Fractional
Differential
Calculus
Volume 11, Number 1 (2021), 1–15
doi:10.7153/fdc-2021-11-01

SOLUTION OF TIME–SPACE FRACTIONAL BLACK–SCHOLES EUROPEAN OPTION PRICING PROBLEM THROUGH FRACTIONAL REDUCED DIFFERENTIAL TRANSFORM METHOD

MANZOOR AHMAD*, RAJSHREE MISHRA AND RENU JAIN

(*Communicated by E. Nane*)

Abstract. Mathematical model introduced by Black and Scholes express financial derivatives more significantly. This model with fractional derivatives resulting in fractional Black-Scholes (B-S) equation express financial problems in a better way. In this paper, we introduce the fractional reduced differential transform method (FRDTM) to solve the time-space fractional Black-Scholes equation executing European options. This method is a modified version of the original differential transform method (DTM). This method proves to be valid for solving time-space Black-Scholes equation as it reduces the computational work to a greater extent. Moreover, this method helps in finding the solution without linearization or discretization. The efficiency of the method is tested by solving certain examples. The proposed mathematical representation can be useful to understand and solve time-space fractional differential equations arising in financial mathematics and other related fields.

1. Introduction

Option pricing has attained a lot of interest as it plays a central part in financial investments. Because of its growing importance in financial industry, the problem has become both theoretical and practical in its nature. In this connection, Black-Scholes (B-S) equation has fascinated considerable attention from researchers being an important and leading mathematical equation in financial mathematics. This equation is the fundamental equation for pricing options. The B-S equation has lead to B-S model obtained under certain assumptions [1]. To weaken these assumptions and to make the B-S model more practical, various methods were employed from time to time to obtain the modified Black-Scholes model that works closer to the actual financial market [2, 3]. The most notable ones are fractional Black-Scholes model [4, 5, 6, 7, 8, 9], stochastic volatility models [10], jump (Levy-stable) processes [11], Black-Scholes model with transaction costs [12], intrinsic parallel difference methods [13], universal difference method [14].

In past few decades, fractional calculus and fractional differential equations have seen remarkable developments. Various phenomena existing in the field of science, engineering, management and finance are well defined by fractional differential equations

Mathematics subject classification (2020): 91G20, 91G30, 91G80, 35Q91.
Keywords and phrases: fractional time-space Black-Scholes equation; fractional derivatives; fractional reduced differential transform method; European option pricing; Mittag-Leffler function.

* Corresponding author.

Paper FDC-11-01

[17]. Differential equations involving fractional derivatives model a number of systems mainly in the fields of heat conduction, visco-elasticity, acoustics, diffusion equation, electro-magnetic waves and material science [18, 19]. In financial mathematics, fractional B-S models are studied widely and extensive progress has been made in recent years to examine these models as the stock prices exhibit stochastic fractional differential equations. The time fractional Black-Scholes equation for European options was first studied by Wyss [5]. Cartea and Del-Castilo-Negrete derived various space fractional B-S equations to price exotic options with jumps [6]. Sunil *et al.* obtained the analytical solution of time fractional Black-Scholes equation governing European options by extending the application of homotopy analysis method and homotopy perturbation method [15]. Jumarie used the fractional Taylor formula to solve time-space fractional B-S equation and derived the optimal fractional Mertons's portfolio [8]. Yang Xiaozhong *et al.* applied universal difference method to arrive at the solution of space time fractional Black-Scholes equation [14]. Yue Li *et al.* introduced the class of intrinsic parallel difference methods to find the solution of space-time fractional Black-Scholes equation [13]. Akrami *et al.* applied the reconstruction of variational iteration method to time fractional Black-Scholes equation for European options. The analytical solutions of time fractional Black-Scholes option pricing equations were derived in the form of Mittag-Leffler functions [20]. Gandheri and Ranjbar obtained the series solution of time fractional B-S equation through extension of the decomposition method [21]. Kumar *et al.* obtained the analytical solution of fractional B-S equation by coupling the homotopy perturbation and Laplace transform method [22]. The fractional differential transform method (FDTM) and fractional modified differential transform method (MFDTM) were applied by Kanth and Aruna to derive the solution of time fractional B-S equation [23]. The fractional reduced differential transform method has been applied to solve space-time fractional order heat and wave equations [16].

The use of differential transform method (DTM) was pioneered by Zhou in electrical circuit analysis [24]. Since then DTM has been widely used by researchers and its applications have been extended to obtain the solution of linear and non-linear differential equations. DTM has been successfully used in various fields like linear and non linear Schrodinger equations [25], partial differential equations [26], space-time heat and wave equations [16], one dimensional Volterra integral equations and integro-differential equations [27] etc. Despite many advantages of DTM like it doesn't require discretization or linearization, some level of difficulty is still met while handling non linear equations or differential equations having variable co-efficients. This provides the room to modify the DTM in many forms by various researchers [28, 29].

In this work, we extend the application of fractional reduced differential transform method to obtain the analytical and approximate solution of time-space fractional Black-Scholes equation governing European options. The rest of the paper is outlined as follows: The basic definitions, mathematical preliminaries and main notations of fractional calculus are inclined in section 2. A brief investigation of fractional reduced differential transform method (FRDTM) is given in section 3. In section 4, FRDTM is employed to solve some time space fractional models governing European options. The results and discussion are presented in section 5. Moreover, the graphical presentation for interpretation of results is given in this section. Finally, the concluding remarks are presented in section 6.

2. Fractional calculus: definitions, notations and preliminaries

This section presents a brief introduction of fractional calculus with respect to its definitions, notations and preliminaries [30, 31, 32]. The important definitions of Riemann Liouville and Caputo fractional order integrals and derivatives used in this work are also given. The gamma function of $f(t)$ is given as:

$$\Gamma(z)=\int_0^{\infty} t^{z-1}e^{-t}dt,\ \Re(z)\in N, \tag{1}$$

with

$$z!=\Gamma(z+1),\quad \left(\Gamma\left(\frac{1}{2}\right)\right)^2=\pi.$$

Suppose

$$\begin{aligned} f(z)&=z^k,\\ Df(z)&=\frac{df(z)}{dz}=kz^{k-1},\\ D^2f(z)&=\frac{d^2f(z)}{dz^2}=k(k-1)z^{k-2}=\frac{k!}{(k-2)!}z^{k-2}, \end{aligned}$$

In general

$$D^m f(z)=\frac{d^m f(z)}{dz^m}=\frac{k!}{(k-m)!}z^{k-m}.$$

This can be expressed in gamma function as:

$$D^{\alpha}f(z)=\frac{d^{\alpha}f(z)}{dz^{\alpha}}=\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}z^{k-\alpha},$$

where $D^{\alpha}f(z)$ is the fractional derivative of $f(z)$ of order α, $\alpha\in\mathbb{R}$.

DEFINITION 1. The fractional integral operator $J_a^{\alpha}f(z)$ of a function $f\in U_{\eta}$, $\eta\geqslant -1$ in Riemann-Liouville sense is defined as [32, 33]:

$$J_a^{\alpha}f(z)=\frac{1}{\Gamma\alpha}\int_a^z (z-t)^{\alpha-1}f(t)dt,\quad \alpha>0,\ z>0. \tag{2}$$

DEFINITION 2. The Riemann-Liouville definition of fractional differential operator $D_a^{\alpha}f(z)$ of order $\alpha>0$ is given as [32, 33]:

$$D_a^{\alpha}f(z)=\left\{\begin{array}{ll} \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_a^z (z-t)^{m-\alpha-1}f(t)dt, & m-1<\alpha<m\in\mathbb{N}\\ \left(\frac{d}{dz}\right)^m f(z), & \alpha=m\in N \end{array}\right\}. \tag{3}$$

DEFINITION 3. The Caputo definition of fractional order derivative $D_a^{\alpha}(\alpha > 0)$ of $f(z)$ is given as [34]:

$$CD_a^{\alpha} f(z) = \left\{ \begin{array}{ll} \frac{1}{\Gamma(m-\alpha)} \int_a^z (z-t)^{m-\alpha-1} f^{(m)}(t)dt, & \\ & m-1 < \alpha < m \in \mathbb{N} \\ \left(\frac{d}{dz}\right)^m f(z), & \alpha = m \in \mathbb{N} \end{array} \right\}, \tag{4}$$

where a is the initial value of function f and α defines the order of the derivative.

DEFINITION 4. The Caputo definition of time fractional derivative of order $\alpha > 0$ is given as [16]:

$$CD_t^{\alpha} f(x,y,z,t) = \frac{\partial^{\alpha} f(x,y,z,t)}{\partial t^{\alpha}} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\xi)^{(m-\alpha-1)} \frac{\partial^m f(x,y,z,\xi)}{\partial \xi^m} d\xi, & \\ & m-1 < \alpha < m \\ \frac{\partial^m f(x,y,z,t)}{\partial t^m}, & \alpha = m \in N. \end{cases} \tag{5}$$

DEFINITION 5. The space fractional derivative of order $\beta > 0$ in Caputo's sense is given as [16]:

$$CD_x^{\beta} f(x,y,z,t) = \frac{\partial^{\beta} f(x,y,z,t)}{\partial x^{\beta}} = \begin{cases} \frac{1}{\Gamma(m-\beta)} \int_0^x (t-\xi)^{(m-\beta-1)} \frac{\partial^m f(\xi,y,z,t)}{\partial \xi^m} d\xi, & \\ & (m-1) < \beta < m \\ \frac{\partial^m f(x,y,z,t)}{\partial x^m}, & \beta = m \in N. \end{cases} \tag{6}$$

REMARK 1. The Riemann-Liouville fractional differential operator and Caputo fractional differential operator for $\alpha \in R$, $n-1 < \alpha < n$ are related as:

$$CD_a^{\alpha} t^m = D^{\alpha} t^m - \sum_{m=0}^{n-1} \frac{t^{m-\alpha}}{\Gamma(m+1-\alpha)} f^{(m)}(0).$$

DEFINITION 6. The generalization of exponential function e^z is given as [35]:

$$E_{\alpha}(z) = \sum_{p=0}^{\infty} \frac{z^p}{\Gamma(\alpha p+1)}, \qquad \alpha > 0, \ \alpha \in \mathbb{R}, \ z \in \mathbb{C}. \tag{7}$$

$E_{\alpha}(z)$ is known as the Mittag-Leffler function in one parameter.

The Mittag-Leffler function in two parameter type is given as [36]:

$$E_{\alpha,\beta}(z) = \sum_{p=0}^{\infty} \frac{z^p}{\Gamma(\alpha p+\beta)}, \qquad (\alpha > 0, \ \beta > 0). \tag{8}$$

3. Fractional reduced differential transform method

Let $f(x,t)$ be an analytic and continuously differentiable function with respect to time and space, then the Taylor series expansion of $f(x,t)$ with respect to $t = t_0$ is given by [16]:

$$f(x,t) = \sum_{h=0}^{\infty} \frac{1}{\Gamma(\alpha h+1)}[(D_t^{\alpha})^h f(x,t)]_{t=t_0}\ (t-t_0)^{\alpha h}. \tag{9}$$

The fractional reduced differential transform $F_h^{\alpha}(x)$ of $f(x,t)$ at $t = t_0$ is given as:

$$F_h^{\alpha}(x) = \frac{1}{\Gamma(\alpha h+1)}[(D_t^{\alpha})^h f(x,t)]_{t=t_0}, \tag{10}$$

where $0 < \alpha \leqslant 1$ and D_t^{α} represents the fractional differential operator with respect to time of order α. The fractional reduced differential inverse transform of $F_h^{\alpha}(x)$ is defined as [16]:

$$f(x,t) = \sum_{h=0}^{\infty} F_h^{\alpha}(x)(t-t_0)^{\alpha h}. \tag{11}$$

Substituting equation (10) in Eq. (11), we get:

$$f(x,t) = \sum_{h=0}^{\infty} \frac{(t-t_0)^{\alpha h}}{\Gamma(\alpha h+1)}[(D_t^{\alpha})^h f(x,t)]_{t=t_0}. \tag{12}$$

In real application, the function $f(x,t)$ can be approximated by a finite series:

$$f_m^*(x,t) = \sum_{h=0}^{m} F_h^{\alpha}(x)(t-t_0)^{\alpha h}, \tag{13}$$

where m denotes the order of the approximation. Therefore, the exact solution can be obtained as follows:

$$f(x,t) = \lim_{m\to\infty} f_m^*(x,t) = \sum_{h=0}^{\infty} F_h^{\alpha}(x)(t-t_0)^{\alpha h}. \tag{14}$$

In above Eq. (14), if we put $\alpha = 1$, the fractional reduced differential transform method converts to regular differential transform method [24].

From above definitions, some basic properties and fundamental operations of FRDTM are listed below [37, 38]:

THEOREM 1. *If* $v(X,t) = u(X,t) + w(X,t)$, *then* $V_h^{\alpha}(X) = U_h^{\alpha}(X) + W_h^{\alpha}(X)$.

THEOREM 2. *If* $v(X,t) = \lambda u(X,t)$, *then* $V_h^{\alpha}(X) = \lambda U_h^{\alpha}(X)$.

THEOREM 3. *If* $v(X,t) = \frac{\partial u(X,t)}{\partial x}$, *then* $V_h^{\alpha}(X) = \frac{\partial U_h^{\alpha}(X)}{\partial x}$.

THEOREM 4. *If* $v(X,t) = D_{t_0}^{\alpha} u(X,t)$, *then* $V_h^{\alpha}(X) = \frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} U_{h+1}^{\alpha}(X)$.

THEOREM 5. *If* $v(X,t) = x_1^{l_1} x_2^{l_2} x_3^{l_3} t^m$, *then* $V_h^{\alpha}(X) = x_1^{l_1} x_2^{l_2} x_3^{l_3} \delta(\alpha h - m)$ *where* $\delta(\alpha h - m) = \begin{Bmatrix} 1 & \text{if } \alpha h = m \\ 0 & \text{if } \alpha h \neq m \end{Bmatrix}$

THEOREM 6. *If* $v(X,t) = D_{t_0}^{N\alpha} u(X,t)$, *then* $V_h^{\alpha}(X) = \frac{\Gamma(\alpha h+N\alpha+1)}{\Gamma(\alpha h+1)} U_{h+N}^{\alpha}(X)$.

THEOREM 7. *If* $v(X,t) = (\frac{\partial^{\beta}}{\partial x_i{}^{\beta}}) u(X,t)$, *then* $V_h^{\alpha}(X) = (\frac{\partial^{\beta}}{\partial x_i{}^{\beta}}) U_h^{\alpha}(X)$.

4. Time-space fractional Black-Scholes equations

In this section the proposed method is applied on some time-space fractional Black-Scholes models governing European options.

MODEL 4.1. Consider the following time-space fractional Black-Scholes equation for European options given as:

$$\frac{\partial^{\alpha}}{\partial \tau^{\alpha}} v(x,\tau) - \left(r + \frac{1}{2}\sigma^{\beta}\cos\left(\frac{\beta\pi}{2}\right)\right)\frac{\partial}{\partial x} v(x,\tau) + \frac{1}{2}\sigma^{\beta}\cos\left(\frac{\beta\pi}{2}\right)\frac{\partial^{\beta}}{\partial_{+} x^{\beta}} v(x,\tau) = -rv(x,\tau), \tag{15}$$

subject to the initial condition:

$$v(x,0) = e^x - 1, \tag{16}$$

where $0 < \alpha \leqslant 1$, $0 < \beta \leqslant 2$, $\frac{\partial^{\beta}}{\partial_{+} x^{\beta}}$ is the left fractional derivative in Riemann-Liouville sense, $v(x,\tau)$ represents the option price or option premium at asset price x and at time τ, σ represents the volatility, r represents the risk free interest rate.

When $\alpha \to 1$ and $\beta \to 2$, equation (15) coincides with the regular Black-Scholes equation:

$$\frac{\partial \upsilon}{\partial \tau} = \frac{\partial^2 \upsilon}{\partial x^2} + (k-1)\frac{\partial \upsilon}{\partial x} - k\upsilon, \qquad \text{where } k = \frac{2r}{\sigma^2}. \tag{17}$$

Equation (15) can be written as:

$$\frac{\partial^{\alpha}}{\partial \tau^{\alpha}} v(x,\tau) = -p\frac{\partial^{\beta}}{\partial_{+} x^{\beta}} v(x,\tau) + (r+p)\frac{\partial}{\partial x} v(x,\tau) - rv(x,\tau), \tag{18}$$

where $p = \frac{1}{2}\sigma^{\beta}\cos(\frac{\beta\pi}{2})$.

Applying the fractional reduced differential transform method on equation (18), we have:

$$V_{h+1}^{\alpha}(x) = \frac{\Gamma(\alpha h+1)}{\Gamma(\alpha(h+1)+1)}\left\{ -p\frac{\partial^{\beta}}{\partial_{+} x^{\beta}} V_h^{\alpha}(x) + (r+p)\frac{\partial}{\partial x} V_h^{\alpha}(x) - rV_h^{\alpha}(x)\right\}, \tag{19}$$

with initial condition given as:

$$V_0^{\alpha}(x) = (e^x - 1). \tag{20}$$

For $h = 0,1,2,3,\dots$ using recurrence relation (19) and initial condition (20), we get:

$$\begin{aligned}
V_1^{\alpha}(x) &= \frac{1}{\Gamma(\alpha+1)}\Big\{ -p\frac{\partial^{\beta}}{\partial_+ x^{\beta}}V_0^{\alpha}(x) + (r+p)\frac{\partial}{\partial x}V_0^{\alpha}(x) - rV_0^{\alpha}(x)\Big\} \\
&= \frac{1}{\Gamma(\alpha+1)}\Big\{ -p\frac{\partial^{\beta}}{\partial_+ x^{\beta}}(e^x-1) + (r+p)\frac{\partial}{\partial x}(e^x-1) - r(e^x-1)\Big\} \\
&= \frac{1}{\Gamma(\alpha+1)}\Big\{ -p\Big[\frac{\partial^{\beta}}{\partial_+ x^{\beta}}(e^x) - \frac{\partial^{\beta}}{\partial_+ x^{\beta}}(1)\Big] + (r+p)\frac{\partial}{\partial x}(e^x-1) - r(e^x-1)\Big\} \\
&= \frac{1}{\Gamma(\alpha+1)}\Big\{ -p\Big[D_*^{\beta}(e^x) + \sum_{k=0}^{m-1}\frac{x^{k-\beta}}{\Gamma(k+1-\beta)} - \frac{1}{\Gamma(1-\beta)}(x)^{-\beta}\Big] + pe^x + r\Big\} \\
&= \frac{1}{\Gamma(\alpha+1)}\Big\{ -p\Big[x^{3-\beta}E_{1,4-\beta}(x) + \frac{x^{2-\beta}}{\Gamma(3-\beta)} + \frac{x^{1-\beta}}{\Gamma(2-\beta)} + \frac{x^{-\beta}}{\Gamma(1-\beta)} \\
&\quad - \frac{x^{-\beta}}{\Gamma(1-\beta)}\Big] + pe^x + r\Big\} \\
&= \frac{1}{\Gamma(\alpha+1)}\{-p[x^{1-\beta}E_{1,2-\beta}(x)] + pE_{1,1}(x) + r\} \\
&= \frac{1}{\Gamma(\alpha+1)}\{p[E_{1,1}(x) - x^{1-\beta}E_{1,2-\beta}(x)] + r\}
\end{aligned} \tag{21}$$

where $D_*^{\beta}(e^x)$ is the Caputo fractional derivative.

$$\begin{aligned}
V_2^{\alpha}(x) &= \frac{1}{\Gamma(2\alpha+1)}\Big\{ -p\frac{\partial^{\beta}}{\partial_+ x^{\beta}}V_1^{\alpha}(x) + (r+p)\frac{\partial}{\partial x}V_1^{\alpha}(x) - rV_1^{\alpha}(x)\Big\} \\
&= \frac{1}{\Gamma(2\alpha+1)}\left\{\begin{array}{c} -p\frac{\partial^{\beta}}{\partial_+ x^{\beta}}[pE_{1,1}(x) - px^{1-\beta}E_{1,2-\beta}(x) + r] \\ +(r+p)\frac{\partial}{\partial x}[pE_{1,1}(x) - px^{1-\beta}E_{1,2-\beta}(x) + r] \\ -r[pE_{1,1}(x) - px^{1-\beta}E_{1,2-\beta}(x) + r] \end{array}\right\} \\
&= \frac{1}{\Gamma(2\alpha+1)}\left\{\begin{array}{c} p^2[x^{1-2\beta}E_{1,2-2\beta}(x) - x^{-2\beta}E_{1,1-2\beta}(x) + x^{-1-\beta}E_{1,\beta}(x) \\ -x^{-\beta}E_{1,1-\beta}(x)] - pr[x^{-2\beta}E_{1,1-2\beta}(x) - x^{-1-\beta}E_{1,\beta}(x) \\ -x^{1-\beta}E_{1,2-\beta}(x) + E_{1,1}(x)] \end{array}\right\}
\end{aligned} \tag{22}$$

Using fractional reduced differential transform we have:

$$v(x,\tau) = \sum_{h=0}^{\infty} V_h^{\alpha}(x)\tau^{\alpha h}. \tag{23}$$

The solution through fractional reduced differential transform method is given as:

$$v(x,\tau)=\begin{cases}E_{1,1}(x)-1+\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\{p[E_{1,1}(x)-x^{1-\beta}E_{1,2-\beta}(x)]+r\}\\ +\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}\{p^2[x^{1-2\beta}E_{1,2-2\beta}(x)-x^{-2\beta}E_{1,1-2\beta}(x)+x^{-1-\beta}E_{1,\beta}(x)\\ -x^{-\beta}E_{1,1-\beta}(x)]-pr[x^{-2\beta}E_{1,1-2\beta}(x)-x^{-1-\beta}E_{1,\beta}(x)\\ -x^{1-\beta}E_{1,2-\beta}(x)+E_{1,1}(x)]\}+\ldots\end{cases} \tag{24}$$

where $E_{\alpha,\beta}(x)$ is the Mittag-Leffler function in two parameter [36]. By putting $p=1$, $r=-k$, $\alpha\to 1$, $\beta\to 2$ in equation (24) and using the properties of Mittag-Leffler function, most of the terms in equation (24) get canceled and the simplified solution is obtained as:

$$v(x,\tau)=e^{x}-e^{-k\tau}, \tag{25}$$

which is the exact solution of the Black-Scholes equation (15).

MODEL 4.2. Consider the following time-space fractional Black-Scholes equation for European options given as:

$$\frac{\partial^{\alpha}v}{\partial\tau^{\alpha}}+x^2\frac{\partial^{\beta}v}{\partial x^{\beta}}+0.5x\frac{\partial v}{\partial x}-v=0,\quad 0<\alpha\leqslant 1,\quad 0<\beta\leqslant 2, \tag{26}$$

subject to the initial condition:

$$v(x,0)=\max(x^3-0)=\begin{cases}x^3, & \text{for } x>0\\ 0, & \text{for } x<0\end{cases} \tag{27}$$

Taking the fractional reduced differential transform method of equations (26), we get:

$$V^{\alpha}_{h+1}(x)=\frac{\Gamma(\alpha h+1)}{\Gamma(\alpha(h+1)+1)}\Big[-x^2\frac{\partial^{\beta}V^{\alpha}_{h}(x)}{\partial_{*}x^{\beta}}-0.5x\frac{\partial V^{\alpha}_{h}(x)}{\partial x}+V^{\alpha}_{h}(x)\Big], \tag{28}$$

where $\frac{\partial^{\beta}}{\partial_{*}x^{\beta}}$ is the fractional derivative in Caputo sense.

With initial condition:

$$V^{\alpha}_{0}(x)=x^3. \tag{29}$$

For $h=0,1,2,3,\ldots$ using the recurrence relation (28) and initial condition (29), we get:

$$
\begin{aligned}
V_1^{\alpha}(x) &= \frac{1}{\Gamma(\alpha+1)}\Big[-x^2\frac{\partial^{\beta}V_0^{\alpha}(x)}{\partial_* x^{\beta}} - 0.5x\frac{\partial V_0^{\alpha}(x)}{\partial x} + V_0^{\alpha}(x)\Big] \\
&= \frac{1}{\Gamma(\alpha+1)}\Big[-\frac{6}{\Gamma(4-\beta)}x^{5-\beta} - 0.5x^3\Big] \\
V_2^{\alpha}(x) &= \frac{1}{\Gamma(2\alpha+1)}\Big[-x^2\frac{\partial^{\beta}V_1^{\alpha}(x)}{\partial_* x^{\beta}} - 0.5x\frac{\partial V_1^{\alpha}(x)}{\partial x} + V_1^{\alpha}(x)\Big] \\
&= \frac{1}{\Gamma(2\alpha+1)}\Big[\frac{6\Gamma(6-\beta)}{\Gamma(4-\beta)\Gamma(6-2\beta)}x^{7-2\beta} + \frac{(12-3\beta)}{\Gamma(4-\beta)}x^{5-\beta} + 0.25x^3\Big] \\
V_3^{\alpha}(x) &= \frac{1}{\Gamma(3\alpha+1)}\Big[-x^2\frac{\partial^{\beta}V_2^{\alpha}(x)}{\partial_* x^{\beta}} - 0.5x\frac{\partial V_2^{\alpha}(x)}{\partial x} + V_2^{\alpha}(x)\Big] \\
&= \frac{1}{\Gamma(3\alpha+1)}\left\{\begin{array}{c} -\frac{6\Gamma(6-\beta)\Gamma(8-2\beta)}{\Gamma(4-\beta)\Gamma(6-2\beta)\Gamma(8-3\beta)}x^{9-3\beta} - \frac{(27-9\beta)\Gamma(6-\beta)}{\Gamma(4-\beta)\Gamma(6-2\beta)}x^{7-2\beta} \\ -\frac{(1.5\beta^2-10.5\beta+19.5)}{\Gamma(4-\beta)}x^{5-\beta} - 0.125x^3 \end{array}\right\} \\
&\ \ \vdots
\end{aligned} \tag{30}
$$

Using fractional reduced differential inverse transform we have:

$$
\begin{aligned}
v(x,\tau) &= \sum_{h=0}^{\infty} V_h^{\alpha}(x)\tau^{\alpha h} \\
&= \begin{cases} x^3 + \frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\big[-\frac{6}{\Gamma(4-\beta)}x^{5-\beta} - 0.5x^3\big] \\ +\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}\big[\frac{6\Gamma(6-\beta)}{\Gamma(4-\beta)\Gamma(6-2\beta)}x^{7-2\beta} + \frac{(12-3\beta)}{\Gamma(4-\beta)}x^{5-\beta} + 0.25x^3\big] \\ +\frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)}\big[-\frac{6\Gamma(6-\beta)\Gamma(8-2\beta)}{\Gamma(4-\beta)\Gamma(6-2\beta)\Gamma(8-3\beta)}x^{9-3\beta} \\ -\frac{(27-9\beta)\Gamma(6-\beta)}{\Gamma(4-\beta)\Gamma(6-2\beta)}x^{7-2\beta} - \frac{(1.5\beta^2-10.5\beta+19.5)}{\Gamma(4-\beta)}x^{5-\beta} - 0.125x^3\big] + \dots \end{cases}
\end{aligned} \tag{31}
$$

By putting $\alpha \to 1$, $\beta \to 2$ in equation (31) and using the properties of Mittag-Leffler function, most of the terms in equation (31) get canceled and the simplified solution is obtained as:

$$v(x,\tau) = x^3 e^{-6.5\tau}, \tag{32}$$

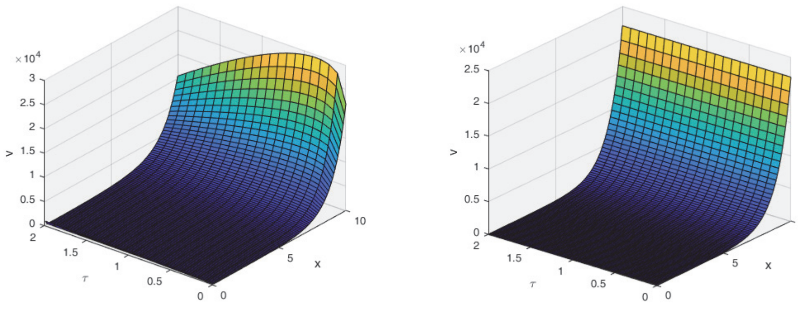
which is the exact solution of the Black-Scholes equation (26).

5. Results and discussion

In this section, the series solutions of above time space fractional European option models are computed through MATLAB.

Figure $1(a)$ shows the surface plot of model 4.1 with order $\alpha = 0.5$, $\beta = 1.5$, $0 \leqslant x \leqslant 10$ and $0 \leqslant \tau \leqslant 2$. The results demonstrate that the value of v increases significantly when the value of x increases. With increasing x from $0-7$ the value of

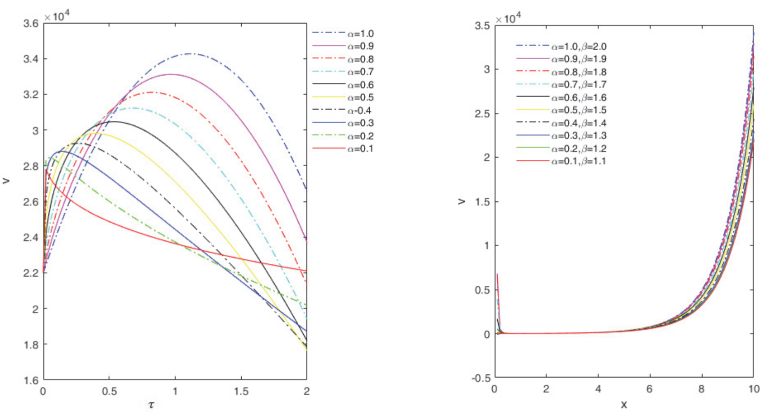
v reaches to zero. After that the solution increases significantly. Figure $1(b)$ shows the exact solution of model 4.1.



(a) (b)

Figure 1: (a) *demonstrates the solution of model* 4.1. *for* $\alpha = 0.5$, $\beta = 1.5$, $r = 1$ *and* $p = 1$ *and* (b) *exact solution for model* 4.1 *at* $k = 1$.

Figure $2(a)$ shows the surface plot of the model 4.1 with different orders of $\alpha = 0.1, 0.2, \ldots, 1.0$ and $\beta = 2$. By setting $x = 10$, $0 \leqslant \tau \leqslant 2$, the solution of model 4.1 is plotted in figure $2(a)$. Figure $2(b)$ shows the solution of model 4.1 by setting $\tau = 1$, $0 \leqslant x \leqslant 10$ with order $\alpha = 0.1, 0.2, 0.3, \ldots 1.0$ and order $\beta = 1.1, 1.2, \ldots, 2.0$.

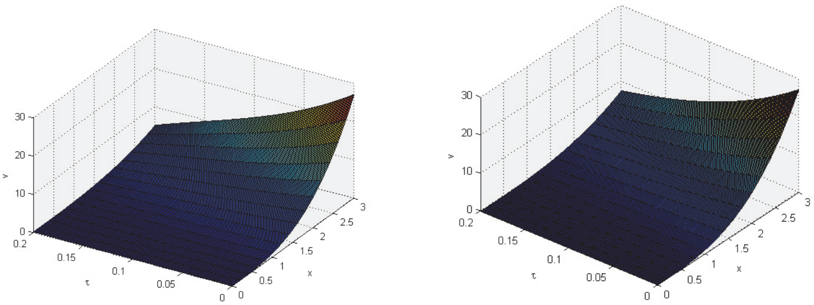


(a) (b)

Figure 2: (a) *demonstrates the solution of model* 4.1 *for different values of* α *and* (b) *demonstrates the solution of model* 4.1 *for different values of* α *and* β.

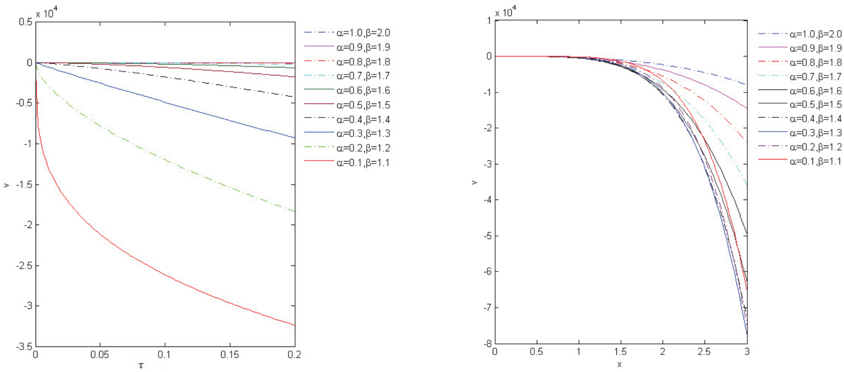
The solution of model 4.2 is plotted in Figures $3-4$. Figure $3(a)$ shows the surface plot of the solution of model 4.2 by setting $\alpha = 1$, $\beta = 2$, $0 \leqslant x \leqslant 3$ and $0 \leqslant \tau \leqslant 0.2$. Figure $3(b)$ shows the surface plot of the exact solution of model 4.2. Figure $4(a)$ shows the solution v with respect to τ for different orders of $\alpha = 0.1.0.2, \ldots 1.0$ and $\beta = 1.1, 1.2, \ldots, 2.0$ with $x = 3$ and $0 \leqslant \tau \leqslant 0.2$. In figure $4(b)$, the solution v

is shown by setting $\tau = 3$, $0 \leqslant x \leqslant 3$ with different orders of $\alpha = 0.1, 0.2, \ldots, 1.0$ and $\beta = 1.1, 1.2, \ldots, 2.0$.



(a) (b)

Figure 3: (a) *demonstrates the solution of model* 4.2 *for* $\alpha = 1$, $\beta = 2$ *and* (b) *exact solution for model* 4.2.

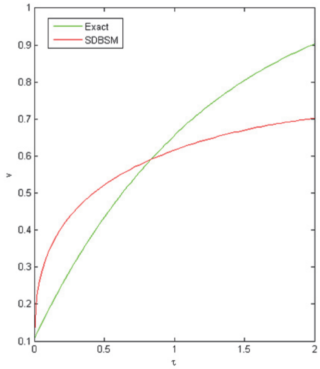


(a) (b)

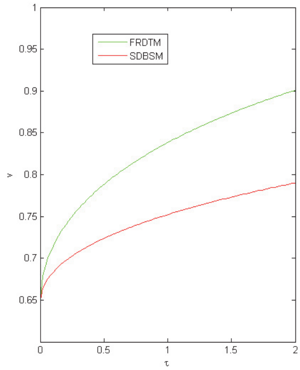
Figure 4: (a) *demonstrates the solution of model* 4.2 *for different values of* α, β *with respect to* τ *and* (b) *demonstrates the solution of model* 4.2 *for different values of* α, β *with respect to* x.

In figure $5(a,b)$, the results are compared to the time fractional Black-scholes formula in subdiffusive regime (SDBSM) [39]. The exact solution of model 4.1 is compared to SDBSM and the results are shown in figure $5(a)$. While the FRDTM results of model 4.1 are compared to SDBSM and the comparison analysis are shown in figure $5(b)$.

The approximate numerical solutions of model 4.1 for different values of α, β, x and τ are shown in table 1. While the numerical solutions of model 4.2 for different values of α, β, x and τ are shown in table 2.



(a)



(b)

Figure 5: (a) *demonstrates the comparison of exact solution with SDBSM presented in* [39] *of model* 4.1 *and* (b) *demonstrates the comparison of FRDTM solution with SDBSM* [39] *of model* 4.1.

Table 1: *Comparison of numerical results of model* 4.1 *through FRDTM for different values of* α *and* β.

τ	x	$\alpha = 0.5$, $\beta = 1.5$	$\alpha = 1$, $\beta = 1.5$	$\alpha = 1.5$, $\beta = 2$	$\alpha = 1$, $\beta = 2$
0.25	0.40	4.5667	1.1260	0.6420	1.3357
	0.50	3.3585	1.1286	0.7814	1.2329
	0.60	2.8797	1.2354	0.9514	1.3014
	0.75	2.7491	1.5003	1.2530	1.5504
	1.00	3.2504	2.1351	1.8838	2.1980
0.50	0.40	8.4117	2.6808	1.1628	3.3215
	0.50	5.8084	2.1751	1.1522	2.4114
	0.60	4.6499	2.0404	1.2629	2.1285
	0.75	4.0509	2.1508	1.5416	2.1679
	1.00	4.3620	2.7616	2.2086	2.7789
0.75	0.40	12.2189	5.1563	2.3099	6.4494
	0.50	8.2154	3.7880	1.8784	4.1841
	0.60	6.3727	3.2372	1.8103	3.3034
	0.75	5.2985	3.0683	1.9915	2.9691
	1.00	5.4044	3.5887	2.6710	3.4608
1.00	0.40	16.0064	8.5523	4.3582	10.7191
	0.50	10.6003	5.9674	3.0973	6.5511
	0.60	8.0711	4.8257	2.6686	4.8260
	0.75	6.5179	4.2530	2.6352	3.9541
	1.00	6.4109	4.6193	3.2796	4.2437

Table 2: *Comparison of numerical results of model* 4.2 *through FRDTM for different values of* α *and* β.

τ	x	$\alpha = 0.5$ $\beta = 1.5$	$\alpha = 1$ $\beta = 1.5$	$\alpha = 1.5$ $\beta = 2$	$\alpha = 1$ $\beta = 2$
0.25	0.40	−0.2800	0.0264	0.0313	−0.0013
	0.50	−0.7736	0.0448	0.0611	−0.0025
	0.60	−1.7671	0.0658	0.1055	−0.0043
	0.75	−4.8402	0.0950	0.2061	−0.0084
	1.00	−17.6879	−0.0289	0.4886	−0.0199
0.50	0.40	−0.9274	−2.7831	−0.0051	−0.1722
	0.50	−2.4980	−0.0960	−0.0101	−0.3363
	0.60	−5.6155	−0.2408	−0.0174	−0.5811
	0.75	−15.1463	−0.7125	−0.0339	−1.1349
	1.00	−54.5141	−2.7831	−0.0804	−2.6901
0.75	0.40	−1.8069	−0.1772	−0.0411	−0.7233
	0.50	−4.8342	−0.4936	−0.0804	−1.4127
	0.60	−10.8171	−1.1351	−0.1389	−2.4412
	0.75	−29.0351	−3.1355	−0.2712	−4.7679
	1.00	−103.9540	−11.5877	−0.6429	−11.3018
1.00	0.40	−2.8744	−0.4935	−0.1341	−1.9293
	0.50	−7.6723	−1.3443	−0.2618	−3.7682
	0.60	−17.1266	−3.0486	−0.4524	−6.5115
	0.75	−45.8505	−8.3091	−0.8839	−12.7178
	1.00	−163.6740	−30.2951	−2.0946	−30.1458

6. Conclusion

The time space fractional Black-Scholes model is the generalization of regular Black-Scholes model. As a result the fractional B-S model proves to be more adequate and competent than the regular B-S model. In this study, the fractional reduced differential transform method (FRDTM) is applied to solve time space fractional option models with boundary conditions governing European options. The solutions are obtained in convergent series forms. The obtained results converge faster when compared to their exact form of solutions. Theoretical analysis prove that FRDTM is a powerful technique to find the approximate analytical solution of time space fractional Black-Scholes equation as it needs less computation with no linearization or perturbation required. A few examples are solved to test the efficiency and performance of the proposed method and the numerical results are in strong agreement with the theoretical analysis. Thus, it can be concluded that FRDTM is an appropriate method to solve time space fractional Black-Scholes equation and other linear and nonlinear stochastic differential equations existing in the field of financial mathematics.

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(Received March 16, 2020)

Manzoor Ahmad
School of Mathematics and Allied Sciences
Jiwaji University Gwalior-474011, India
e-mail: ju_somaas.mz@jiwaji.edu*

Rajshree Mishra
Govt. Model Science College
Gwalior-474011, India
e-mail: ju_somaas.rm@jiwaji.edu

Renu Jain
School of Mathematics and Allied Sciences
Jiwaji University Gwalior-474011, India
e-mail: ju_somaas.rj@jiwaji.edu