uniqueness of the solution to the second forward problem and the formula (10) is established based on the above reasoning.

3. First inverse problem

LEMMA 3. *Given* ρ_0 *from the interval* $0 < \rho_0 < 1$, *there exists a number* $T_0 = T_0(\lambda_0, \rho_0)$, *such that for all* $t_0 \geqslant T_0$ *and* $\lambda \geqslant \lambda_0$ *the function* $e_\lambda(\rho) = E_\rho(-\lambda t_0^\rho)$ *is positive and monotonically decreasing with respect to* $\rho \in [\rho_0, 1]$ *and*

$$e_\lambda(1) \leqslant e_\lambda(\rho) \leqslant e_\lambda(\rho_0).$$

Proof. Let us denote by $\delta(1;\beta)$ a contour oriented by non-decreasing $\arg\zeta$ consisting of the following parts: the ray $\arg\zeta = -\beta$ with $|\zeta| \geqslant 1$, the arc $-\beta \leqslant \arg\zeta \leqslant \beta$, $|\zeta| = 1$, and the ray $\arg\zeta = \beta$, $|\zeta| \geqslant 1$. If $0 < \beta < \pi$, then the contour $\delta(1;\beta)$ divides the complex ζ-plane into two unbounded parts, namely $G^{(-)}(1;\beta)$ to the left of $\delta(1;\beta)$ by orientation, and $G^{(+)}(1;\beta)$ to the right of it. The contour $\delta(1;\beta)$ is called the Hankel path.

Let $\beta = \frac{3\pi}{4}\rho$, $\rho \in [\rho_0, 1)$. Then by the definition of this contour $\delta(1;\beta)$, we arrive at (note, $-\lambda t_0^\rho \in G^{(-)}(1;\beta)$, see [6], p. 27)

$$E_\rho(-\lambda t_0^\rho) = \frac{1}{\lambda t_0^\rho \Gamma(1-\rho)} - \frac{1}{2\pi i \rho \lambda t_0^\rho} \int\limits_{\delta(1;\beta)} \frac{e^{\zeta^{1/\rho}}\zeta}{\zeta + \lambda t_0^\rho} d\zeta = f_1(\rho) + f_2(\rho). \tag{21}$$

To prove the lemma it suffices to show that the derivative $\frac{d}{d\rho} e_\lambda(\rho)$ is negative for all $\rho \in [\rho_0, 1)$, since the positivity of $e_\lambda(\rho)$ follows from the inequality $e_\lambda(1) = e^{-\lambda t} > 0$.

It is not hard to estimate the derivative $f_1'(\rho)$. Indeed, let $\Psi(\rho)$ be the logarithmic derivative of the gamma function $\Gamma(\rho)$ (for the definition and properties of Ψ see [34]). Then $\Gamma'(\rho) = \Gamma(\rho)\Psi(\rho)$, and therefore,

$$f_1'(\rho) = -\frac{\ln t_0 - \Psi(1-\rho)}{\lambda t_0^\rho \Gamma(1-\rho)}.$$

Since

$$\frac{1}{\Gamma(1-\rho)} = \frac{1-\rho}{\Gamma(2-\rho)}, \quad \Psi(1-\rho) = \Psi(2-\rho) - \frac{1}{1-\rho},$$

the function $f_1'(\rho)$ can be represented as follows

$$f_1'(\rho) = -\frac{1}{\lambda t_0^\rho} \frac{(1-\rho)[\ln t_0 - \Psi(2-\rho)] + 1}{\Gamma(2-\rho)}.$$

If $\gamma \approx 0,57722$ is the Euler-Mascheroni constant, then $-\gamma < \Psi(2-\rho) < 1-\gamma$. By virtue of this estimate we may write

$$-f_1'(\rho) \geqslant \frac{(1-\rho)[\ln t_0 - (1-\gamma)] + 1}{\Gamma(2-\rho)\lambda t_0^\rho} \geqslant \frac{1}{\lambda t_0^\rho}, \tag{22}$$

provided $\ln t_0 > 1-\gamma$ or $t_0 \geqslant 2$.

To estimate the derivative $f_2'(\rho)$, we denote the integrand in (21) by $F(\zeta,\rho)$:

$$F(\zeta,\rho)=\frac{1}{2\pi i\rho\lambda t_0^\rho}\cdot\frac{e^{\zeta^{1/\rho}}\zeta}{\zeta+\lambda t_0^\rho}.$$

Note, that the domain of integration $\delta(1;\beta)$ also depends on ρ. To take this circumstance into account when differentiating the function $f_2'(\rho)$, we rewrite the integral (21) in the form:

$$f_2(\rho)=f_{2+}(\rho)+f_{2-}(\rho)+f_{21}(\rho),$$

where

$$f_{2\pm}(\rho)=e^{\pm i\beta}\int_1^\infty F(se^{\pm i\beta},\rho)\,ds,$$

$$f_{21}(\rho)=i\int_{-\beta}^{\beta}F(e^{iy},\rho)\,e^{iy}dy=i\beta\int_{-1}^{1}F(e^{i\beta s},\rho)\,e^{i\beta s}ds.$$

Let us consider the function $f_{2+}(\rho)$. Since $\beta=\frac{3\pi}{4}\rho$ and $\zeta=se^{i\beta}$, then

$$e^{\zeta^{1/\rho}}=e^{\frac{1}{2}(i-1)s^{\frac{1}{\rho}}}.$$

The derivative of the function $f_{2+}(\rho)$ has the form

$$f_{2+}'(\rho)=\frac{1}{2\pi i\rho\lambda t_0^\rho}\int_1^\infty\frac{e^{\frac{1}{2}(i-1)s^{1/\rho}}se^{2ia\rho}\left[-\frac{i-1}{2\rho^2}s^{1/\rho}\ln s+2ia-\frac{1}{\rho}-\ln t_0-\frac{iase^{ia\rho}+\lambda t_0^\rho\ln t_0}{se^{ia\rho}+\lambda t_0^\rho}\right]}{se^{ia\rho}+\lambda t_0^\rho}ds,$$

where $a=\frac{3\pi}{4}$. By virtue of the inequality $|se^{ia\rho}+\lambda t_0^\rho|\geqslant\lambda t_0^\rho$ we arrive at

$$|f_{2+}'(\rho)|\leqslant\frac{C}{\rho(\lambda t_0^\rho)^2}\int_1^\infty e^{-\frac{1}{2}s^{1/\rho}}s\left[\frac{1}{\rho^2}s^{1/\rho}\ln s+\ln t_0\right]ds.$$

LEMMA 4. *Let* $0<\rho\leqslant1$ *and* $m\in\mathbb{N}$. *Then*

$$K(\rho)=\frac{1}{\rho}\int_1^\infty e^{-\frac{1}{2}s^{\frac{1}{\rho}}}s^{\frac{m}{\rho}+1}ds\leqslant C_m.$$

Proof. Set $r=s^{\frac{1}{\rho}}$. Then

$$s=r^\rho,\quad ds=\rho r^{\rho-1}dr.$$

Therefore,

$$K(\rho)=\int_1^\infty e^{-\frac{1}{2}r}r^{m-1+2\rho}dr\leqslant\int_1^\infty e^{-\frac{1}{2}r}r^{m+1}dr=C_m.$$

Since $\ln s^{\frac{1}{\rho}}<s^{\frac{1}{\rho}}$, then by virtue of Lemma 4,

$$|f'_{2+}(\rho)|\leqslant\frac{C}{(\lambda t_0^\rho)^2}\Big[\frac{C_2}{\rho}+C_0\ln t_0\Big]\leqslant\frac{C}{(\lambda t_0^\rho)^2}\Big[\frac{1}{\rho}+\ln t_0\Big].$$

Function $f'_{2-}(\rho)$ has exactly the same estimate.

Now consider the function $f_{21}(\rho)$. It is not hard to verify that

$$f'_{21}(\rho)=\frac{a}{2\pi\lambda t_0^\rho}\int_{-1}^{1}\frac{e^{e^{ias}}e^{2ia\rho s}\Big[2ias-\ln t_0-\frac{iase^{ia\rho s}+\lambda t_0^\rho\ln t_0}{e^{ia\rho s}+\lambda t_0^\rho}\Big]}{e^{ia\rho s}+\lambda t_0^\rho}ds.$$

Therefore,

$$|f'_{21}(\rho)|\leqslant C\frac{\ln t_0}{(\lambda t_0^\rho)^2}.$$

Taking into account estimate (22) and the estimates of $f'_{2\pm}$ and f'_{21}, we have

$$\frac{d}{d\rho}e_\lambda(\rho)<-\frac{1}{\lambda t_0^\rho}+C\frac{1/\rho+\ln t_0}{(\lambda t_0^\rho)^2}. \tag{23}$$

In other words, this derivative is negative if

$$t_0^\rho>C\frac{1/\rho+\ln t_0}{\lambda}$$

for all $\rho\in[\rho_0,1)$ or

$$t_0^{\rho_0}>C\frac{1/\rho_0+\ln t_0}{\lambda}. \tag{24}$$

Thus, there exists a number $T_0=T_0(\lambda_0,\rho_0)$ such, that for all $t_0\geqslant T_0$ we have the estimate

$$\frac{d}{d\rho}e_\lambda(\rho)<0,\quad \lambda\geqslant\lambda_0,\quad \rho\in[\rho_0,1].$$

Since

$$U(t,\rho)=|\hat{u}(\xi_0,t)|=E_\rho(-A(\xi_0)t^\rho)|\hat{\varphi}(\xi_0)|=E_\rho(-\lambda_0t^\rho)|\hat{\varphi}(\xi_0)|,$$

Lemma 1 follows immediately from Lemma 3. Theorem 2 is an easy consequence of these two lemmas.

In conclusion, we make the following remark. If the elliptic polynomial $A(\xi)$ is nonhomogeneous, that is $A(\xi)=\sum_{|\alpha|\leqslant m}a_\alpha\xi^\alpha$ and moreover, $A(\xi)\geqslant\lambda_0>0$, then from Lemma 3 it follows:

If $t_0\geqslant T_0$ and T_0 is as above, then $E_\rho(-A(\xi)t^\rho)$, as a function of ρ, is positive and decreases monotonically in $\rho\in[\rho_0,1]$ for any $\xi\in\mathbb{R}^N$.

Therefore, in this case you can also consider various options for the function $U(t,\rho)$. Examples $U(t,\rho)=||Au(x,t)||^2$ and $U(t,\rho)=||u(x,t)||^2$.

4. Second inverse problem

To prove Theorem 3, we first find the unknown parameter ρ. Suppose, as required by Theorem 3, that d_0 satisfies condition (7) with $\lambda_0 = A(\xi_0) = 1$. Then, as it follows from Lemma 3, for all $t_0 \geqslant T_0(1,\rho_0)$ the equation

$$V(\xi_0,t_0,\rho,\sigma) = |\hat{v}(\xi_0,t_0)| = E_\rho(-t^\rho)|\hat{\varphi}(\xi_0)| = d_0$$

has the unique solution $\rho^\star \in [\rho_0, 1]$.

Now let us define $\sigma^\star \in [\sigma_0, 1]$, which corresponds to the already found $\rho^\star$ and satisfies condition (12).

We first assume, that $\rho^\star < 1$ and let $\beta = \frac{3\pi}{4}\rho^\star$. Then formula (21) will have the form

$$E_{\rho^\star}(-\lambda^\sigma t_1^{\rho^\star}) = \frac{1}{\lambda^\sigma t_1^{\rho^\star}\Gamma(1-\rho^\star)} - \frac{1}{2\pi i\rho^\star\lambda^\sigma t_1^{\rho^\star}} \int_{\delta(1;\beta)} \frac{e^{\zeta^{1/\rho^\star}}\zeta}{\zeta+\lambda^\sigma t_1^{\rho^\star}} d\zeta = g_1(\sigma)+g_2(\sigma). \tag{25}$$

One has

$$g_1'(\sigma) = -\frac{\ln\lambda}{\lambda^\sigma t_1^{\rho^\star}\Gamma(1-\rho^\star)}$$

and

$$g_2'(\sigma) = \frac{(1+t_1^{\rho^\star})\ln\lambda}{2\pi i\rho^\star\lambda^\sigma t_1^{\rho^\star}} \int_{\delta(1;\beta)} \frac{e^{\zeta^{1/\rho^\star}}\zeta}{\zeta+\lambda^\sigma t_1^{\rho^\star}} d\zeta.$$

It is easy to check that $g_2'(\sigma)$ has an estimate (it is proved similarly to the estimate for $f_{2\pm}'$)

$$|g_2'(\sigma)| \leqslant \frac{(1+t_1^{\rho^\star})\ln\lambda}{\pi(\lambda^\sigma t_1^{\rho^\star})^2}\left(C_0+\frac{4}{3}\pi\right) < \frac{5\ln\lambda}{\lambda^{2\sigma}t_1^{\rho^\star}}.$$

Therefore, for all $t_1 \geqslant 1$ we have

$$\frac{d}{d\sigma}E_{\rho^\star}(-\lambda^\sigma t_1^{\rho^\star}) < -\frac{\ln\lambda}{\lambda^\sigma t_1^{\rho^\star}\Gamma(1-\rho^\star)} + \frac{5\ln\lambda}{\lambda^{2\sigma}t_1^{\rho^\star}}. \tag{26}$$

Hence this derivative is negative if

$$\lambda^\sigma \geqslant \lambda^{\sigma_0} \geqslant 5\Gamma(1-\rho^\star).$$

Thus, if $\lambda_1 \geqslant \Lambda_1 = \Lambda_1(\rho^\star,\sigma_0)$, and (see (12))

$$\Lambda_1 = e^n, \quad n \geqslant \frac{\ln(5\Gamma(1-\rho^\star))}{\sigma_0}, \tag{27}$$

then $E_{\rho^\star}(-\lambda^\sigma t_1^{\rho^\star})$, as a function of $\sigma \in [\sigma_0, 1]$, strictly decreases for all $t_1 \geqslant 1$.

Now let $\rho^\star = 1$. Then $E_{\rho^\star}(-\lambda^\sigma t_1^{\rho^\star}) = e^{-\lambda^\sigma t_1}$ and the derivative (26) is negative for all $\lambda > 1$ and $t_1 \geqslant 1$.

Since the function $E_{\rho^\star}(-\lambda^{\sigma} t_1^{\rho^\star})$ is decreasing, then the following estimates hold

$$E_{\rho^\star}(-\lambda_1 t_1^{\rho^\star}) \leqslant E_{\rho^\star}(-\lambda_1^{\sigma} t_1^{\rho^\star}) \leqslant E_{\rho^\star}(-\lambda_1^{\sigma_0} t_1^{\rho^\star}), \quad \lambda_1 \geqslant \Lambda_1, \quad \sigma \in [\sigma_0, 1].$$

The last estimate shows that if d_1 satisfies condition (13), then, assuming $\rho^\star$ has already been found, we can uniquely determine the parameter $\sigma^\star$ from equality (12), that is, from

$$E_{\rho^\star}(-\lambda_1^{\sigma} t_1^{\rho^\star})|\hat{\varphi}(\xi_1)| = d_1.$$

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Ravshan Ashurov
Institute of Mathematics
Uzbekistan Academy of Science
Tashkent, 81 Mirzo Ulugbek str. 100170
e-mail: ashurovr@gmail.com

Rakhim Zunnunov
Institute of Mathematics
Uzbekistan Academy of Science
Tashkent, 81 Mirzo Ulugbek str. 100170
e-mail: zunnunov@mail.ru